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Well-posedness of the Cauchy problem for the Maxwell–Dirac system in one space dimension

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Abstract.

We determine the range of Sobolev regularity for the Maxwell–Dirac system in $1+1$ space time dimensions to be well-posed locally. The well-posedness follows from the null form estimates. Outside the range for the well-posedness, we show either the flow map is not continuous or not twice differentiable at zero.

§1. Introduction

In this note, we study the Cauchy problem of the Maxwell–Dirac (M–D) system in $1+1$ dimensions;

$$(-i\alpha^\mu \partial_\mu + m\beta)\psi = A_\mu \alpha^\mu \psi, \tag{1}$$
$$\Box A_\mu = -\langle \alpha_\mu \psi, \psi \rangle, \tag{2}$$
$$\partial^\mu A_\mu = 0, \tag{3}$$
$$\psi(0) = \psi_0,\ A_\mu(0) = a_\mu,\ \partial_t A_\mu(0) = \dot{a}_\mu \tag{4}$$

where $\partial_0 = \partial_t$, $\partial_1 = \partial_x$, $\Box = -\partial_t^2 + \partial_x^2$, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{C}^2$, $\psi = \psi(t,x)$ is a $\mathbb{C}^2$ valued unknown function, $A_\mu = A_\mu(t,x)$ are real valued unknown functions, and m is a nonnegative constant. We are concerned with the Minkowski space with the metric $g^{\mu\nu} = \mathrm{diag}(1,-1)$ and the summation convention is used for summing over repeated indices. Thus $\alpha^\mu \partial_\mu = \sum_{\mu=0}^{1} \alpha^\mu \partial_\mu$, where α^μ given $\alpha^0 = I_2$, $\alpha = \alpha^1 = \gamma^0\gamma^1$, and $\beta = \gamma^0$, where I_2 denotes the identity matrix of

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size 2. We take the matrices γ^μ as follows

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The constraint (3) is the Lorenz gauge condition. The M–D system describes an electron self-interacting with its own electromagnetic field. The system in $1+1$ dimensions is the prototype model in the quantum field theory.

In the one dimensional case, the equations (2) and (3) require the initial data to satisfy the following two compatibility conditions:

$$\text{(5)} \qquad \partial_x \dot{a}_1(x) = |\psi_0(x)|^2 + \partial_x^2 a_0(x), \quad \dot{a}_0(x) = \partial_x a_1(x).$$

The Lorenz gauge condition (3) restricts the behavior of the solutions at the spatial infinity, though wave equations have finite speed propagation. Indeed, if $\partial_x a_0$ and $\dot{a}_1$ vanish at $x = \pm\infty$, then (5) implies that

$$\int_{-\infty}^{\infty} |\psi_0|^2 = \|\psi_0\|_{L^2}^2 = 0,$$

which excludes the nontrivial case, this was pointed out in [19]. It is a difficulty of the one dimensional case. Let f be a real valued function in $C^\infty(\mathbb{R})$ satisfying the following assumption

$$f(x) = \frac{c_0}{2} x \text{ on } |x| \le \frac{2}{5}, \quad f(x) = \operatorname{sgn} x \cdot \frac{c_0}{2} \text{ on } |x| \ge \frac{3}{5},$$

$c_0 := \|\psi_0\|_{L^2}^2$. In this note, we consider the case $s \ge 0$ and the initial data $\dot{a}_1 - f$ vanishing at $\pm\infty$. This condition for the initial data $\dot{a}_1$ of the spatial infinity does not unnatural condition physically. Replacing $A_1(t,x)$ with $A_1(t,x) + tf(x)$, we rewrite (1)–(4) as follows.

$$\text{(6)} \qquad (-i\alpha^\mu \partial_\mu + m\beta)\psi = A_\mu \alpha^\mu \psi + tf\alpha\psi,$$

$$\text{(7)} \qquad \Box A_\mu = -\langle \alpha_\mu \psi, \psi \rangle - \mu t \partial_x^2 f,$$

$$\text{(8)} \qquad \partial^\mu A_\mu = -t\partial_x f,$$

$$\text{(9)} \qquad \psi(0) = \psi_0, \ A_\mu(0) = a_\mu, \ \partial_t A_\mu(0) = \dot{a}_0.$$

