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Strichartz estimates for Schrödinger equations with variable coefficients and unbounded potentials

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Abstract.

This note is concerned with Strichartz estimates for solutions to Schrödinger equations with asymptotically flat metrics and subquadratically growing potentials. We prove the estimates outside a large compact set centered at origin. Under the non-trapping condition, we also prove global-in-space estimates.

§1. Introduction

Let (M, g) be a complete, connected and smooth Riemannian manifold with a metric g and consider formally self-adjoint Schrödinger operators $-(1/2)\Delta_g + V(x)$, where Δ_g is the Laplace–Beltrami operator associated with g and $V(x)$ is a smooth and real-valued function on M. We denote by H the unique self-adjoint realization of $-\frac{1}{2}\Delta_g + V(x)$ on $L^2(M)$ and consider the following time-dependent Schrödinger equation:

$$\partial_t u(t) = Hu(t),\ t \in \mathbb{R}; \quad u|_{t=0} = u_0 \in L^2(M).$$

The solution is given by $u(t) = e^{-itH}u_0$, where e^{-itH} is a one-parameter strongly continuous unitary group generated by H and is called a propagator. The distribution kernel of the propagator e^{-itH} is called the fundamental solution (FDS for short) of the above Cauchy problem.

In the flat case, that is the case with $H = -\frac{1}{2}\Delta + V(x)$ on the Euclidean space $(\mathbb{R}^d, \delta_{jk})$, the construction of the FDS has been extensively studied by many authors. It is well known that the FDS of the

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free evolution $e^{it\Delta/2}$ is explicitly given by

$$\frac{1}{(2\pi it)^{d/2}}e^{i|x-y|^2/2t} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - it|\xi|^2/2}d\xi.$$

For the case with potential perturbations, it was shown by Fujiwara [5] that if the potential V is real-valued, smooth and of quadratic type, that is $|\partial_x^\alpha V(x)| \le C_\alpha$ for all $|\alpha| \ge 2$, then, for sufficiently small $0 < |t| \ll 1$, the FDS can be written in terms of an oscillatory integral form. Under the same condition, Kitada and Kumano-go [8] constructed the FDS as a Fourier integral operator:

$$e^{-itH}u_0(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(\Psi(t,x,\xi)-y\cdot\xi)}b(t,x,\xi)u_0(y)dyd\xi,$$

where $\Psi(t,x,\xi)$ is real-valued and smooth, and $b(t,x,\xi)$ is uniformly bounded with respect to (t,x,ξ). We also refer to [12] for the case with magnetic fields and singular potentials. It follows from the above representations that for any $u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, the solution $e^{-itH}u_0(x)$ satisfies the $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ estimates:

$$||e^{-itH}u_0||_{L^\infty(\mathbb{R}^d)} \le C|t|^{-d/2}||u_0||_{L^1(\mathbb{R}^d)}, \quad 0 < |t| \ll 1.$$

Since e^{-itH} is bounded on $L^2(\mathbb{R}^d)$, applying the TT^*-argument due to Ginibre and Velo [6], we see that e^{-itH} satisfies the so-called (local-in-time) Strichartz estimates:

$$||e^{itH}u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T||u_0||_{L^2(\mathbb{R}^d)},$$

where (p,q) satisfies the following admissible condition:

$$p \ge 2, \quad 2/p + d/q = d/2, \quad (d,p,q) \ne (2,2,\infty).$$

For the proof of the endpoint estimate $(p,q) = (2, 2d/(d-2))$, we also refer to Keel and Tao [7]. Strichartz estimates imply that, for any $u_0 \in L^2(\mathbb{R}^d)$, $e^{-itH}u_0 \in L^q(\mathbb{R}^d)$ for a.e. $t \in \mathbb{R}$ and for above q. These estimates hence can be regarded as L^q-type smoothing properties of Schrödinger equations. The estimates have been widely used in the study of nonlinear Schrödinger equations (see, *e.g.*, Cazenave [3]). More recently, the construction of (microlocal) parametrices and Strichartz estimates for Schrödinger equations on manifolds have been studied by many authors under suitable assumptions on the geometry. For example, Bouclet and Tzvetkov [2] studied Schrödinger equations on $(\mathbb{R}^d, g)$, where g is a long-range perturbation of the flat metric in the following sense:

