Advanced Studies in Pure Mathematics 66, 2015
Singularities in Geometry and Topology 2011
pp. 93–110

On classes in the classification of curves on rational surfaces with respect to logarithmic plurigenera

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Dedicated to Professor Shigeru Iitaka on his seventieth birthday

Abstract.

Let C be a nonsingular curve on a rational surface S. In the case when the logarithmic 2 genus of C is equal to two, Iitaka proved that the geometric genus of C is either zero or one and classified such pairs (S, C). In this article, we prove the existence of these classes with geometric genus one in Iitaka's classification. The curve in the class is a singular curve on $\mathbb{P}^2$ or the Hirzebruch surface Σ_d and its singularities are not in general position. For this purpose, we provide the arrangement of singular points by considering invariant curves under a certain automorphism of Σ_d.

§1. Introduction

In this article, we study the existence of curves on rational surfaces which appear in the classification of curves with respect to logarithmic plurigenera. Here, we use the word *curves* and *surfaces* to mean irreducible varieties of dimension one and two, respectively. First, we recall basic notions of birational geometry of plane curves (see [5], [6] and [7]).

Let S be a complex surface and C a curve on S. A pair (S, C) is *birationally equivalent* to another pair (W, D) if there exists a birational map $h\colon S \to W$ such that the proper image of C by h coincides with D. A pair (S, C) is called a *nonsingular pair* if S and C are nonsingular. Let K_S be the canonical divisor of S. For a nonsingular pair (S, C) and a positive integer m, we denote the dimension of $H^0(S, \mathcal{O}_S(m(K_S+C)))$ by $P_m[C]$ and call it the *logarithmic m genus* of (S, C). It is easy to see

Received May 23, 2012.
Revised September 28, 2012.
2010 *Mathematics Subject Classification.* 14H45, 14J26, 14E20.
Key words and phrases. Plane curve, rational surface, double cover.

that $P_m[C]$ is birational invariant of a pair (S, C). Given a pair (W, D), there exists a nonsingular pair (S, C) which is birationally equivalent to (W, D). We define $P_m[D]$ to be $P_m[C]$. If we assume that S is rational, then $P_1[C]$ coincides with the geometric genus $g(C)$ of C.

By using the value of $P_1[C] = g(C)$, we obtain the classification of curves with respect to the topological type of C. We infer that the invariant $P_m[C]$ is useful to characterize algebraic curves on rational surfaces. For example, we know the following:

Theorem 1. (Coolidge [1] (cf. Iitaka [7, Theorem 1])) *Let S be a complex rational surface and C a curve on S. If $P_2[C] = 0$, then (S, C) is birationally equivalent to $(\mathbb{P}^2, L)$, where L is a projective line.*

A singular point of multiplicity m is called a *m-ple point.*

Theorem 2. (Coolidge [1] (cf. Iitaka [7, Theorem 1])) *Let S be a complex rational surface and C a curve on S. If $P_2[C] = 1$, then (S, C) is birationally equivalent to one of the following pairs*:
(i) $(\mathbb{P}^2, C_1)$, *and*
(ii) $(\mathbb{P}^2, C'_m)$ $(m \geq 2)$,
where C_1 is an elliptic curve and C'_m is a plane curve of degree $3m$ with nine m-ple points and one double point. (*These points may be infinitely near points.*)

From the above theorems, we see that pairs (S, C) with $P_2[C] \leq 1$ are classified into three types.

Let $pr_d \colon \Sigma_d \to \mathbb{P}^1$ be the d-th Hirzebruch surface, Δ_∞ the minimal section of Σ_d and F a fiber of pr_d. The symbol $\sim$ means the linear equivalence between divisors.

Iitaka classified pairs (S, C) with $P_2[C] = 2$ into ten classes.

Theorem 3. (Iitaka [8, Theorems 4 and 10], [7, pp. 290–291]) *Let S be a complex rational surface and C a curve on S. If $P_2[C] = 2$, then $g(C)$ is either 0 or 1. Moreover,*
(a) *if $g(C) = 1$, then C is birationally equivalent to one of the following curves*:
(i) *a plane curve C_m $(m \geq 2)$ of degree $3m$ with nine m-ple points,*
(ii) $D_8 \sim 8\Delta_\infty + (8 + 4d)F$ $(d = 0, 1, 2)$ *on Σ_d, where D_8 has seven quadruple points and two triple points,*
(iii) $D_6 \sim 6\Delta_\infty + (6 + 3d)F$ $(d = 0, 1, 2)$ *on Σ_d, where D_6 has seven triple points and three double points, and*
(iv) $D_4 \sim 4\Delta_\infty + (5 + 2d)F$ $(d = 0, 1, 2)$ *on Σ_d, where D_4 has eleven double points,*
(b) *if $g(C) = 0$, C is birationally equivalent to one of the following curves*:

(i) $E_{12} \sim 12\Delta_\infty + (12+6d)F$ $(d=0,1,2)$ *on* Σ_d, *where* E_{12} *has seven sextuple points, a quintuple point and a quadruple point,*
(ii) $E_{10} \sim 10\Delta_\infty + (11+5d)F$ $(d=0,1,2)$ *on* Σ_d, *where* E_{10} *has nine quintuple points,*
(iii) $D'_8 \sim 8\Delta_\infty + (8+4d)F$ $(d=0,1,2)$ *on* Σ_d, *where* D'_8 *has seven quadruple points, two triple points and a double point,*
(iv) $E_6 \sim 6\Delta_\infty + (7+3d)F$ $(d=0,1,2)$ *on* Σ_d, *where* E_6 *has ten triple points,*
(v) $D'_6 \sim 6\Delta_\infty + (6+3d)F$ $(d=0,1,2)$ *on* Σ_d, *where* D'_6 *has seven triple points and four double points, and*
(vi) $D'_4 \sim 4\Delta_\infty + (5+2d)F$ $(d=0,1,2)$ *on* Σ_d, *where* D'_4 *has twelve double points,*
where these singular points may be infinitely near singular points.

By [2, Proposition 1], a plane curve of degree $3m$ with nine m-ple points for integer $m \geq 2$ is realized as a general member of a Halphen pencil (see also [3, Theorem 2.1, Remark 2.6]). On the other hand, it is unknown that there exist the other pairs in Theorem 3. The aim of this article is to prove the existence of classes in the case that the geometric genus of C is equal to one, i.e., we show the following:

Theorem 4. *Under the same notation as in Theorem* 3, *there exist curves* D_8, D_6 *and* D_4, *i.e., there exist all classes in Iitaka's classification of pairs* $(S,\ C)$ *with* $P_2[C]=2$ *and* $g(C)=1$.