The initial datum ψ_0, a_μ, and $\dot{a}_\mu$ of the Cauchy problem will be taken in a Sobolev space $H^s = H^s(\mathbb{R})$ defined by the norm $\|u\|_{H^s} := \|\langle\cdot\rangle^s \widehat{u}\|_{L^2}$, where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$ and $\widehat{u}$ denotes the Fourier transform of u. For $1+n$ dimensions, the M–D system with $m = 0$ is invariant under the scaling

$$\psi(t,x) \to \frac{1}{\lambda^{3/2}} \psi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \ A_\mu(t,x) \to \frac{1}{\lambda} A_\mu \left(\frac{t}{\lambda}, \frac{x}{\lambda}\right),$$

hence the scaling invariant data space is

$$\psi_0 \in \dot{H}^{n/2-3/2}(\mathbb{R}^n),\ a_\mu \in \dot{H}^{n/2-1}(\mathbb{R}^n),$$

where $\dot{H}^s(\mathbb{R}^n)$ denotes a homogeneous Sobolev space. One does not expect the well-posedness below this regularity.

There are not many results on the $1+1$ dimensional case unlike the higher dimensional case. Chadam [5] obtained the global existence of solution in $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$. In the case $m = 0$, Huh [12] proved the global well-posedness in $L^2(\mathbb{R}) \times C_b(\mathbb{R}) \times C_b(\mathbb{R})$. Note that the wave data a_μ and $\dot{a}_\mu$ are taken in the same space $C_b(\mathbb{R})$ and $\partial_t A_\mu \in C_b(\mathbb{R})$ is not proved in [12]. Usually, we assume that the regularity of $\dot{a}_\mu$ is one derivative less than a_μ, and for the well-posedness, we have to prove the solution stays in the same space as the initial data, which is called the "persistency". Recently, the well-posedness for the M–D system in $1+3$ and $1+2$ dimensions has intensively been studied by D'Ancona, Foschi and Selberg [7] and D'Ancona and Selberg [9] (see also [6]). Especially, the three dimensional result obtained by D'Ancona, Foschi, and Selberg [7] is optimal with respect to the scaling except for the critical case $L^2(\mathbb{R}^3) \times H^{1/2}((\mathbb{R}^3)$.

We describe two new ingredients of the proof by D'Ancona, Foschi, and Selberg [7] and the difference between the higher dimensional and the one dimensional cases. The first one is they have uncovered an additional null form in the Dirac equation. We here explain null forms and null form estimates. In the 3-dimension case, the quadratic forms in first derivatives

$$Q_0(f,g) = -\partial_t f \partial_t g + \sum_{j=1}^{3} \partial_j f \partial_j g,$$

$$Q_{\mu\nu}(f,g) = \partial_\mu f \partial_\nu g - \partial_\nu f \partial_\mu g,\ 0 \leq \mu < \nu \leq 3,$$

are said to be null forms. The space-time estimates for null forms were first proved in Klainerman and Machedon [13]. They were used to improve the classical local existence theorem for nonlinear wave equations with the null forms. Using the classical method, i.e., energy estimates and the embedding theorems, one can prove that the M–D system in 1+3 dimensions is locally well-posed in $H^2(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. Roughly speaking, the use of the Strichartz inequality allows us to improve classical local existence theorems by $1/2$ derivative. However, the Strichartz inequality method does not take into account the special structure of the nonlinearities that come up in the equations. Using the null form estimates, Bournaveas [3] proved local well-posed in $H^{1/2+\varepsilon}(\mathbb{R}^3) \times H^{1+\varepsilon}(\mathbb{R}^3)$ for

$\varepsilon > 0$. D'Ancona, Foschi, and Selberg [6], [7] have uncovered the full null structure which can not be seen directly. The null structure found in [6], [7] is not the usual bilinear null structure that may be seen in bilinear terms of each individual component equation of a system. But one can find the special property depends on the structure of the system as a whole. Hence, they call it system null structure. In the $1+1$ dimensional case, we can find the system null structure by employing the argument in [7]. Thus, our task is to prove the one dimensional null form estimates.

The second one is the fact that the M–D system in Lorenz gauge with space being 3-dimension or 2-dimension can be rewritten the system of the fields $(\boldsymbol{B}, \boldsymbol{E})$ and the spinor ψ, instead of the potentials A_μ and the spinor ψ. In this case, the worst part of A_μ, that has no better structure, can be neglected. The observation plays a crucial role in the proof of [7] and [9]. On the other hand, in $1+1$ dimensions the electromagnetic fields $(\mathbf{B}, \mathbf{E})$ are not necessarily converted to the potential fields A_μ decaying near the spatial infinity. We directly consider the system of the potentials A_μ and the spinor ψ, and we must estimate the worst part of A_μ.