$$|\partial_x^\alpha(g_{jk}(x) - \delta_{jk})| \le C_\alpha\langle x\rangle^{-\mu-|\alpha|},$$

with some positive constant $\mu > 0$, where $\langle x \rangle = \sqrt{1+|x|^2}$. They then showed that, for appropriate symbols $\chi^{\pm}(x,\xi)$ supported in frequency localized outgoing and incoming regions and $h \in (0,1]$, $e^{-itH}\chi^{\pm}(x,hD)$ can be brought to the following form

$$e^{-itH}\chi^{\pm}(x,hD)u_0(x) \sim U^h(S^{\pm},a_h^{\pm})e^{it\Delta/2}U^h(S^{\pm},b_h^{\pm})^*$$

for $0 \le \pm t \lesssim 1$, respectively, where $S^{\pm}(x,\xi)$ solves the Eikonal equation

$$p(x,\partial_x S^{\pm}(x,\xi)) = |\xi|^2/2 \text{ on a neighborhood of } \operatorname{supp}\chi^{\pm},$$

$a_{\pm}^h$ and $b_{\pm}^h$ are h-dependent, smooth, uniformly bounded amplitudes and $U^h(\varphi,w)$ is a semiclassical FIO:

$$U^h(\varphi,w) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^{2d}} e^{\frac{i}{h}(\varphi(x,\xi)-y\cdot\xi)}w(x,\xi)u_0(y)dyd\xi,$$

for suitable phase function φ and amplitude w. Such a representation is called the (semiclassical) Isozaki and Kitada (IK for short) parametrix for e^{-itH}. Moreover they proved that (i) local-in-time Strichartz estimates, outside a large compact set centered at origin, by using the IK parametrix, (ii) global-in-space Strichartz estimates under the nontrapping condition on the kinetic energy. If the potential V is bounded at least, then there are also several results for the case on manifolds (e.g., see [1], [9], [10], [11] and references therein). However, to my best knowledge, there is no result for the case where both of non-flat geometries and unbounded potentials in the spatial variable are present.

In this note we consider Schrödinger equations with an asymptotically flat metric and subquadratically growing potentials. We then construct a (short-time) microlocal approximation of e^{-itH} in terms of two kinds of Fourier integral operators, and prove local-in-time Strichartz estimates for any admissible pair (p,q) including the end point.

§2. Main results

Let $d \ge 1$ and consider $\mathbb{R}^d$ equipped with a smooth Riemannian metric $g = (g_{jk})$. Let $-\frac{1}{2}\Delta_g + V(x)$ be a Schrödinger operator on $(\mathbb{R}^d, g)$, where Δ_g is the Laplace–Beltrami operator associated to g and $V(x)$ is a real-valued and smooth function. Let $(g^{jk}) = (g_{jk})^{-1}$ and $k(x,\xi) := \frac{1}{2}\sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k$ the corresponding kinetic energy. We assume that $k(x,\xi) \ge c|\xi|^2$ for all $(x,\xi) \in \mathbb{R}^{2d}$ with some constant $c > 0$. We also assume that there exists a constant $\mu > 0$ such that for all $\alpha \in Z_+^d$, there exists $C_\alpha > 0$ such that

$$\left|\partial_x^\alpha (g^{jk}(x) - \delta_{jk})\right| \leq C_\alpha \langle x \rangle^{-\mu-|\alpha|},$$
$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{2-\mu-|\alpha|}.$$

It is known that $-\frac{1}{2}\Delta_g + V(x)$ is then essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and we denote by H the unique self-adjoint extension on $L^2(\mathbb{R}^d)$.