It may be well-known that there exists a plane curve of degree $3m$ with nine m-ple points and one double point. However, for lack of a suitable reference, we prove its existence.

Theorem 5. *Under the same notation as in Theorem* 2, *there exists a curve* C'_m *for* $m \geq 2$, *i.e., there exist all classes in Coolidge's classification of pairs* $(S,\ C)$ *with* $P_2[C]=0,1$.

For simplicity, we use the notation of types of curves. Let C be a curve on $\mathbb{P}^2$ or Σ_d and ν a succession of r blowing-ups which resolves the singularity of C. Let m_i be the the multiplicity of i-th center of the blowing-up appeared in ν. We can assume that $m_1 \geq m_2 \geq \cdots \geq m_r$ by rearranging in a suitable order of these blowing-ups.

Definition 1.1. (Iitaka [7, p. 291]) For an above plane curve C, if the degree of C is equal to α, then we say C is of type $[\alpha\,;m_1, m_2, \ldots, m_r]$. In the case that C is nonsingular, we put $r=1$ and $m_r=1$.

Definition 1.2. (Iitaka [7, p. 294]) For an above curve C on Σ_d, if C is linearly equivalent to $\alpha\Delta_\infty + \beta F$, then we say C is of type

$[\alpha * \beta,\, d\,;\, m_1, m_2, \ldots, m_r]$. In the case that C is nonsingular, we put $r = 1$ and $m_r = 1$.

Whenever $m_i = m_{i+1} = \cdots = m_{i+k-1}$, for simplicity, we denote $[\alpha,\, d\,;\, m_1, m_2, \ldots, m_r]$ by $[\alpha,\, d\,;\, m_1, \ldots, m_{i-1}, m_i^k, m_{i+k} \ldots, m_r]$ and denote $[\alpha * \beta,\, d\,;\, m_1, m_2, \ldots, m_r]$ by $[\alpha * \beta,\, d\,;\, m_1, \ldots, m_{i-1}, m_i^k, m_{i+k} \ldots, m_r]$. With this notation, C_1, C_m, C'_m, D_8, D_6 and D_4 are of types $[3;\, 1]$, $[3m;\, m^9]$, $[3m\,;\, m^9, 2]$, $[8*(8+4d),\, d\,;\, 4^7, 3^2]$, $[6*(6+3d),\, d\,;\, 3^7, 2^3]$ and $[4*(5+2d),\, d\,;\, 2^{11}]$, respectively.

Curves as in Theorem 3 have many singular points. In order to construct desired curves, we provide the arrangement of singularities.

In Section 2, we recall Iitaka's classification of nonsingular pairs $(S,\, C)$ with $P_2[C] = 2$.

In Sections 3 and 4, we give a curve of type $[6*(6+3d),\, d\,;\, 3^7, 2^3]$ for $d = 0, 1, 2$. Let $f\colon \Sigma_d \to \Sigma_{2d}$ be the double cover branched along $\Delta_\infty + \Delta$, where Δ is a section of Σ_{2d} such that $\Delta_\infty \cdot \Delta = 0$. We construct desired curves C which are inverse images of certain curves on Σ_{2d} by f. In other words, our curves are invariant under the automorphism with order two induced by f, which implies that singular points are in special position. To complete the proof, we shall give the defining polynomials of these quotient curves.

The similar technique used in this section is applied in [11, Proposition 3.1] and [9, Section 2]. This method is also applied in the later section.

In Sections 5 and 6, we show the existence of curves of types $[8*(8+4d),\, d\,;\, 4^7, 3^2]$ and $[4*(5+2d),\, d\,;\, 2^{11}]$ for $d = 0, 1, 2$.

In Section 7, we show that there exists a curve of type $[3m\,;\, m^9, 2]$, which may be well-known. We choose nine points $a_1, a_2, \ldots, a_8$ and a_9 on $\mathbb{P}^2$ such that $m(a_1 + a_2 + \cdots + a_9) = 0_E$, where 0_E is the zero element with respect to the group operation $+$ on an elliptic curve E. Then the surface obtained by the succession of blowing ups at all the a_i's has the structure of an elliptic surface with one multiple fiber of multiplicity m (see [3, Theorem 2.1]). By imposing another condition for a_i's, we shall prove the existence of a plane curve with degree $3m$ which has m-ple points at a_i's and a double point.

§2. Classification of pairs with logarithmic 2 genus two

In this section, we recall the classification of pairs with logarithmic 2 genus two due to Iitaka.

Let $(S,\, C)$ be a pair of a complex rational surface S and a curve C on S. Denote the degree of the Hilberto polynomial of $\bigoplus_{m \geq 0} H^0(S,$

$\mathcal{O}_S(m(K_S + C)))$ by $\kappa[C]$ and call it the *Kodaira dimension* of (S, C). If $P_m[C] = 0$ for any $m > 0$, then we put $\kappa[C] = -\infty$.

In the case that $\kappa[C] = 0, 1$, then a pair (S, C) is classified as in the following:

Proposition 6. (Iitaka [7, pp. 290–291]) *Let S be a complex rational surface and C a curve on S with $g(C) > 0$.*
(i) *If $\kappa[C] = 0$, then (S, C) is birationally equivalent to $(\mathbb{P}^2, C_1)$, where C_1 is of type* $[3\,;\,1]$.
(ii) *If $\kappa[C] = 1$, then (S, C) is birationally equivalent to one of the following pairs*:
(a) $(\mathbb{P}^2, B_m)$ $(m \geq 2)$, *and*
(b) $(\mathbb{P}^2, C_m)$ $(m \geq 4)$,
where B_m is of type $[m\,;\,m-2]$ *and C_m is of type* $[3m\,;\,m^9]$.

Proposition 7. (Iitaka [6, Proposition 2]) *Let S be a complex rational surface and C a curve on S with $g(C) = 0$.*
(i) *If $\kappa[C] = 0$, then (S, C) is birationally equivalent to $(\mathbb{P}^2, C_2')$, where C_2' is of type* $[6\,;\,2^{10}]$.
(ii) *If $\kappa[C] = 1$, then (S, C) is birationally equivalent to $(\mathbb{P}^2, C_m')$ $(m \geq 3)$, where C_m' is of type* $[3m\,;\,m^9, 2]$.

By an easy calculation, we obtain $P_2[C_1] = 1$, $P_2[B_m] = 2m - 5$, $P_2[C_m] = 2$ and $P_2[C_m'] = 1$. Thus, if $P_2[C] = 2$ and $\kappa[C] \leq 1$ then (S, C) is birationally equivalent to $(\mathbb{P}^2, C_m)$. In particular, we obtain $g(C) = 1$.

