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# Mathematical, numerical and experimental study of solitary waves

**Jeongwhan Choi, Dal-Soo Lee, Sangho Oh, Shu-Ming Sun and Sung-Im Whang**

**Abstract.**

The note discusses the motion of solitary waves on the free surface of a layer of water. The rigorous results for the existence of solitary-wave solutions of exact governing equations are given. To generate such surface waves, a moving bump placed at the bottom or a pressure source on the free surface is used. A model equation, called forced Korteweg–de Vries (FKdV) equation, is numerically studied and multi-solitary-wave solutions are obtained. Then, the numerical solutions are compared with experimental results using a water tank with a moving bump at the bottom.

## §1. Introduction

Solitary waves have been studied since its discovery by Scott Russell in 1834. In 1895, Korteweg and de Vries [1] derived a model equation for the problem, which was also obtained by Boussinesq [2] earlier and is now called the Korteweg–de Vries (KdV) equation, and gave the solitary-wave solutions and periodic (also called cnoidal wave) solutions of the KdV equation. In 1960s, the solutions of KdV equation that consist of several solitary-wave solutions with different amplitude and different traveling speeds were found using an inverse scattering method. However, it is easy to show that the KdV equation has no solutions that consist of several solitary-wave solutions with same amplitude. A series of solitary waves generated by a moving bump placed at bottom were first studied numerically in [3]. Then, a forced KdV (FKdV) equation was used to model such waves where the moving bump was considered as the forcing (see [4] and references therein). Recently, Choi, et al [4] gave

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a very detailed discussion of steady and unsteady solutions of FKdV equation numerically as well as theoretically.

The mathematical proof of the existence of solitary waves for exact governing equations was first given by Lavrentiev [5], who showed that the exact governing equations have a solitary-wave solution and the first-order approximation of this solution is the solitary-wave solution of the KdV equation. When the surface tension is included, then the situation is quite different. Hunter and Vanden-Broeck [6] numerically found that for large surface tension there are solitary-wave solutions of the exact equations decaying to zero at infinity, while for small surface tension there are solitary waves with small oscillations at infinity. The rigorous proof of these existence results was also obtained for large and small surface tension (see [7] and references therein). However, the time-evolution of these waves can only be studied using model equations or experiments.

In this note, a brief discussion on the derivation of FKdV equation from the exact equations governing the fluid flows will be given in Section 2. In addition, rigorous results on the existence of solitary-wave solutions for the exact equations will be provided. In Section 3, numerical solutions of FKdV equation are obtained. In Section 4, the wave phenomena from experiments are discussed to illustrate the numerical results obtained in Section 3.

## §2. Mathematical theory

We consider two-dimensional surface waves on a fluid flow of constant density. The fluid is assumed to be inviscid and incompressible, and bounded above by a free surface and below by an obstruction of compact support over a rigid horizontal bottom. Moreover, there exists a pressure source on the free surface.

The Cartesian coordinates $(x^*, y^*)$ are chosen such that $x^*$-axis is aligned along the longitudinal direction and $y^*$-axis is the vertical direction opposite to the gravity. Let the fluid be in the region $\Omega^* = \{(x^*, y^*) \mid x^* \in (-\infty, +\infty), y^* \in (-h + b^*(x^*, t^*), \eta^*(x^*, t^*))\}$. Assume that $(u^*, v^*)$ is the velocity vector, $y^* = \eta^*(x^*, t^*)$ is the equation of the free surface, $\rho^*$ is the constant density of the fluid, $g$ is the gravitational acceleration constant, $p^*$ is the pressure, $h$ is the constant depth of the fluid at far upstream, $b^*(x^*, t^*)$ is the obstruction with finite support on the rigid horizontal bottom, $P_s^*$ is a pressure function on the free surface, and $T$ is the surface tension coefficient on the free surface. The following dimensionless variables are used: $\epsilon^{1/2} = (h/L), t = \epsilon^{3/2}\sqrt{g/h}\,t^*$,

