

Blow up points and the Morse indices of solutions to the Liouville equation: inhomogeneous case

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Abstract.

Let us consider the Liouville equation

$$-\Delta u = \lambda V(x)e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in \mathbb{R}^2 , $V(x) > 0$ is a given function in $C^1(\bar{\Omega})$, and $\lambda > 0$ is a constant. Let $\{u_n\}$ be an m -point blowing up solution sequence for $\lambda = \lambda_n \downarrow 0$, in the sense that

$$\lambda_n \int_{\Omega} V(x)e^{u_n} dx \rightarrow 8\pi m \quad \text{as } n \rightarrow \infty$$

for $m \in \mathbb{N}$. We prove that the number of blow up points m is less than or equal to the Morse index of u_n for n sufficiently large. This extends the main result of the recent paper [13] to an inhomogeneous ($V \not\equiv 1$) case.

§1. Introduction

In this paper we study the Liouville equation

$$(1) \quad \begin{cases} -\Delta u = \lambda V(x)e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 , $V(x) > 0$ is a given function in $C^1(\bar{\Omega})$, and $\lambda > 0$ is a constant.

The purpose of this note is to extend the main result of the recent paper [13], where only the case of $V \equiv 1$ was considered, to the present case.

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The Liouville equation appears in several fields of mathematics and physics, and the study of it has a rather long history; see for example, [3], [4], [12], and the references therein.

Let $\{\lambda_n\}$ be a sequence of positive numbers with $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. One of the interesting issues of this problem is the study of asymptotic behavior of solutions as $n \rightarrow \infty$. Concerning this, Ma and Wei [10] proved the following fact, which extends the former result by Nagasaki and Suzuki [11] where the case of $V \equiv 1$ was considered.

Theorem 1. (Ma and Wei [10]) *For any solution sequence $\{u_n\}$ of (1) for $\lambda = \lambda_n \downarrow 0$, there exists a subsequence (denoted by u_n again) such that it holds*

$$\lambda_n \int_{\Omega} V(x)e^{u_n} dx \rightarrow 8\pi m, \quad \text{for some } m \in \{0\} \cup \mathbb{N} \cup \{+\infty\},$$

and according to the cases, the solution sequence $\{u_n\}$ behaves as

- (i) uniform convergence to 0: $\|u_n\|_{L^\infty(\Omega)} \rightarrow 0$, when $m = 0$,
- (ii) entire blow-up: $u_n(x) \rightarrow +\infty$ as $n \rightarrow \infty$ for any $x \in \Omega$ when $m = +\infty$,
- (iii) m -points blow-up: there exists an m -points set $\mathcal{S} = \{a_1, \dots, a_m\}$, called blow up set, such that each a_i is an interior point of Ω , $\|u_n\|_{L^\infty(K)} = O(1)$ for any compact set $K \subset \bar{\Omega} \setminus \mathcal{S}$, $u_n|_{\mathcal{S}} \rightarrow +\infty$, and

$$(2) \quad u_n \rightarrow 8\pi \sum_{i=1}^m G(\cdot, a_i) \quad \text{in } C_{loc}^2(\bar{\Omega} \setminus \mathcal{S})$$

as $n \rightarrow \infty$ when $m \in \mathbb{N}$. Furthermore, any blow up point $a_i \in \mathcal{S}$ must satisfy the condition

$$\frac{1}{2} \nabla R(a_i) + \sum_{j=1, j \neq i}^m \nabla_x G(a_i, a_j) + \frac{1}{8\pi} \nabla \log V(a_i) = 0$$

for $i = 1, 2, \dots, m$. Here, $G = G(x, y)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition with a pole $y \in \Omega$, and $R(x) = [G(x, y) + \frac{1}{2\pi} \log |x - y|]_{y=x}$ denotes the Robin function.

Later, the existence of multiple blowing up solutions with a prescribed blow up set is established; see [6], [7].

Let $i_M(u)$ denote the Morse index of a solution u of (1), that is, the number of negative eigenvalues of the linearized operator $L_u = -\Delta - \lambda V(x)e^u$ acting on $H_0^1(\Omega)$. In this note, we prove the following, which is an extension of the main theorem in [13] to the inhomogeneous case.