Next, we consider the case that $\kappa[C] = 2$. Then, we have $P_2[C] = (C + K_S)^2 + 2g(C) - 1$ (see [8, Proposition 2]). By [7, Proposition 3], if $g(C) \geq 2$, then we obtain $P_2[C] \geq 3$. In particular, $g(C)$ is either 0 or 1. In the case that $P_2[C] = 2$ and $g(C) = 0, 1$, by [8, Theorems 4 and 10], pairs (S, C) are classified into nine types. By the above argument, we obtain Theorem 3.

§3. Construction of a curve of type $[6 * 6,\ 0\,;\ 3^7, 2^3]$

Let $pr_d\colon \Sigma_d \to \mathbb{P}^1$ be the d-th Hirzebruch surface, Δ_∞ the minimal section of Σ_d and F a fiber of pr_d. Now we recall the elementary transformation. Let p be a point on Σ_d. By blowing up at p, we obtain the birational morphism $\sigma\colon S_1 \to \Sigma_d$. Then the self-intersection number of the proper transform of the fiber contained p is equal to -1. By contracting this (-1)-curve into nonsingular point, we obtain the birational morphism $\sigma'\colon S_1 \to S_2$. We call $\sigma \circ {\sigma'}^{-1}$ the *elementary transformation* centered at p. If $p \in \Delta_\infty$, then $S_2 = \Sigma_{d+1}$ and we denote $\sigma \circ {\sigma'}^{-1}$ by

$I_+(p)$. If $p \not\in \Delta_\infty$, then $S_2 = \Sigma_{d-1}$ and we denote $\sigma \circ {\sigma'}^{-1}$ by $I_-(p)$. We denote the intersection multiplicity of divisors D_1 and D_2 at p by $m_p(D_1, D_2)$.

In this section, we construct a curve of type $[6*6,\ 0\ ;\ 3^7, 2^3]$. By considering elementary transformations and a certain double cover of Σ_0, we show the following:

Lemma 8. *Let $a_1, a_2, \ldots, a_4$ and a_5 be points on Σ_0 such that $pr_0(a_1) = pr_0(a_3)$. If there exist three curves D, Q and R on Σ_0 satisfying the following conditions*:
(i) *$D \sim 3\Delta_\infty + F$, $Q \sim \Delta_\infty + F$ and $R \sim \Delta_\infty + 2F$,*
(ii) *$m_{a_1}(D, Q) = 1$ and $m_{a_2}(D, Q) = 2$,*
(iii) *$m_{a_3}(D, R) = 1$ and $m_{a_4}(D, R) = m_{a_5}(D, R) = 2$,*
(iv) *D meets Q and R transversally except for $a_1, a_2, \ldots, a_4$ and a_5, and*
(v) *$D \cap Q \cap R = \emptyset$,*
then there exist a curve of type $[6*6,\ 0\ ;\ 3^7, 2^3]$.

Proof. From $Q \cdot R = 3$, we assume that Q meets R at b_1, b_2 and b_3. Note that these points may be infinitely near points. By abuse of notations a_i and b_j, we use the same notations to describe the images of points by birational maps. Let $\nu = I_-(a_3) \circ I_+(b_3) \circ I_-(b_2) \circ I_+(b_1)$. Note that ν is the birational map from Σ_0 to Σ_0. Let D_1, Q_1 and R_1 be the proper transforms of D, Q and R by ν, respectively.

Since $b_i \not\in D$, it follows that D_1 has three triple points, which may be infinitely near points. Furthermore, since $a_3 \in D$, there exists a node on D_1, say c. Then, from $pr_0(a_1) = pr_0(a_3)$, we obtain $m_c(D_1, Q_1) = 3$. By hypothesis $Q^2 = 2$ and $R^2 = 4$, we derive ${Q_1}^2 = {R_1}^2 = 0$, which implies that $Q_1 \sim R_1 \sim \Delta_\infty$. From $D_1 \cdot Q_1 = 6$, we obtain $D_1 \sim 3\Delta_\infty + 6F$ (see Fig. 1).

In Fig. 1, thin curves denote Q, R, Q_1 and R_1. Broken lines denote fibers of pr_d and thick curves denote D and D_1. (In Figs. 3, 5 and 9, curves are represented in a similar manner as Fig. 1.)

Let $f\colon \Sigma_0 \to \Sigma_0$ be the double cover of Σ_0 branched along $Q_1 + R_1$ and C the inverse image of D by f. Since one of analytic branches of D_1 at c meets Q_1 tangentially, the singular point of C induced by c is an ordinary triple point. Therefore, C has seven triple points, which may be infinitely near points. Moreover, D_1 meets $Q_1 + R_1$ tangentially at a_2, a_4 and a_5, which implies that singular point of C induced by these points are nodes (see Fig. 2). In Fig. 2, thin curves denote Q_1, R_1 and these inverse image by f. Broken lines denote fibers of pr_d and thick curves denote D_1 and C. (In Figs. 4, 7 and 10, curves are represented in a similar manner as Fig. 2.)

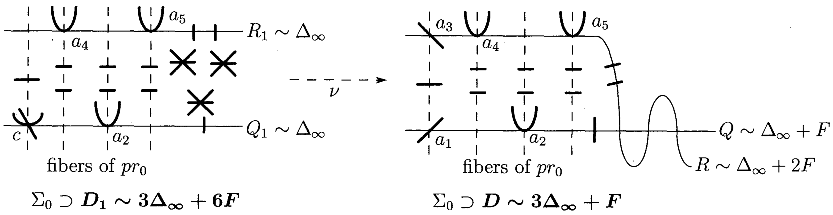

Fig. 1. The arrangement of D_1, Q_1 and R_1

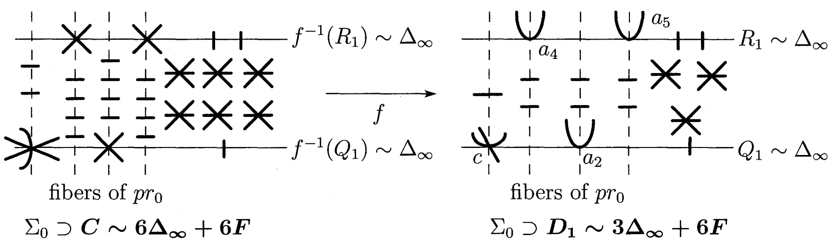

Fig. 2. Singular points of C

Since D is irreducible and there exist points at which D_1 meets Q_1+R_1 transversally, C is irreducible. Thus, C is of type $[6{*}6,\ 0\,;\ 3^7, 2^3]$. Q.E.D.