$$(x,y) = (\epsilon^{1/2}x^*/h, y^*/h), \quad (u,v) = (u^*, \epsilon^{-1/2}v^*)/\sqrt{gh},$$
$$p = p^*/(gh\rho^*),\ \eta = \eta^*/h,\ b(x,t) = b^*(x^*,t^*)/(h\epsilon^2),$$
$$P_s(x,t) = P_s^*(x^*,t^*)/(gh\rho^*\epsilon^2),\ \tau = T/(\rho^* C^2)\,,$$

where $L$ is the horizontal length scale and $u^* \to C$ as $x^* \to -\infty$.

The non-dimensional exact governing equations are the following: $u_x + v_y = 0$,

$$\epsilon u_t + uu_x + vu_y = -p_x, \quad \epsilon^2 v_t + \epsilon uv_x + \epsilon vv_y = -p_y - 1; \tag{1}$$

at the free surface $y = \eta$,

$$\epsilon\eta_t + u\eta_x - v = 0, \quad p = \epsilon^2 P_s(x,t) - \tau\epsilon\eta_{xx}(1 + \epsilon\eta_x^2)^{3/2}; \tag{2}$$

at the rigid bottom $y = -1 + \epsilon^2 b(x,t)$,

$$\epsilon^3 b_t + \epsilon^2 ub_x - v = 0; \qquad \text{as } x \to -\infty,\ (u,v) \to (F,0)\,, \tag{3}$$

where $F = C/\sqrt{gh}$. For $\epsilon = 0$ there is a solution $(u,v) = (F,0), p = -y, \eta = \zeta = 0$. To have other solutions, we let $(u,v) = \epsilon(\tilde{u},\tilde{v}) + (F,0)$, $p = \epsilon\tilde{p} - y$, $\eta = \epsilon\tilde{\eta}$, $\zeta = \epsilon\tilde{\zeta}$. Assume that $P_s, b$ have a form $f(x - a(t)F\epsilon^{-1}t)$ and $a(t)$ is a smooth function of $t$. In dimensional variables, $a(t) = 1$ *corresponds to the case that the forcing is moving downstream with a constant speed $C$ in the moving frame* (or is fixed in the laboratory frame) and $a(t) = 0$ *corresponds to the case that the forcing is fixed in the moving frame* (or is moving upstream with a constant speed $C$ in the laboratory frame).

To find the solutions of (1)–(3), we assume that $\tilde{u}$, $\tilde{v}$, $\tilde{p}$, and $\tilde{\eta}$ have the asymptotic expansions, $\tilde{\phi}(x,z,\epsilon) = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + O(\epsilon^3)$, and $F = F_0 + \epsilon\lambda$. By substituting the asymptotic expansions into (1)–(3) and comparing the orders of $\epsilon$, a sequence of equations and boundary conditions for successive approximations are obtained [8] and an equation for $\eta_0$ can be derived as follows (for notational simplicity, replace $\eta_0$ by $\eta$),

$$\begin{aligned} &2\eta_t + 2\lambda\eta_x - 3\eta\eta_x - ((1/3) - \tau)\eta_{xxx} \\ &\quad = [(1 - (a(t)t)_t)b_x + P_{s,x}](x - a(t)(1 + \epsilon\lambda)\epsilon^{-1}t)\,, \end{aligned} \tag{4}$$

where $F_0 = 1$. The equation was studied in [4] when $a(t) = 0 = P_s$.

If $b = P_s = 0$, (4) becomes the well-known KdV equation

$$2\eta_t + 2\lambda\eta_x - 3\eta\eta_x - ((1/3) - \tau)\eta_{xxx} = 0 \tag{5}$$

which has a time-independent solution $S(x) = 2\lambda\,\mathrm{sech}^2[(2\lambda/((1/3-\tau))^{1/2})x/2]$. Since $S(x)$ is a solution of the first-order model equation (5), it is important to know whether the exact equations (1)–(3) have such solitary-wave solutions. The rigorous existence results for solitary-wave solutions of (1)–(3) can be stated as follows. The references for these results can be found in [7].