Theorem 2. *Let $\{u_n\}$ be a solution sequence of (1) for $\lambda = \lambda_n$ satisfying*

$$\lambda_n \int_{\Omega} V(x) e^{u_n} dx \rightarrow 8\pi m$$

for some $m \in \mathbb{N}$. Then $m \leq i_M(u_n)$ for n sufficiently large.

In the homogeneous ($V \equiv 1$) case [13], we used the fact that $w(x) = (x - a) \cdot \nabla u_n(x) + 2$ satisfies the equation $-\Delta w = \lambda_n e^{u_n} w$ (except for the boundary condition) for $a \in \mathbb{R}^2$. This is no longer true when V is not a constant, and we need another method. The proof presented here works also for the homogeneous case and the main idea originates from [1].

§2. Proof of Theorem 2

In this section, we prove Theorem 2 along the line of [13].

Let $\{u_n\}$ be a solution sequence to (1) for $\lambda = \lambda_n$ with $\lambda_n \int_{\Omega} V(x) e^{u_n} dx \rightarrow 8\pi m$ for some $m \in \mathbb{N}$. Theorem 1 implies that the existence of the blow up set $\mathcal{S} = \{a_1, \dots, a_m\} \subset \Omega$. Also we have a sufficiently small $\rho > 0$ and m sequences of local maximum points $\{x_n^i\}$ such that for each $a_i \in \mathcal{S}$,

$$u_n(x_n^i) = \max_{B_\rho(x_n^i)} u_n(x) \rightarrow \infty, \quad x_n^i \rightarrow a_i \quad (i = 1, \dots, m),$$

as $n \rightarrow \infty$.

Now we recall the following local pointwise estimate for the blowing-up solutions to (1) thanks to YanYan Li [8]: For a fixed $\rho \in (0, 1)$, there exists a constant $C > 0$ independent of $i = 1, \dots, m$ and $\lambda_n > 0$ such that

$$(3) \quad \left| u_n(x) - \log \frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n}{8} V(x_n^i) e^{u_n(x_n^i)} |x - x_n^i|^2\right)^2} \right| \leq C \text{ for } x \in B_\rho(x_n^i)$$

holds true.

Here we show a proof for the reader's convenience. Define $v_n(x) = u_n(x) + \log \lambda_n$. Then v_n satisfies

$$-\Delta v_n = V(x) e^{v_n} \quad \text{in } \Omega, \quad v_n = \log \lambda_n \quad \text{on } \partial\Omega.$$

Furthermore, by the assumption $\lambda_n \int_{\Omega} V(x) e^{u_n} dx \rightarrow 8\pi m$ and $0 < \exists a \leq V(x) \leq \exists b < +\infty$, we see that $\int_{\Omega} e^{v_n} dx = O(1)$ as $n \rightarrow \infty$.

Now, we claim that $v_n(x_n^i) \rightarrow +\infty$ as $n \rightarrow \infty$ for any $i \in \{1, \dots, m\}$. Indeed, assume the contrary that there exists $i \in \{1, \dots, m\}$ and a subsequence (denoted by the same symbol) such that

- (i) $v_n(x_n^i) \rightarrow -\infty$, or
- (ii) $v_n(x_n^i) \rightarrow C$ for some $C \in \mathbb{R}$.

When (i) happens, we see

$$\int_{B_\rho(x_n^i)} V(x)e^{v_n(x)} dx \leq e^{v_n(x_n^i)} \int_{B_\rho(x_n^i)} V(x) dx \rightarrow 0$$

as $n \rightarrow \infty$. However, this contradicts the fact that

$$\lim_{n \rightarrow \infty} \int_{B_\rho(x_n^i)} V(x)e^{v_n} dx \geq 8\pi,$$

see, for example, Li and Shafrir [9].

Also if (ii) happens, a result of Brezis and Merle ([2]: Theorem 3) implies that $\{v_n\}$ is bounded in $L_{loc}^\infty(\Omega)$. On the other hand, (2) in Theorem 1 implies that $v_n = u_n + \log \lambda_n \rightarrow -\infty$ on any compact set in $\bar{\Omega} \setminus \mathcal{S}$. Thus again we have a contradiction and we have proved the claim.