By the above lemma, it suffices to construct curves D, Q and R on Σ_0 satisfying conditions (i), (ii), (iii), (iv) and (v). Let (x, y) be the affine coordinate of $\mathbb{P}^1 \setminus \{\infty\} \times \mathbb{P}^1 \setminus \{\infty\} \subset \Sigma_0$. By giving the defining equations in x and y, we show the following:

Proposition 9. *For* $d = 0, 1$, *there exist a curve of type* $[6 * 6 + 3d,\ d;\ 3^7, 2^3]$.

Proof. We give divisors D, Q and R by the following equations:

$$D\colon x^3 - x^2y - x - 2y = 0,$$
$$Q\colon 8xy + 75x - 122y = 0,$$
$$R\colon 8xy^2 + 35xy - 50y^2 + 20x - 56y - 20 = 0.$$

It is clear that D, Q and R satisfy the condition (i) in Lemma 8. Put $a_1 = (0,0)$, $a_2 = (4, 10/3)$, $a_3 = (1,0)$, $a_4 = (2,1)$ and $a_5 = (-2,-1)$. Then, by calculating of partial derivatives of variables x and y of these defining polynomials, we can verify that conditions (ii), (iii), (iv) and (v) in Lemma 8 are satisfied.

Suppose that D is not irreducible. Since $D \sim 3\Delta_\infty + F$, one of irreducible components of D is linearly equivalent to F or Δ_∞. But the defining polynomial of D can not be divided by a polynomial in a variable x or y. Therefore, we see that D is irreducible. By the similar argument, Q and R are irreducible. Hence, there exists a curve C of type $[6 * 6,\ 0\,;\ 3^7, 2^3]$.

Let p be one of triple points of C. Then the proper transform of C by $I_+(p)$ is of type $[6 * 9,\ 1\,;\ 3^7, 2^3]$. Q.E.D.

Remark 10. Let Q and R be curves as in the proof of Proposition 9. Then, Q meets R transversally.

§4. Construction of a curve of type $[6 * 12,\ 2\,;\ 3^7, 2^3]$

In the previous section, we construct curves of types $[6 * (6+3d),\ d\,;\ 3^7, 2^3]$ $(d = 0, 1)$. In this section, we construct a curve of type $[6 * 12,\ 2\,;\ 3^7, 2^3]$ by a similar method as in the previous section.

Lemma 11. *Let $a_1, a_2, \ldots, a_4$ and a_5 be points on Σ_0 such that $pr_0(a_1) = pr_0(a_2)$. If there exist three curves D, Q and R on Σ_0 satisfying the following conditions*:
(i) *$D \sim 3\Delta_\infty + F$, $Q \sim \Delta_\infty$ and $R \sim \Delta_\infty + 3F$,*
(ii) *$m_{a_1}(D, Q) = 1$,*
(iii) *$m_{a_2}(D, R) = 1$ and $m_{a_3}(D, R) = m_{a_4}(D, R) = m_{a_5}(D, R) = 2$,*
(iv) *D meets R transversally except for a_3, a_4 and a_5, and*
(v) *$D \cap Q \cap R = \emptyset$,*
then there exists a curve of type $[6 * 12,\ 2\,;\ 3^7, 2^3]$.

Proof. From $Q \cdot R = 3$, we assume that Q meets R at b_1, b_2 and b_3, which may be infinitely near points. To simplify the notation, we use the same notations to describe the images of points by birational maps.

Since $a_1, b_1, b_2, b_3 \in Q$ and $Q \sim \Delta_\infty$, the self-intersection number of the proper transform of Q by a succession of elementary transformations centered at some points of a_1, b_1, b_2 and b_3 is negative. Hence, this proper transform of Q coincides with Δ_∞. This implies that the succession of elementary transformations centered at a_1, b_1, b_2 and b_3 is the birational map from Σ_4 to Σ_0. Let $\nu = I_+(a_1) \circ I_+(b_3) \circ I_+(b_2) \circ I_+(b_1)$. Let D_1, Q_1 and R_1 be the proper transforms of D, Q and R by ν, respectively.

Since $b_i \notin D$, it follows that D_1 has three triple points, which may be infinitely near points. Furthermore, since $a_1 \in D$ and $pr_0(a_1) = pr_0(a_2)$, we see that D_1 has a node c with $m_c(D_1, Q_1) = 3$. From $Q^2 = 0$ and $R^2 = 6$, we derive ${Q_1}^2 = -4$ and ${R_1}^2 = 4$, which imply that $Q_1 = \Delta_\infty$ and $R_1 \sim \Delta_\infty + 4F$. Since $D_1 \cdot Q_1 = 12$, we have $D_1 \sim 3\Delta_\infty + 12F$ (see Fig. 3).

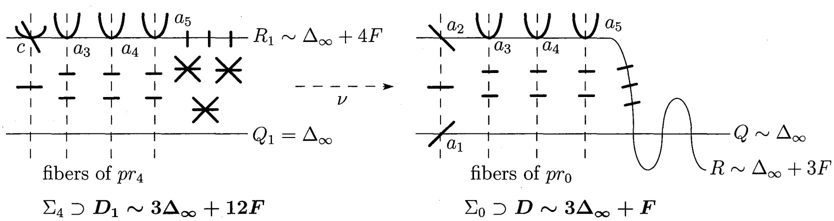

Fig. 3. The arrangement of D_1, Q_1 and R_1

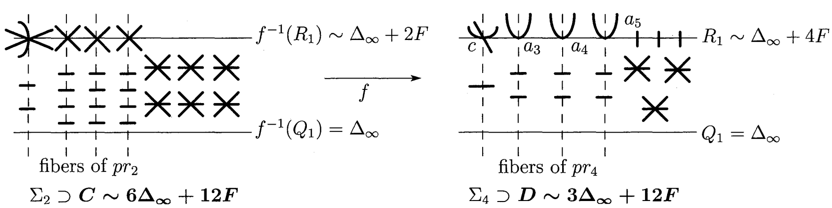

Fig. 4. Singular points of C

Let $f \colon \Sigma_2 \to \Sigma_4$ be the double cover of Σ_4 branched along $Q_1 + R_1$ and C the inverse image of D by f.

By the same argument as in the proof of Lemma 8, C is of type $[6 * 12,\, 2\,;\, 3^7, 2^3]$ (see Fig. 4). Q.E.D.

By the above lemma, it suffices to construct curves D, Q and R on Σ_0 satisfying conditions (i), (ii), (iii), (iv) and (v). We construct required curves by giving the defining equations.