**Theorem.** *Assume that $P_s = b = 0$ and $F = 1 + \lambda\epsilon$.*

(1) *For $\tau = 0$ and $\lambda > 0$, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the time independent equations* (1)–(3) *have a solution with $\eta(x) = \epsilon S(x) + \epsilon^2 R(x,\epsilon)$ where $R(x) \in C^n(R)$ is even, bounded and decays exponentially as $|x| \to \infty$.*

(2) *For $\tau > 1/3$ and $\lambda < 0$, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the time independent equations* (1)–(3) *have a solution with $\eta(x) = \epsilon S(x) + \epsilon^2 R(x,\epsilon)$ where $R(x) \in C^n(R)$ is even, bounded and decays exponentially as $|x| \to \infty$.*

(3) *For $0 < \tau < 1/3$ and $\lambda > 0$, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the time independent equations* (1)–(3) *have a solution with $\eta(x) = \epsilon S(x) + \epsilon^2 R(x,\epsilon) + AR^+(x,\epsilon)$ where $R(x) \in C^n(R)$ is even, bounded, decays exponentially as $|x| \to \infty$, and $R^+(x)$ is periodic. Here, $A$ can be of order of $\epsilon^m$ for any $m > 0$ or $\epsilon^{-c_0/\epsilon}$ for some $c_0 > 0$. Moreover, there are no solitary-wave solutions that decay to zero at infinity if $\tau < 1/3$ is near $1/3$.*

The time dependent problem of (1)–(3) is much more difficult to study mathematically for the time evolution of solitary-wave solutions. The global well-posedness is still a very challenging open problem. Therefore, we shall only study the time-dependent problem for the FKdV equation (4) numerically and then verify the numerical results with experiments. Our goal is to generate the solitary or multi-solitary waves using a moving bump placed at the bottom by letting $a(t)$ in (4) be zero and then let the bump stop moving by letting $a(t)$ be one so that we can see how these waves move freely afterwards.

## §3. Numerical solutions

For the sake of simplicity, we assume $P_s = 0$ and $\tau = 0$. For our calculation, we let $b(x)$ be chosen as a semi-circle. Thus, the equation is

$$\begin{aligned} &2\eta_t + 2\lambda\eta_x - 3\eta\eta_x - (1/3)\eta_{xxx} \\ &\qquad = (1-(a(t)t)_t)b_x(x - a(t)(1+\epsilon\lambda)\epsilon^{-1}t)\,, \end{aligned} \tag{6}$$

where $b(x) = \sqrt{1-x^2}$ for $|x| \leq 1$ and $b(x) = 0$ for $x > 1$. In the following, we calculate the solution of (6) with $\lambda > 0$ and the function $a(t)$ chosen as $a(t) = 0$ for $0 \leq t \leq T$ and $a(t) = 1$ for $t \geq T+\delta$. For $T \leq t \leq T+\delta$, $a(t) = \sin^2((\pi/2\delta)(t-T))$, where $\delta$ is a constant measuring the time period used for the moving bump to stop completely. $\epsilon$ is chosen as 1/3, which is used to set up the experiments to be discussed later. The numerical solutions are displayed in Figs. 1–3 and the amplitude of solitary-wave solutions is calculated.