Once we have the claim, we are in the same situation of Theorem 0.3 in [8] (setting that $\Omega = B_\rho(x_n^i)$, $0 = x_n^i$ there). Note that

$$\max_{\partial B_\rho(x_n^i)} v_n(x) - \min_{\partial B_\rho(x_n^i)} v_n(x) = \max_{\partial B_\rho(x_n^i)} u_n(x) - \min_{\partial B_\rho(x_n^i)} u_n(x) = O(1)$$

as $n \rightarrow \infty$. Thus by Theorem 0.3 in [8], we have

$$\left| v_n(x) - \log \frac{e^{v_n(x_n^i)}}{\left(1 + \frac{1}{8} V(x_n^i) e^{v_n(x_n^i)} |x - x_n^i|^2\right)^2} \right| \leq C \quad \text{for } x \in B_\rho(x_n^i),$$

which is equivalent to (3).

Now, let us define

$$\begin{aligned} (\delta_n^i)^2 \lambda_n e^{u_n(x_n^i)} &= 1, \\ \tilde{u}_n^i(y) &= u_n(\delta_n^i y + x_n^i) - u_n(x_n^i), \quad y \in B_{\rho/\delta_n^i}(0) \end{aligned}$$

for $i \in \{1, \dots, m\}$. By the above claim, we easily see that $\delta_n^i = o(1)$ as $n \rightarrow \infty$. The scaled function \tilde{u}_n^i satisfies

$$\begin{cases} -\Delta \tilde{u}_n^i = V(\delta_n^i y + x_n^i) e^{\tilde{u}_n^i} & \text{in } B_{\rho/\delta_n^i}(0), \\ \tilde{u}_n^i(0) = 0, \tilde{u}_n^i(x) \leq 0, \quad \forall x \in B_{\rho/\delta_n^i}(0), \\ \int_{B_{\rho/\delta_n^i}(0)} V(\delta_n^i y + x_n^i) e^{\tilde{u}_n^i} dy = O(1), \quad (n \rightarrow \infty). \end{cases}$$

Moreover, by an argument in [13], we obtain

$$(4) \quad \tilde{u}_n^i \rightarrow U^i(y) = -2 \log \left(1 + \frac{V(a_i)}{8} |y|^2 \right) \quad \text{for } i = 1, \dots, m$$

in $C_{loc}^1(\mathbb{R}^2)$ as $n \rightarrow \infty$, where U^i is a unique ([5]) solution of

$$\begin{cases} -\Delta U^i = V(a_i) e^{U^i} & \text{in } \mathbb{R}^2, \\ U^i(0) = 0, U^i(y) \leq 0, & \forall y \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{U^i} dy < +\infty. \end{cases}$$

Now, we define two elliptic operators

$$L_n := -\Delta_x - \lambda_n V(x) e^{u_n(x)} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega),$$

$$\tilde{L}_n^i := -\Delta_y - V(\delta_n^i y + x_n^i) e^{\tilde{u}_n^i(y)} : H_0^1(B_{\rho/\delta_n^i}(0)) \rightarrow H^{-1}(B_{\rho/\delta_n^i}(0)).$$

These two operators are related to each other by the formula

$$(\delta_n^i)^2 L_n \Big|_{u_n(x) = \tilde{u}_n^i(y) + u_n(x_n^i)} = \tilde{L}_n^i,$$

where $x = \delta_n^i y + x_n^i$ for $x \in B_\rho(x_n^i)$ and $y \in B_{\rho/\delta_n^i}(0)$. Also for a domain $D \subset B_\rho(x_n^i)$, we have

$$(5) \quad (\delta_n^i)^2 \lambda_j(L_n, D) = \lambda_j(\tilde{L}_n^i, D_n^i), \quad D_n^i = \frac{D - x_n^i}{\delta_n^i},$$

where $\lambda_j(L_n, D)$, $\lambda_j(\tilde{L}_n^i, D_n^i)$ ($j \in \mathbb{N}$) denote the j -th eigenvalue of elliptic operators L_n, \tilde{L}_n^i acting on $H_0^1(D), H_0^1(D_n^i)$ respectively.