Proposition 12. *There exists a curve of type* $[6 * 12,\, 2\,;\, 3^7, 2^3]$.

Proof. Let ζ be a real number satisfying $5\sqrt{21}\zeta^2 + 91\zeta - 210 = 0$. Put $a_1 = (\zeta, 2 + \sqrt{21}/15)$, $a_2 = (2 + \sqrt{21}/15, 2 + \sqrt{21}/15)$, $a_3 =$

$(0,0)$, $a_4 = (2,3)$ and $a_5 = (3,2)$. For these points, it is easy to see that the divisors defined by the following equations satisfy conditions in Lemma 11:

$$D\colon\ -210x + 384x^2 - 150x^3 + (210 - 91x - 150x^2 + 75x^3)y = 0,$$
$$Q\colon\ x = \zeta,$$
$$R\colon\ -210y + 384y^2 - 150y^3 + (210 - 91y - 150y^2 + 75y^3)x = 0.$$

Since the defining polynomial of D can not be divided by a polynomial in a variable x or y, we see that D is irreducible. Furthermore, since the defining polynomial of R translate into the defining polynomial of D by transposing variables x and y, R is also irreducible. Thus, we have a curve C of type $[6 * 12,\ 2\,;\ 3^7, 2^3]$. Q.E.D.

Remark 13. Let Q and R be curves as in the proof of Proposition 12. Then, Q meets R transversally.

§5. Construction of a curve of type $[8 * (8 + 4d),\ d\,;\ 4^7, 3^2]$

In this section, we construct a curve of type $[8*8,\ 0\,;\ 4^7, 3^2]$ similarly as in Section 3. We call a point $p \in C$ a *2-fold m-ple point* if it turns into an ordinary m-ple point after blowing up at p.

Lemma 14. *Let $a_1, a_2, \ldots, a_6$ and a_7 be points on Σ_0 such that $pr_0(a_i) = pr_0(a_{i+3})$ $(i = 1, 2, 3)$. If there exist three curves D, Q and R on Σ_0 satisfying the following conditions*:
(i) *$D \sim 4\Delta_\infty + 3F$, $Q \sim \Delta_\infty + F$ and $R \sim \Delta_\infty + 2F$,*
(ii) *$a_1, \ldots, a_5$ and a_6 are nodes of D,*
(iii) *$m_{a_2}(D,Q) = m_{a_3}(D,Q) = m_{a_4}(D,Q) = 2$ and $m_{a_7}(D,Q) = 1$,*
(iv) *$m_{a_1}(D,R) = m_{a_5}(D,R) = m_{a_6}(D,R) = 2$ and $m_{a_7}(D,R) = 1$,*
(v) *D meets R transversally except for a_1, a_5 and a_6, and*
(vi) *$D \cap Q \cap R = \{a_7\}$,*
then there exists a curve of type $[8 * 8,\ 0\,;\ 4^7, 3^2]$.

Proof. Under the assumption, since $Q \cdot R = 3$, we assume that Q meets R at b_1 and b_2 except for a_7. Note that b_2 may be an infinitely near point of b_1. To simplify the notation, we use the same notations to describe the images of points by birational maps. First, we consider $\nu = I_+(a_7) \circ I_-(b_2) \circ I_+(b_1)$. Note that ν is the birational map from Σ_1 to Σ_0. Let D_1, Q_1 and R_1 be the proper transforms of D, Q and R by ν, respectively.

Since $a_7 \in D$ and $b_1, b_2 \notin D$, it follows that D_1 has a triple point and two quadruple points, which may be infinitely near points. By hypothesis $Q^2 = 2$ and $R^2 = 4$, we obtain ${Q_1}^2 = -1$ and ${R_1}^2 = 1$. Therefore, $Q_1 = \Delta_\infty$ and $R_1 \sim \Delta_\infty + F$. From $Q_1 \cdot R_1 = 10$, we obtain $D_1 \sim 4\Delta_\infty + 10F$ (see Fig. 5).

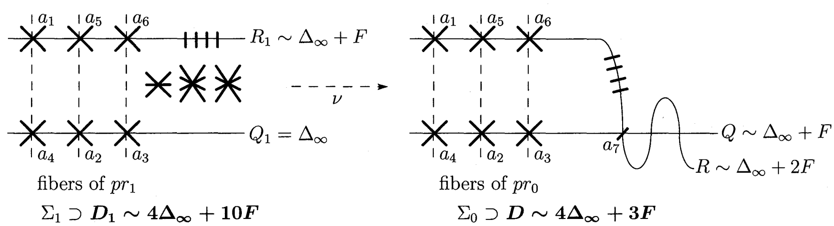

Fig. 5. The arrangement of D_1, Q_1 and R_1

Next, we consider $\mu = I_-(a_4) \circ I_+(a_5) \circ I_-(a_6)$, which is a birational map from Σ_0 to Σ_1. Let D_2, Q_2 and R_2 be the proper transforms of D_1, Q_1 and R_1 by μ, respectively.

For $i = 1, 2, 3$, since a_i and a_{i+3} are double points of D_1 with $pr_1(a_i) = pr_1(a_{i+3})$, the elementary transformation centered at a_{i+3} gives a 2-fold double point c_i of D_2 such that $m_{c_i}(D_2, Q_2 + R_2) = 4$.

Moreover, ${Q_1}^2 = -1$ and ${R_1}^2 = 1$ imply that ${Q_2}^2 = 0$ and ${R_2}^2 = 0$, i.e., $Q_2 \sim R_2 \sim \Delta_\infty$. Hence, we obtain $D_2 \sim 4\Delta_\infty + 8F$ (see Fig. 6).

In Fig. 6, thin curves denote Q_i and R_i. Broken lines denote fibers of pr_d and thick curves denote D_i.

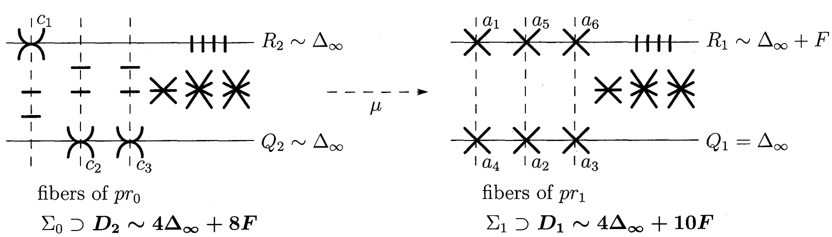

Fig. 6. The arrangement of D_2, Q_2 and R_2

Let $f\colon \Sigma_0 \to \Sigma_0$ be the double cover of Σ_0 branched along $Q_2 + R_2$ and C the inverse image of D_2 by f. Since the analytic branches of D_2 at c_i is tangent to the branch divisor of f for each i, there exist three ordinary quadruple points on C. Moreover, since D_2 has two quadruple points and a triple point which are not contained in $Q_2 + R_2$, it follows that C has seven quadruple points and two triple points (see Fig. 7).