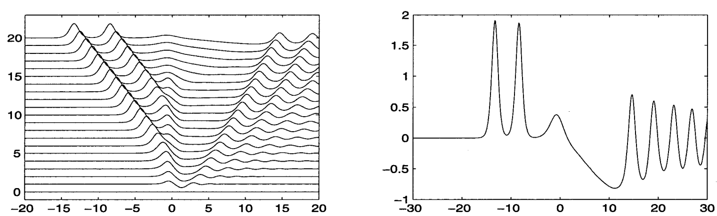


Fig. 1. The solution of (6) with $\lambda = 0.1299$ and $T = 20$. A snap shoot at $t = 20$

In Fig. 1, $\lambda = 0.1299$ and if the motion of the bump is stopped at time $T = 14$ with $\delta = 0.5$, the solution of (6) for a zero initial condition is given. It can be seen that after the bump is stopped, two solitary waves generated by the bump are propagating freely. The amplitude of the solitary waves is about 1.88. Fig. 2 gives the solution of (6) for

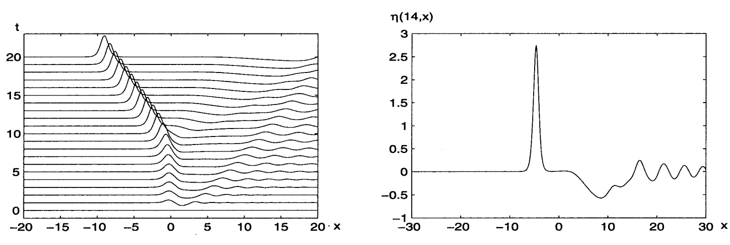


Fig. 2. The solution of (6) with $\lambda = 0.6385$ and $T = 10$. A snap shoot at $t = 14$

$\lambda = 0.6385$ and $T = 10$. It looks that the similar phenomena happen although the amplitude becomes larger (around 2.75) and the time period to generate one solitary wave becomes longer. In Fig. 3, we use

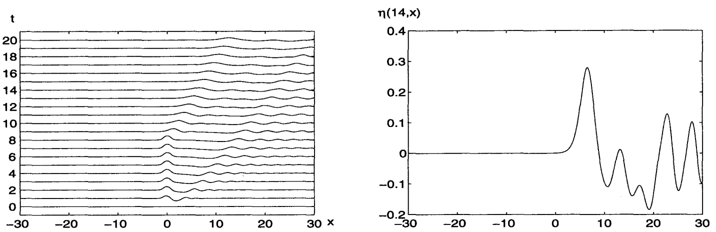


Fig. 3. The solution of (6) with $\lambda = 1.108$ and $T = 8$. A snap shoot at $t = 14$

$\lambda = 1.108$ and $T = 8$. If the initial profile of the wave is zero, then for a slightly large time, a steady state wave appears and is stable. After the bump stops moving at $T = 8$, it is seen that the steady state wave quickly becomes a small solitary wave moving to the right with a constant speed approximately $c = 0.95$ and the amplitude of steady state is reduced to a solitary wave of amplitude 0.28. If the solitary wave is viewed in the laboratory frame, then the wave is moving to the left (or upstream) with a speed $\sqrt{gh}(1 + \epsilon(\lambda - c)) = 1.053\sqrt{gh}$ m/s.

## §4. Experiments

The experiments were conducted in a water tank of 0.7 meter wide and 20 meters long, which was located at Coastal Engineering and Ocean Energy Research Department in Korea Ocean Research and Development Institute. The tank was filled with water of depth 0.06 meter, which gives the critical speed $\sqrt{gh} = 0.7668$ m/s. A moving bump with a uniform cross section was placed at the bottom of the tank and could be moved horizontally by a motor with any given speed. The shape of the bump was given by $b^*(x^*) = h\epsilon^2\sqrt{1 - (\epsilon^{1/2}x^*/h)^2}$ for $|x^*| \le h\epsilon^{-1/2}$ and $b^*(x^*) = 0$ otherwise. Here, $\epsilon = 1/3$ is chosen, i.e, the amplitude of the bump is around 0.0067 meter and the width is 0.208 meter. The relation $C/\sqrt{gh} = 1 + \epsilon\lambda = 1 + (\lambda/3)$ with $C$ being the moving speed of the bump in laboratory frame may give the value of $\lambda$ in the numerical solutions of (6). Three cameras were placed at the locations $L_1$ at 3.32 meters, $L_2$ at 6.43 meters, and $L_3$ at 8.8 meters from the starting position of the bump. The moving bump was stopped around the location $L_2$. Many experiments were conducted and three very typical experiments are report here, which are shown in Figs. 4 to 6.

