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Quadratic nonlinear Klein–Gordon equation in 2d, Cauchy problem

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Abstract.

We consider the Cauchy problem for the two dimensional nonlinear Klein–Gordon equation with a quadratic nonlinearity. In the present paper we find more natural conditions for the initial data than those of previous works to ensure the existence of scattering states.

§1. Introduction

Consider the Cauchy problem for the two dimensional nonlinear Klein–Gordon equation with a quadratic nonlinearity

$$\partial_t^2 v - \Delta v + v = 2\lambda v^2, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^2 \tag{1}$$

with the real-valued data $v(0,x) = v_0(x)$, $v_t(0,x) = v_1(x)$, where $\lambda \in \mathbf{R}$. By changing the dependent variable $u = \left(v + i\langle i\nabla\rangle^{-1} v_t\right)/2$, we find that u satisfies the following Cauchy problem

$$\mathcal{L}u = i\lambda \langle i\nabla\rangle^{-1} (u + \overline{u})^2 \tag{2}$$

with the initial data $u(0,x) = u_0 = \left(v_0 + i\langle i\nabla\rangle^{-1} v_1\right)/2$, where $\mathcal{L} = \partial_t + i\langle i\nabla\rangle$, $\langle i\nabla\rangle = \sqrt{1-\Delta}$. In what follows we study equation (2). Our aim is to find more natural requirements on the initial data u_0, comparing with the previous papers [7], [10].

Our main result is the following.

Theorem 1. *Assume that $u_0 \in \mathbf{H}^{\alpha,1}$ with $\|u_0\|_{\mathbf{H}^{\alpha,1}} \leq \varepsilon$, where $\varepsilon > 0$ is small, $\alpha > 1$. Then there exists $\varepsilon > 0$ such that (2) has a*

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unique global solution $u \in \mathbf{C}\left(\mathbf{R};\mathbf{H}^{\alpha,1}\right)$ *satisfying the time decay estimate* $\|u(t)\|_{\mathbf{L}^4} \le Ct^{-\frac{1}{4}}$. *Furthermore for any* $u_0 \in \mathbf{H}^{\alpha,1}$ *with* $\|u_0\|_{\mathbf{H}^{\alpha,1}} \le \varepsilon$, *there exists a unique final state* $u_+ \in \mathbf{H}^{\alpha,\theta}$ *such that*

$$\lim_{t\to\infty}\left\|e^{i\langle i\nabla\rangle t}u - u_+\right\|_{\mathbf{H}^{\alpha,\theta}} = 0,$$

where $0 < \theta < 1$.

Note that equation (2) has some gain of regularity in the nonlinearity in spite of its critical large time behavior. Using the identity $1 = \left\langle \frac{i}{t}\nabla\right\rangle^{-2} - t^{-2}\Delta\left\langle \frac{i}{t}\nabla\right\rangle^{-2}$, we split the nonlinearity in equation (2)

$$\begin{aligned}\mathcal{L}u &= i\lambda\left\langle \frac{i}{t}\nabla\right\rangle^{-2}\langle i\nabla\rangle^{-1}(u+\overline{u})^2 \\ &\quad -i\lambda t^{-2}\Delta\langle i\nabla\rangle^{-1}\left\langle \frac{i}{t}\nabla\right\rangle^{-2}(u+\overline{u})^2, \end{aligned} \tag{3}$$

where the first term has more gain of regularity and the second one has a better time decay. We apply the method of normal forms by Shatah [11] to remove the quadratic nonlinearity $i\lambda\left\langle it^{-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle^{-1}(u+\overline{u})^2$ from the right-hand side of (3). In order to do it, we define the bilinear operators for $j = 1,2,3$

$$\mathcal{T}_j(\phi,\psi)(x) = \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} e^{ix\cdot(\xi+\eta)}\widehat{\phi}(\xi)\widehat{\psi}(\eta)\frac{d\xi d\eta}{4\pi^2 S_j(\xi,\eta)}$$

with symbols $S_1(\xi,\eta) = \langle\xi+\eta\rangle + \langle\xi\rangle + \langle\eta\rangle$, $S_2(\xi,\eta) = \langle\xi+\eta\rangle - \langle\xi\rangle - \langle\eta\rangle$, $S_3(\xi,\eta) = \langle\xi+\eta\rangle + \langle\xi\rangle - \langle\eta\rangle$. The bilinear operators $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3$ correspond to the nonlinear terms $\overline{u}^2, u^2, |u|^2$, respectively. Then we find from (2)