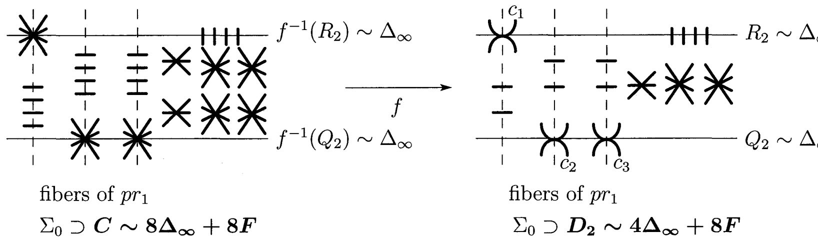

Fig. 7. Singular points of C

Since D is irreducible and there exist points at which D meets $Q_2 + R_2$ transversally, C is irreducible. Thus, C is of type $[8 * 8,\ 0\,;\ 4^7, 3^2]$. Q.E.D.

To prove the existence of a curve of type $[8 * 8,\ 0\,;\ 4^7, 3^2]$, it suffices to construct curves on Σ_0 satisfying conditions (i), (ii), (iii), (iv), (v) and (vi) in Lemma 14. By giving these defining polynomials, we show the following:

Proposition 15. *There exists a curve of type* $[8{*}(8{+}4d),\ d\,;\ 4^7, 3^2]$.

Proof. Required curves D, Q and R are given by the following equations:

$$
\begin{aligned}
D\colon & (-76 + 12\sqrt{2})x^2 + (186 - 50\sqrt{2})y^2 + (300 - 68\sqrt{2})x^2y + (-38 + 6\sqrt{2})y^3 \\
& + (-393 + 93\sqrt{2})x^2y^2 + (21 + 7\sqrt{2})x^2y^3 + (-36 + 16\sqrt{2})x^4y^2 \\
& + (73 - 27\sqrt{2})x^4y^3 = 0, \\
Q\colon & -(2 + 2\sqrt{2})x + \sqrt{2}y + (1 + 2\sqrt{2})xy = 0, \\
R\colon & 200 - 6\sqrt{2} + (-267 + 6\sqrt{2})y + (-45 + 27\sqrt{2})xy + (31 + 30\sqrt{2})y^2 \\
& + (75 - 45\sqrt{2})xy^2 = 0.
\end{aligned}
$$

Put $a_1 = (\infty, 0)$, $a_2 = (\sqrt{2}, 1)$, $a_3 = (-1, 2)$, $a_4 = (0, 0)$, $a_5 = (-\sqrt{2}, 1)$, $a_6 = (1, 2)$ and $a_7 = (-2/3, 2\sqrt{2})$. For these points, it is easy to check that divisors defined by above equations satisfy conditions (i), (ii), (iii), (iv), (v) and (vi) in Lemma 14. Furthermore, since the defining polynomials of Q and R can not be divided by a polynomial in a variable x or y, we see that Q and R are irreducible.

We shall verify that D is irreducible. Let $\Delta_0 = \{0\} \times \mathbb{P}^1 \subset \Sigma_0$ and suppose that $\Delta_\infty = \{\infty\} \times \mathbb{P}^1$. Let $g\colon \Sigma_0 \to \Sigma_0$ be the double cover of Σ_0 branched along $\Delta_0 + \Delta_\infty$. Then D coincides with $g^{-1}(D')$, where D' is the divisor defined by the following:

$$\begin{aligned}&(-76+12\sqrt{2})x + (186-50\sqrt{2})y^2 + (300-68\sqrt{2})xy + (-38+6\sqrt{2})y^3\\ &+(-393+93\sqrt{2})xy^2 + (21+7\sqrt{2})xy^3 + (-36+16\sqrt{2})x^2y^2\\ &+(73-27\sqrt{2})x^2y^3 = 0.\end{aligned}$$

We see that $g(a_2)$ and $g(a_3)$ are double points of D' and that $m_{g(a_1)}(D', \Delta_\infty) = m_{g(a_4)}(D', \Delta_0) = 2$ (see Fig. 8). Note that $g(a_1) = (\infty, 0)$, $g(a_2) = g(a_5) = (2,1)$, $g(a_3) = g(a_6) = (1,2)$, $g(a_4) = (0,0)$ and $g(a_7) = (4/9, 2\sqrt{2})$.

In Fig. 8, thin curves denote Δ_0 and Δ_∞. Broken lines denote fibers of pr_0 and thick curves denote D and D'.

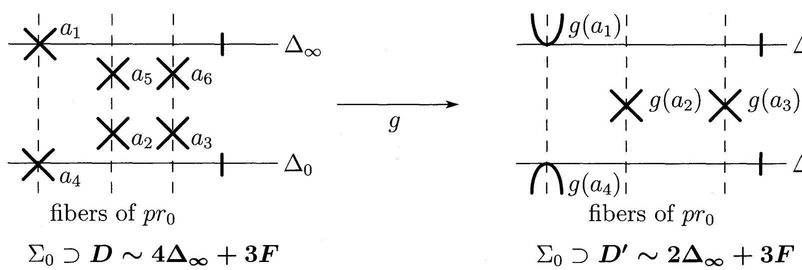

Fig. 8. The arrangement of Δ_0, Δ_∞ and D'

In order to complete the proof, it suffices to show that D' is irreducible and that there exists a point at which D' meets $\Delta_0 + \Delta_\infty$ transversally.

Suppose that D' is not irreducible, i.e., there exist two divisors D'_1 and D'_2 such that $D' = D'_1 + D'_2$. Then one of the following case occurs: (1) $D'_1 \sim \Delta_\infty$, $D'_2 \sim \Delta_\infty + 3F$, or (2) $D'_1 \sim \Delta_\infty + F$, $D'_2 \sim \Delta_\infty + 2F$, or (3) $D'_1 \sim F$, $D'_2 \sim 2\Delta_\infty + 2F$.

In the cases (1) and (2), since we have $D'_1 \cdot \Delta_\infty < 2$, the divisor D'_1 does not pass through $g(a_1)$ and $g(a_4)$. Hence, it follows that D'_2 passes through both $g(a_1)$ and $g(a_4)$. This contradicts to the fact that $D'_2 \cdot F = 1$. In the case (3), by an easy calculation, the coefficients of the defining polynomial of D' with respect to $1, x$ and x^2 have no common divisor. Therefore, we see that D' is irreducible.