Let $(y_0(t), \eta_0(t))$ be the Hamilton flow generated by $k(x,\xi)$:

$$\dot{y}_0(t) = \partial_\xi k(y_0(t), \eta_0(t)),\ \dot{\eta}_0(t) = -\partial_x k(y_0(t), \eta_0(t)),$$

with $(y_0(0), \eta_0(0)) = (x, \xi)$. We recall the *nontrapping condition*:

(1) For any $(x,\xi) \in T^*\mathbb{R}^d$ with $\xi \neq 0$, $|y_0(t,x,\xi)| \to +\infty$ *as* $t \to \pm\infty$.

For a set A, χ_A denotes the characteristic function on A. The main result is the following:

Theorem 1. *Let* $\mu > 0$, $T > 0$, $p \geq 2$, $q < \infty$ *and* $2/p + d/q = d/2$. *Then the followings hold.*

(i) *For* $R > 1$ *large enough,*

$$||\chi_{\{|x|>R\}} e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T ||u_0||_{L^2(\mathbb{R}^d)}. \tag{2}$$

(ii) *For any* $r > 0$,

$$||\chi_{\{|x|<r\}} e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_{T,r} ||u_0||_{H^{1/p}(\mathbb{R}^d)}. \tag{3}$$

Moreover, if we assume in addition that $k(x,\xi)$ *satisfies the non-trapping condition* (1), *then* (3) *holds with* $||u_0||_{H^{1/p}(\mathbb{R}^d)}$ *replaced by* $||u_0||_{L^2(\mathbb{R}^d)}$. *In particular, combining with* (2), *we have the global-in-space estimates:*

$$||e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T ||u_0||_{L^2(\mathbb{R}^d)},$$

under the non-trapping condition.

§3. Construction of parametrix

Let $g_0 = dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2$ be a metric on $T^*\mathbb{R}^d$. We say that $a \in S(g_0)$ if $a \in C^\infty(\mathbb{R}^{2d})$ and, for any $\alpha, \beta \in \mathbb{Z}_+^d$, there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi) \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}, \quad x, \xi \in \mathbb{R}^d.$$

We denote the pseudo-differential operator (PDO) and Fourier integral operator (FIO) by $\mathrm{Op}(a)$ and $U(\psi, a)$, respectively:

$$\mathrm{Op}(a)f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}a(x,\xi)f(y)dyd\xi,$$
$$U(\psi,a)f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(\psi(x,\xi)-y\cdot\xi)}a(x,\xi)f(y)dyd\xi.$$

As we mentioned above, if $g_{jk} \equiv \delta_{jk}$ then e^{-itH} is given by $U(\psi, a)$ with some $\psi(t,x,\xi)$ and $a(t,x,\xi)$. On the other hand, if g is a long-range perturbation of the flat metric and the potential $V \equiv 0$ then $e^{-itH}\,\mathrm{Op}^h(\chi^\pm)$ is given by $U^h(S^\pm, a_h^\pm)e^{it\Delta/2}U^h(S^\pm, b_h^\pm)^*$ modulo some smoothing term, where $\mathrm{Op}^h(\chi)$, $U^h(S,a)$ are semiclassical PDO and FIO, respectively. Our model is regarded as a mixed case of these two models and we construct the semiclassical parametrix for e^{-itH} in terms of a sum of these two representations as follows.

First, consider a partition of unity on $\{x; |x| > R\} \times \mathbb{R}^d$:

$$l(x,\xi) + h^+(x,\xi) + h^-(x,\xi) = 1, \quad |x| > R,\ \xi \in \mathbb{R}^d,$$

where $l, h^\pm \in S(g_0)$ so that

$$\operatorname{supp} l \subset \{(x,\xi) \in \mathbb{R}^{2d}; |x| > R/2,\ |x| > \langle\xi\rangle\},$$
$$\operatorname{supp} h^\pm \subset \{(x,\xi) \in \mathbb{R}^{2d}; R/2 < |x| < \langle\xi\rangle/2,\ \pm x\cdot\xi/|x||\xi| > -1/2\}.$$

We also consider the dyadic partition of unity with respect to the frequency. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}\varphi \subset [1/2, 2]$, $\varphi_j(\xi) = \varphi(\xi/2^j)$ and