Theorem 1. *If $s > 0$, $s \le r \le \min(2s + 1/2, s + 1)$, $r > 1/2$, and $(s, r) \ne (1/2, 3/2)$, then (6)–(9) is locally well-posed in $H^s \times H^r$.*

In the proof of Theorem 1, we will pick out the worst part. The many restrictions in Theorem 1 comes from this part. Thus, we may suppose the well-posedness is broken by this part. We analyze this part in details and obtain the following theorems, which say Theorem 1 is optimal.

Theorem 2. *Suppose $0 \le s < 1/2$, $r > \max(2s + 1/2, 1/2)$. Then there exist sequences $\{u_N\} \subset \mathcal{S}(\mathbb{R})$ and $t_N \searrow 0$ such that $\|u_N\|_{H^s} \to 0$, as $N \to \infty$, and the corresponding solution $(\psi_N, A_{\mu,N})$ to (6)–(7) with initial data $\left(\binom{u_N}{0}, 0, 0\right)$ satisfies*

$$\|A_{0,N}(t_N)\|_{H^r} \to \infty, \text{ as } N \to \infty.$$

Remark 1. The ill-posedness appearing in Theorem 2 is referred to as norm inflation. It says that the flow map of (6)–(9) fails to be continuous at 0, and fails to be bounded in a neighborhood of 0.

Theorem 3. *Suppose $r < s$ or $r > s + 1$ or $r \le 1/2$ or $s = 1/2$, $r \ge 3/2$. Then for any $T > 0$, the flow map of (6)–(9), as a map from the unit ball centered at 0 in $H^s \times H^r \times H^{r-1}$ to $C([-T, T]; H^s) \times \left(C([-T, T]; H^r) \cap C^1([-T, T]; H^{r-1})\right)$, fails to be C^2.*

Remark 2. If $m = 0$, we can prove the norm inflation at $(s, r) = (0, 1/2)$.

Remark 3. Theorem 3 does not imply the ill-posedness but precludes proofs of the well-posedness by the contraction argument. Indeed, if the contraction argument works, the flow map proves to be C^∞ in most cases.

§2. Local well-posedness

As in [7], we decompose A_μ as follows:

$$A_\mu = W(t)[a_\mu, \dot{a}_\mu] + A_\mu^{\text{inh.}} - \mu(tf - W(t)[0, f]),$$
$$A_\mu^{\text{inh.}} = -\Box^{-1}\langle \alpha_\mu \psi, \psi \rangle.$$

Here we use the notations $W(t)[a, b]$ and $\Box^{-1}F$ for the solution of the homogeneous wave equation with initial data a, b and the solution of the inhomogeneous wave equations $\Box u = F$ with vanishing data at time $t = 0$, respectively. According to the linear part of the system, we define the following function spaces.

Definition 1. *For $s, b \in \mathbb{R}$, the function space $X_\pm^{s,b}$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norm*

$$\|u\|_{X_\pm^{s,b}} = \|\langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \widetilde{u}(\tau, \xi)\|_{L^2_{\tau,\xi}},$$

where $\widetilde{u}(\tau, \xi)$ denotes the time-space Fourier transform of $u(t, x)$. The function spaces $H^{s,b}$ and $\mathcal{H}^{s,b}$ are the completion of $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norm

$$\|u\|_{H^{s,b}} := \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{\mathcal{H}^{s,b}} := \|u\|_{H^{s,b}} + \|\partial_t u\|_{H^{s-1,b}},$$

respectively.

These spaces are introduced by Bourgain [2] and Klainerman and Machedon [14].

Remark 4. For $b > 1/2$, we have $X_\pm^{s,b} \hookrightarrow C(\mathbb{R}; H^s)$ and $\mathcal{H}^{s,b} \hookrightarrow C(\mathbb{R}; H^s) \cap C^1(\mathbb{R}; H^{s-1})$.

By a standard argument, the problem obtaining closed estimates for the iterates reduces to proving the nonlinear estimates. Thus, for example, we need to show that

$$\|\Box^{-1}\langle\alpha_\mu\psi_1,\psi_2\rangle\alpha^\mu\psi_3\|_{X^{s,-1/2+2\varepsilon}_{\pm_4}} \lesssim \prod_{j=1}^{3}\|\psi\|_{X^{s,1/2+\varepsilon}_{\pm_j}},$$

where $\pm_0$, $\pm_1$, $\pm_2$, $\pm_3$, and $\pm_4$ denotes independent signs. We omit the detail for the proof this estimate. Since the null structure plays crucial role in the proof, we only consider the null form estimates.