Moreover, D' meets Δ_0 transversally at $(0, (33-4\sqrt{2})/7)$. Thus, D is irreducible, i.e., there exists a curve C of type $[8*8,\ 0\,;\ 4^7, 3^2]$.

We use the same notation as in the proof of Lemma 14. The constructed curve C has two quadruple points on $f^{-1}(Q_2)$, say p_1 and p_2. The proper transform of C by $I_+(p_1)$ is of type $[8*12,\ 1\,;\ 4^7, 3^2]$ and the proper transform of C by $I_+(p_2)\circ I_+(p_1)$ is of type $[8*16,\ 2\,;\ 4^7, 3^2]$.

Q.E.D.

Remark 16. Let Q and R be curves as in the proof of Proposition 15. Then, Q meets R transversally.

§6. Construction of a curve of type $[4*(5+2d),\ d\,;\ 2^{11}]$

In this section, we construct a curve of type $[4*(5+2d),\ d\,;\ 2^{11}]$ by a similar method in Section 3.

Lemma 17. *Let a_1, a_2 and a_3 be points on Σ_0. If there exist three curves D, Q and R on Σ_0 satisfying the following conditions:*
(i) *$D \sim 2\Delta_\infty + F$ and $Q \sim R \sim \Delta_\infty + 2F$,*
(ii) *$m_{a_1}(D, Q) = 2$ and $m_{a_2}(D, Q) = 3$,*
(iii) *$m_{a_3}(D, R) = 2$,*
(iv) *D meets R transversally except for a_3, and*
(v) *$D \cap Q \cap R = \emptyset$,*
then there exists a curve of type $[4*5,\ 0\,;\ 2^{11}]$.

Proof. From $Q \cdot R = 4$, we assume that Q meets R at b_1, b_2, b_3 and b_4, which may be infinitely near points. To simplify the notation, we use the same notations to describe the images by birational maps. Let $\nu = I_-(b_4)\circ I_+(b_3)\circ I_-(b_2)\circ I_+(b_1)$. Note that ν is the birational map from Σ_0 to Σ_0. Let D_1, Q_1 and R_1 be the proper transforms of D, Q and R by ν, respectively.

Since $b_i \notin D$, it follows that D_1 has four double points, which may be infinitely near points. From $Q^2 = 4$ and $R^2 = 4$, we derive ${Q_1}^2 = R_1^2 = 0$, which imply that $Q_1 \sim R_1 \sim \Delta_\infty$. By $D_1 \cdot Q_1 = 5$, we see that $D_1 \sim 2\Delta_\infty + 5F$ (see Fig. 9).

Let $f\colon \Sigma_0 \to \Sigma_0$ be the double cover of Σ_0 branched along $Q_1 + R_1$ and C the inverse image of D_1 by f. The singularities induced by a_1, a_3 and a_2 are nodes and a cusp, respectively. Moreover, since D_1 has four double points which are not contained in $Q_1 + R_1$, it follows that C has eleven double points (see Fig. 10).

Since D is irreducible and there exist points at which D_1 meets $Q_1 + R_1$ transversally, C is irreducible. Thus, C is of type $[4*5,\ 0\,;\ 2^{11}]$.

Q.E.D.

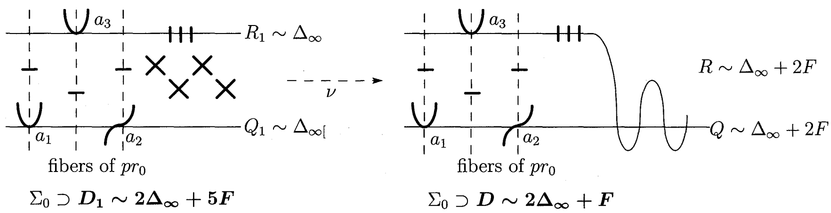

Fig. 9. The arrangement of D_1, Q_1 and R_1

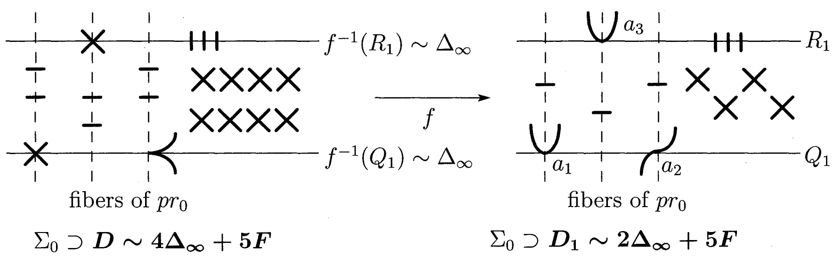

Fig. 10. Singular points of C

In order to prove the existence of a curve of type $[4 * 5,\, 0\,;\, 2^{11}]$, it suffices to construct curves on Σ_0 satisfying conditions (i), (ii), (iii), (iv) and (v) in Lemma 17. We also obtain desired curves by giving defining polynomials as in the following:

Proposition 18. *For* $d = 0, 1, 2$, *there exists a curve of type* $[4 * (5 + 2d),\, d\,;\, 2^{11}]$.

Proof. Put $a_1 = (0, 0)$, $a_2 = (1, 1)$ and $a_3 = (3, -1)$. For these points, we see that the divisors defined by the following equations satisfy conditions (i), (ii), (iii), (iv) and (v) in Lemma 17:

$$D\colon 3x - 2x^2 - 3y + 4xy - 2x^2y = 0,$$
$$Q\colon 3y - 2y^2 - 3x + 4xy - 2xy^2 = 0,$$
$$R\colon 2 + 4x + y + 3xy - 7y^2 + xy^2 = 0.$$

Since $D \sim 2\Delta_\infty + F$ and $Q \sim R \sim \Delta_\infty + 2F$, the irreducibilities of D, Q and R are verified by the similar argument as in the proof of Proposition 9. Note that the defining polynomial of Q translate into the defining polynomial of D by transposing variables x and y. Therefore, we have a curve C of type $[4*5,\, 0\,;\, 2^{11}]$.

We use the same notation as in the proof of Lemma 17. The constructed curve C has two quadruple points on $f^{-1}(Q_1)$, say p_1 and p_2. The proper transform of C by $I_+(p_1)$ is of type $[4*7,\, 1\,;\, 2^{11}]$ and the proper transform of C by $I_+(p_2)\circ I_+(p_1)$ is of type $[4*9,\, 2\,;\, 2^{11}]$. Q.E.D.