We show the following.

Lemma 2.1. *There exists $R > 0$ such that $\lambda_1(L_n, B_{\delta_n^i R}(x_n^i)) < 0$ for n large and for any $i \in \{1, \dots, m\}$. Furthermore, these m balls are disjoint for n large.*

Proof. For $R > 0$, we define

$$w_R(y) = 2 \log \frac{8 + R^2}{8 + |y|^2}.$$

Since $w_R = 0$ on $\partial B_R(0)$, we see $w_R \in H_0^1(B_R(0))$.

We will prove that $(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} < 0$ for $R > 0$ sufficiently large and $B_R(0) \subset B_{\rho/\delta_n^i}(0)$. Indeed,

$$\begin{aligned} (\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} &= \int_{B_R(0)} |\nabla w_R|^2 dy - \int_{B_R(0)} V(\delta_n^i y + x_n^i) e^{\tilde{u}_n^i(y)} w_R^2(y) dy \\ &=: I_1 - I_2. \end{aligned}$$

We observe that

$$I_1 = \int_{B_R(0)} \frac{16|y|^2}{(8+|y|^2)^2} dy = 2\pi \int_0^R \frac{16r^2}{(8+r^2)^2} r dr \leq 32\pi \log R [1 + o_R(1)],$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$. On the other hand, we have

$$\begin{aligned} I_2 &= \int_{B_R(0)} V(\delta_n^i y + x_n^i) e^{\tilde{u}_n^i(y)} w_R^2(y) dy \\ &= V(a_i) \int_{B_R(0)} \frac{1}{\left(1 + \frac{V(a_i)}{8}|y|^2\right)^2} \left\{ 2 \log \frac{8+R^2}{8+|y|^2} \right\}^2 dy + o_n(1) \\ &= 8\pi V(a_i) \int_0^R \frac{r}{\left(1 + \frac{V(a_i)}{8}r^2\right)^2} \left\{ \log(8+R^2) - \log(8+r^2) \right\}^2 dr + o_n(1) \\ &= 8\pi V(a_i) \cdot 8^2 \left\{ \log(8+R^2) \right\}^2 \left[\frac{1}{16V(a_i)} + o_R(1) \right] + o_n(1) \\ &= 32\pi V(a_i) \left\{ \log(8+R^2) \right\}^2 [1 + o_R(1)] + o_n(1), \end{aligned}$$

where we have used (4) and

$$\int_0^R \frac{r}{(8+cr^2)^2} dr = \int_0^\infty \frac{r}{(8+cr^2)^2} dr + o_R(1) = \frac{1}{16c} + o_R(1)$$

for $c > 0$. Thus we obtain

$$\begin{aligned} (\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} &= I_1 - I_2 \\ &\leq -32\pi V(a_i) \left\{ \log(8+R^2) \right\}^2 [1 + o_R(1)] + o_n(1) < 0 \end{aligned}$$

by taking n sufficiently large first, and then $R > 0$ large such that $B_R(0) \subset B_{\rho/\delta_n^i}(0)$. This implies that the first eigenvalue of the operator \tilde{L}_n^i on B_R is negative: $\lambda_1(\tilde{L}_n^i, B_R) < 0$. By this and the scaling formula (5) proves the first half part of the Lemma.

The fact that these balls $B_{\delta_n^i R}(x_n^i)$ are disjoint for large n follows directly from the discreteness of \tilde{S} . Q.E.D.

By Lemma 2.1, we have m open balls B^1, \dots, B^m , $B^i = B_{\delta_n^i R}(x_n^i)$, which are disjoint, and

$$\lambda_1(L_n, B^i) < 0 \quad \text{for } i = 1, \dots, m.$$

On the other hand, it is easy to see that

$$\lambda_m(L_n, \Omega) \leq \sum_{i=1}^m \lambda_1(L_n, B^i)$$

holds; see for example, [13]. Combining these inequalities, we have $\lambda_m(L_n, \Omega) < 0$. Therefore by the definition of the Morse index of u_n , we have $m \leq i_M(u_n)$. This proves Theorem 2. Q.E.D.

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