We define the null structure

$$\theta_{jk} = \theta(e_j, e_k) = \begin{cases} 1, & e_j e_k < 0, \\ 0, & e_j e_k > 0. \end{cases}$$

In higher dimensions, θ_{ij} denotes the angle between e_i and e_j (see [7], [9]). The following Proposition is the 1-dimensional null form estimates.

Propositon 1. *Suppose $s_0, s_1, s_2 \in \mathbb{R}$, $b_0, b_1, b_2 \geq 0$. We define $A := b_0 + b_1 + b_2$, $B = \min(b_0, b_1, b_2)$, and $s = s_0 + s_1 + s_2$). If*

$$s_0 + s_1 \geq 0,\ s_0 + s_2 \geq 0,\ A > 1/2,$$
$$s_1 + s_2 + A > 1/2,\ s + A > 1,$$
$$s_1 + s_2 + B \geq 0,\ s + B \geq 1/2,$$

we then have

$$\|\mathfrak{B}_{\theta_{12}}(u_1, u_2)\|_{X^{-s_0,-b_0}_{\pm_0}} \lesssim \|u_1\|_{X^{s_1,b_1}_{\pm_1}}\|u_2\|_{X^{s_2,b_2}_{\pm_2}},$$

where

$$\mathcal{F}[\mathfrak{B}_{\theta_{12}}(u_1, u_2)](X_0) = \iint \theta(X_1, X_2)\widetilde{u_1}(X_1)\overline{\widetilde{u_2}(X_2)}d\mu^{12}_{X_0}.$$

If the bilinear form has no null structure, the following estimate holds.

Propositon 2. *Suppose $s_0, s_1, s_2 \in \mathbb{R}$, $b_0, b_1, b_2 \geq 0$, and $b_0 + b_1 + b_2 > 1/2$. If*

$$s_0 + s_1 + s_2 \geq \max(s_0, s_1, s_2),\ s_0 + s_1 + s_2 \geq 1/2$$

and we do not allow both to be equalities, we then have

$$\|u_1\overline{u_2}\|_{X^{-s_0,-b_0}_{\pm_0}} \lesssim \|u_1\|_{X^{s_1,b_1}_{\pm_1}}\|u_2\|_{X^{s_2,b_2}_{\pm_2}}.$$

Remark 5. By the null structure, Proposition 1 permits $s_0 + s_1 + s_2 < 1/2$, while Proposition 2 requires $s_0 + s_1 + s_2 > 1/2$. Roughly speaking, in Proposition 1, we can replace s_j by $s_j + b_j$.

§3. Ill-posedness

Let S_m and W be the free evolution operator of the massive Dirac equation and wave equation, respectively. We set

$$\widehat{u}_N(\xi) = N^{-2s+r/2-3/4}(\chi_{[N,N+N^{2s-r+3/2}]}(\xi) + \chi_{[-N-N^{2s-r+3/2},-N]}(\xi)),$$

where χ_A is the characteristic function of A. Then we have

$$\|u_N\|_{H^{s'}} \leq N^{-2s+r/2-3/4}N^{s'+s-r/2+3/4} = N^{s'-s}.$$

We split the proof into four steps and omit the details.

Step 1. We now prove

$$\left\| \int_0^t W(t-s)|S_m(s)\psi_{0,N}|^2 ds \right\|_{H^r} \gtrsim tN^{\sigma}, \ \sigma := -s + r/2 - 1/4 > 0$$

for $t \gtrsim 1/N$. Thus the desired result holds provided $u_{0,N}$ is replaced by $S_m(t)\psi_{0,N}$, where $\psi_{0,N} = \binom{u_N}{0}$.

Step 2. When $0 < s < 1/2$ and $2s + 1/2 < r < \min(14s/11 + 19/22, 14s/3 + 1/2)$, we prove

$$\left\| \int_0^t W(t-s)(|\psi_N(s)|^2 - |S_m(s)\psi_{0,N}|^2) ds \right\|_{L_t^\infty H^r(S_T)} \lesssim N^{\sigma/2}.$$

Step 3. We obtain $\|A_{0,N}(t)\|_{H^r} \gtrsim tN^{\sigma}$, if $0 < s < 1/2$ and $2s+1/2 < r < \min(14s/11 + 19/22, 14s/3 + 1/2)$, and $t \gtrsim 1/N$.

Step 4. When $0 \leq s < 1/2$ and $r > 2s+1/2$, we have $\|A_{0,N}(t)\|_{H^r} \geq CtN^{\alpha}$, for some $\alpha > 0$ and $t \gtrsim 1/N$.

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