Remark 19. Let Q and R be curves as in the proof of Proposition 18. Then, Q meets R transversally.

§7. Construction of a curve of type $[3m\,;\, m^9, 2]$

In this section, we shall show that there exists a curve of type $[3m\,;\, m^9, 2]$. Let E be an elliptic curve and 0_E the zero element with respect to a group law of E. Let $\iota\colon E \to \mathbb{P}^2$ be the embedding by $|3\,0_E|$.

First, we determine the arrangement of singular points of a required curve. It is well-known that E is isomorphic to $(\mathbb{R}/\mathbb{Z})^2$ as a group. When points on E are regarded as elements on $(\mathbb{R}/\mathbb{Z})^2$, we take a_1, a_2, $,\ldots, a_8 \in E$ which are linearly independent over $\mathbb{Q}$. Let a_9 be a point of E such that the order of the sum of $a_1, a_2, \ldots, a_9$ with respect to the group law of E is equal to m. Then it is clear that $m(a_1 + a_2 + \cdots + a_9)$ is linearly equivalent to $9m\,0_E$. Suppose that we have $\sum_j m_j a_j \sim \sum_j m_j\, 0_E$ such that $\sum_j m_j a_j$ is not multiples of $m(a_1 + a_2 + \cdots + a_9)$. Since we have $m(a_1 + a_2 + \cdots + a_9) \sim 9m\,0_E$, we can eliminate the term of a_9. This contradicts to the choice of a_i's. Therefore, all divisors of E supported in $\{a_1, a_2, \ldots, a_9\}$ except for multiples of $m(a_1 + a_2 + \cdots + a_9)$ are not linearly equivalent to multiples of 0_E. In particular, $a_1, a_2, \ldots, a_9$ satisfy the following properties:

(i) $m(a_1 + a_2 + \cdots + a_9) \sim 9m\,0_E$, and

(ii) there exist no plane curves such that $E \cap C \subset \{a_1, a_2, \ldots, a_9\}$ except for curves whose degrees are multiples of $3m$.

Remark 20. The condition (i) is necessary for the existence of a plane curve of degree $3m$ which has m-ple points at a_i's. In the later argument, we use the condition (ii) to prove the irreducibility of a required curve.

We prove the existence of a plane curve of degree $3m$ which has m-ple points at a_i's and a double point.

Proposition 21. *For integer $m \geq 2$, there exist a curves of type $[3m\,;\, m^9, 2]$.*

Proof. Take distinct points $a_1, a_2, \dots, a_8$ and a_9 on E as above. Let $\nu\colon S \to \mathbb{P}^2$ be the succession of blowing-ups at a_i's and ϵ_i the pullbak to S of a_i. Let H be the projective line. The proper transform of E by ν is linearly equivalent to $-K_S$ and we denote it by $\bar{E}$. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_S(-mK_S - \bar{E}) \to \mathcal{O}_S(-mK_S) \to \mathcal{O}_{\bar{E}}(-mK_S) \to 0.$$

By the condition (ii) and $(m-1)\bar{E} \in |-(m-1)K_S|$, we see that $h^0(S,\, \mathcal{O}_S(-mK_S - \bar{E})) = h^0(S,\, \mathcal{O}_S(-(m-1)K_S)) = 1$. Since the negative divisor $-\bar{E}$ is linearly equivalent to K_S, we have $h^2(S,\, \mathcal{O}_S(-mK_S - \bar{E})) = h^0(S,\, \mathcal{O}_S(mK_S)) = 0$. Thus, the Riemann–Roch theorem gives us $h^1(S,\, \mathcal{O}_S(-mK_S - \bar{E})) = 0$, i.e., we obtain

$$h^0(S,\, \mathcal{O}_S(-mK_S)) = h^0(S,\, \mathcal{O}_S(-mK_S - \bar{E})) + h^0(\bar{E},\, \mathcal{O}_{\bar{E}}(-mK_S)).$$

Moreover, $h^0(\bar{E},\, \mathcal{O}_{\bar{E}}(-mK_S)) = h^0(\bar{E},\, \mathcal{O}_{\bar{E}}) = 1$. Therefore, we obtain $h^0(S,\, \mathcal{O}_S(-mK_S)) = 2$ (see also [2, Proposition 1.(1)]). From $K_S^2 = 0$, the complete linear system $|-mK_S| = |3m\nu^*H - \sum_{i=1}^{9} m\epsilon_i|$ is base point free, i.e., the anti-pluricanonical map $\Phi_{|-mK_S|}$ gives the structure of an elliptic surface over $\mathbb{P}^1$ with multiple fiber $m\bar{E}$ (see [3, Theorem 2.1]).

Let D be a fiber of $\Phi_{|-mK_S|}$ which is not $m\bar{E}$. We shall show that D are irreducible. It suffices to show that $\nu(D)$ is irreducible. Since D is a member of $|3m\nu^*H - \sum_{i=1}^{9} m\epsilon_i|$, $\nu(D)$ is a divisor of degree $3m$ such that a_i's are m-ple points of $\nu(D)$. In particular, we have $\iota^*\nu(D) = \sum_{i=1}^{9} ma_i$. Suppose that $\nu(D) = D_1 + D_2$, where D_1 and D_2 are divisors in $\mathbb{P}^2$. Then $\iota^*(D_1) \sim 3(\deg D_1)\, 0_E$ and $\mathrm{Supp}(\iota^*(D_1)) \subset \{a_1, a_2, \dots, a_9\}$, which contradicts to the condition (ii).

The Euler characteristic of S is equal to 12. The sum of the Euler characteristics of singular fibers is equal to the Euler characteristic of an elliptic surface. Since the unique multiple fiber is equal to $m\bar{E}$, its Euler characteristic is equal to zero. It implies that there exist other singular fibers which are not multiple fibers. Since all fibers of $\Phi_{|-mK_S|}$ are irreducible, the image of a singular fiber of $\Phi_{|-mK_S|}$ by ν is of type I_1 or II (see [10, Theorem 6.2]). Hence, this image has nine m-ple poitns $a_1, a_2, \dots, a_9$ and one double point, i.e., it follows that there exists a curve of type $[3m\,;\, m^9, 2]$. Q.E.D.

From Propositions 9, 12, 15, 18 and 21, we obtain Theorem 4. From Proposition 21, we obtain Theorem 5.

Acknowledgements. The author expresses his thanks to Professor Shigeru Iitaka for valuable discussions and advice. Thanks are also due to Professors Tadashi Ashikaga and Mizuho Ishizaka for encouragements and advice. The author is also grateful to the referee for suggesting improvements to this paper.

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