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \quad |\xi| \geq 1,$$

and set $h_j^\pm(x,\xi) = h^\pm(x,\xi)\varphi_j(\xi)$. We then have the following theorem:

Theorem 2. *Suppose $\mu > 0$. Let $R \gg 1$ be large enough and $0 < \delta \ll 1$ small enough. The followings then hold:*
(1) There exists a time-dependent phase function $\psi \in C^\infty((-\delta,\delta) \times \mathbb{R}^{2d}; \mathbb{R})$ such that, for $x, \xi \in \mathbb{R}^d$ and $|t| < \delta$,

$$|\partial_x^\alpha \partial_\xi^\beta(\psi(t,x,\xi) - x\cdot\xi + tk(x,\xi) + tV(x))| \leq C_{\alpha\beta}|t|^2\langle x\rangle^{2-|\alpha+\beta|}.$$

For sufficiently large $j_0 \geq 1$, there exists a family of time-independent phase functions $\{S_j^\pm \in C^\infty(\mathbb{R}^{2d}; \mathbb{R}); j \geq j_0\}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta(S_j^\pm(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}\langle x\rangle^{1-\mu-|\alpha|},$$

uniformly with respect to $x, \xi \in \mathbb{R}^d$ *and* $j \geq j_0$.
(2) *For any* $N \geq 0$, *there exist bounded families of amplitudes*

$$\{a_N(t) \in S(g_0); |t| < \delta\},$$
$$\{b^{\pm}_{N,j} \in S(g_0); j \geq j_0\},$$
$$\{c^{\pm}_{N,j} \in S(g_0); j \geq j_0\},$$

such that, for all $s \in \mathbb{R}$ *with* $N > 2s + 1$,

$$\operatorname{Op}(l)e^{-itH}\operatorname{Op}(l)^* = U(\psi(t), a_N(t)) + O_{H^{-s}\to H^s}(|t|), \quad |t| < \delta,$$
$$\operatorname{Op}(h^{\pm}_j)e^{-itH}\operatorname{Op}(h^{\pm}_j)^* = U(S^{\pm}_j, b^{\pm}_{N,j})e^{it\Delta/2}U(S^{\pm}_j, c^{\pm}_{N,j})^* + O_{H^{-s}\to H^s}(2^{-j(N-2s-1)}), \quad 0 \leq \pm t < \delta.$$

Moreover, if we choose $N > 2d + 1$ *then the distribution kernels of* $\operatorname{Op}(l)e^{-itH}\operatorname{Op}(l)^*$ *and* $\operatorname{Op}(h^{\pm}_j)e^{-itH}\operatorname{Op}(h^{\pm}_j)^*$ *satisfy*

$$|K(t, x, y)| \leq C|t|^{-d/2}, \quad x, \xi \in \mathbb{R}^d,\ 0 < |t| < \delta,$$
$$|K^{\pm}_j(t, x, y)| \leq C|t|^{-d/2}, \quad x, \xi \in \mathbb{R}^d,\ 0 < \pm t < \delta,\ j \geq j_0,$$

respectively.

For the proof of Theorem 2, we refer to [11].

§4. Sketch of the proof of Theorem 1

We here give an outline of the proof of Theorem 1 by using Theorem 2. Observe that Theorem 2, combined with a trick by using a duality argument due to Bouclet and Tzvetkov [2, Lemma 4.3] and the Keel and Tao theorem, imply Strichartz estimates for $\operatorname{Op}(l)e^{-itH}$ and $\operatorname{Op}(h^{\pm}_j)e^{-itH}$:

$$\|\operatorname{Op}(l)e^{-itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \tag{4}$$
$$\|\operatorname{Op}(h^{\pm}_j)e^{-itH}e^{-itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \tag{5}$$

uniformly with respect to $j \geq j_0$. We consider a 4-adic partition of unity $\widetilde{f}(H) + \sum_{j\geq 0} f_j(H) = \mathrm{Id}$, where $\widetilde{f}, f \in C^\infty_0(\mathbb{R})$ and $f_j(\lambda) = f(\lambda/2^{2j})$. Then, we can prove that