In Fig. 4, the bump at the bottom was moving with a constant speed 0.8 m/s ($\lambda = 0.1299$ in (6)). At $L_1$, the first solitary wave appeared and

Fig. 4. The speed of the bump is 0.8 m/s.

second one was about to be generated. At $L_2$, the second solitary wave with same amplitude was fully generated before the bump was stopped. Then, it was observed that these two solitary waves were continuously moving with a constant speed afterwards. The amplitude of the solitary wave was about 0.04 meter.

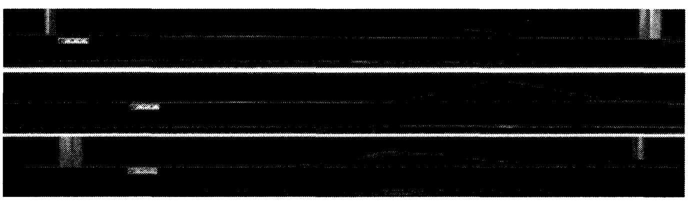

Fig. 5. The speed of the bump is 0.93 m/s.

In Fig. 5, the moving speed of the bump was 0.93 m/s (or $\lambda = 0.6385$). At $L_1$, no full solitary wave had been generated yet, which implies that the time period to generate one solitary wave became longer. At $L_2$, just before the time that the bump was stopped, only one solitary wave was about to be generated. However, its amplitude was so large at about 0.062 m that the maximum height of the solitary wave was almost reached. The wave near the maximum height was unstable and the top of the solitary wave broke. After the bump was stopped, the wave became a bore-like wave at $L_3$. Here, note that for large amplitude waves, (6) cannot be used. Also, it was found that there was a critical speed $C_0$ around 0.95 m/s such that when $C$ was between $C_b \sim 0.93$ and $C_0$, the amplitude of the forced solitary waves always approached to the maximum amplitude and the waves generated were unstable.

Fig. 6 is for the moving speed of the bump at 0.97 m/s. At the locations $L_1$ and $L_2$, it was seen that a steady state wave appeared on

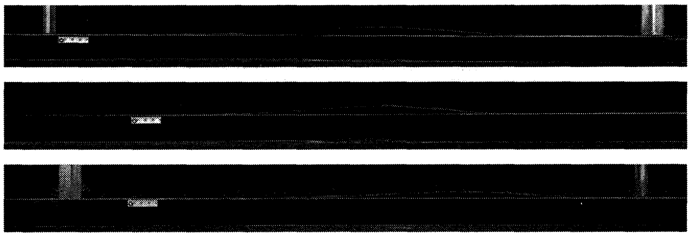

Fig. 6. The speed of the bump is 0.97 m/s.

the top of the bump and was very stable, as predicted by the theory and numerical study [4]. At $L_3$, after the bump stopped, a single hump wave was continuously moving by itself with a constant speed and an amplitude about 0.01655 m, which was a classical solitary wave.

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J. W. Choi
*Department of Mathematics*
*Korea University*
*Seoul, Korea*

D. S. Lee
*Coastal Engineering & Ocean Energy Research Department*
*Korea Ocean Research & Development Institute*
*Ansan, Korea*

S. H. Oh
*Coastal Engineering & Ocean Energy Research Department*
*Korea Ocean Research & Development Institute*
*Ansan, Korea*

S. M. Sun
*Department of Mathematics*
*Virginia Polytechnic Institute and State University*
*Blacksburg, VA*
*U.S.A.*

S. I. Whang
*Department of Mathematics*
*Ajou University Suwon, Korea*

*E-mail address*: sun@math.vt.edu