(a) L^q-functional calculus. For any $2 \leq q \leq \infty$, $\operatorname{Op}(h^{\pm}_j)\widetilde{f}(H)$ and $\operatorname{Op}(h^{\pm}_j)f_j(H)$ satisfy

$$\|\operatorname{Op}(h^{\pm}_j)\widetilde{f}(H)\|_{L^2\to L^q} + \|\operatorname{Op}(h^{\pm}_j)f_j(H)\|_{L^2\to L^q} \leq C_q 2^{jd(1/2-1/q)},\ j \geq 0.$$

(b) Littlewood–Paley estimates. For $2 \le q < \infty$ and $1 \ll j_0 < \infty$, there exists $C_{q,j_0} > 0$ such that

$$|| \operatorname{Op}(h^{\pm})u||_{L^q} \le C_{q,j_0}||u||_{L^2} + C_{q,j_0} \left(\sum_{j \ge j_0} (|| \operatorname{Op}(h_j^{\pm}) f_j(H)u||_{L^q}^2 \right)^{1/2}.$$

By (5), Littlewood–Paley estimates with $u = e^{-itH}u_0$ and Minkowski inequality, we have Strichartz estimates for $\operatorname{Op}(h^{\pm})e^{-itH}$. Since $\operatorname{Op}(l) + \sum_{\pm} \operatorname{Op}(h^{\pm}) = \mathrm{Id}$, we obtain (2).

Strichartz estimates with the derivative loss $1/p$ can be proved by a standard method using the WKB parametrix for $e^{-itH} \operatorname{Op}(\chi_h)$ for $|t| \ll h$, where $\chi_h(x, \xi) = \chi(x, h\xi)$ with $\chi \in C_0^{\infty}(\mathbb{R}^{2d})$ (see, *e.g.*, [2]). Finally, if $k(x, \xi)$ satisfies the non-trapping condition (1), then the missing $1/p$ derivative can be recovered thanks to the local smoothing effects due to Doi [4]. For the details of the proof, we refer to [11].

References

[1] J.-M. Bouclet, Strichartz estimates on asymptotically hyperbolic manifolds, Anal. PDE, **4** (2011), 1–84.

[2] J.-M. Bouclet and N. Tzvetkov, Strichartz estimates for long range perturbations, Amer. J. Math., **129** (2007), 1565–1609.

[3] T. Cazenave, Semilinear Schrödinger Equations, Courant Lect. Nates Math., **10**, New York Univ., Courant Inst. Math. Sci., New York, NY, 2003.

[4] S. Doi, Smoothness of solutions for Schrödinger equations with unbounded potentials, Publ. Res. Inst. Math. Sci., **41** (2005), 175–221.

[5] D. Fujiwara, Remarks on convergence of the Feynman path integrals, Duke Math. J., **47** (1980), 559–600.

[6] J. Ginibre and G. Velo, The global Cauchy problem for the nonlinear Schrödinger equation revisited, Ann. lnst. H. Poincaré Anal. Non Linéaire, **2** (1985), 309–327.

[7] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math., **120** (1998), 955–980.

[8] H. Kitada and H. Kumano-go, A family of Fourier integral operators and the fundamental solution for a Schrödinger equation, Osaka J. Math., **18** (1981), 291–360.

[9] J. Marzuola, J. Metcalfe and D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations, J. Funct. Anal., **255** (2008), 1497–1553.

[10] H. Mizutani, Strichartz estimates for Schrödinger equations on scattering manifolds, Comm. Partial Differential Equations, **37** (2012), 169–224.

[11] H. Mizutani, Strichartz estimates for Schrödinger equations with variable coefficients and potentials at most linear at spatial infinity, preprint, arXiv:1108.2103.

[12] K. Yajima, Schrödinger evolution equation with magnetic fields, J. Analyse Math., **56** (1991), 29–76.

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