$$\mathcal{L}(u+\mathcal{N}_1) = \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4, \tag{4}$$

where

$$\begin{aligned}
\mathcal{N}_1 &= -\lambda \left\langle \frac{i}{t}\nabla \right\rangle^{-2} \langle i\nabla\rangle^{-1} \left(\mathcal{T}_1\left(\overline{u},\overline{u}\right) + \mathcal{T}_2\left(u,u\right) + 2\mathcal{T}_3\left(u,\overline{u}\right)\right), \\
\mathcal{N}_2 &= -i\lambda t^{-2}\Delta \langle i\nabla\rangle^{-1} \left\langle \frac{i}{t}\nabla \right\rangle^{-2} \left(u+\overline{u}\right)^2, \\
\mathcal{N}_3 &= 2t^{-3}\Delta \left\langle \frac{i}{t}\nabla \right\rangle^{-2} \mathcal{N}_1, \\
\mathcal{N}_4 &= -2\lambda \left\langle \frac{i}{t}\nabla \right\rangle^{-2} \langle i\nabla\rangle^{-1} \left(\mathcal{T}_1\left(\overline{u},\overline{\mathcal{L}u}\right)\right. \\
&\quad \left. +\mathcal{T}_2\left(u,\mathcal{L}u\right) + \mathcal{T}_3\left(u,\overline{\mathcal{L}u}\right) + \mathcal{T}_3\left(\mathcal{L}u,\overline{u}\right)\right).
\end{aligned}$$

The first and second terms in the right-hand side of (4) are the quadratic nonlinearities with an explicit additional time decay, whereas the third term is a cubic nonlocal nonlinearity since in view of equation (2) $\mathcal{T}_1\left(\overline{u},\overline{\mathcal{L}u}\right) = -i\lambda\mathcal{T}_1\left(\overline{u},\langle i\nabla\rangle^{-1}\left(u+\overline{u}\right)^2\right)$, and so on. If we could apply the Hölder inequality to the bilinear operators $\mathcal{T}_j$, we would get the desired result easily. Unfortunately it is impossible, so we encounter the derivative loss difficulty applying Proposition 2 below. The higher order of the derivative loss implies the smoothness of the initial data. This is the reason why sufficiently smooth initial data were required in paper [10]. The derivation of equation (4) is similar to that of papers [11] and [10]. Another nonlinear transformation was proposed by Kosecki [7] for a single equation and refined by Sunagawa [12] to a system of nonlinear Klein–Gordon equations. Note that the derivative loss does not occur in the one dimensional case (see [6]).

Also an important tool of papers [7], [10], [12] is obtaining the time decay estimates through the vector $\Lambda = (\partial_t, \nabla, \mathcal{P}, \Omega)$ with $\mathcal{P} = t\nabla + x\partial_t$ and $\Omega = x_1\partial_{x_2} - x_2\partial_{x_1}$, which was found by [8] and improved by [2]. Roughly speaking, the proof of [2] requires the estimates of Λ^4, and the proof of [8] needs estimates of Λ^2 with a compact support condition. To improve the regularity conditions on the initial data we use the time decay estimates from [4] and estimates of the bilinear operators from [5] comparing with papers [2] and [10]. So we can reduce to 2 the order 4 of the vector Λ used in [2] and the 4-th order of the derivative loss in [10] we reduce to a small order $\delta > 0$. In paper [10], the 37-th order of a vector Λ was used, though their results include the nonlinearities containing the derivatives of the unknown functions (see Remark 1). Our method with the splitting argument (3) can not be applied to the nonlinearities from [10] and systems from [1], [12], directly since that nonlinearities do not have a gain of regularity. In order to treat the nonlinearities containing the derivatives of the unknown function such as $a(\nabla u)^2 + b(\partial_t u)^2$, we need the $\mathbf{L}^\infty$-time decay estimates for the solutions with a requierement $u_0 \in \mathbf{H}_1^3$ at least, which follows by the estimate $\left\| e^{-i\langle i\nabla\rangle t} u_0 \right\|_{\mathbf{H}_\infty^1} \leq Ct^{-1} \|u_0\|_{\mathbf{H}_1^3}$. Note that our method (time decay and bilinear estimates) without (3) still works for general nonlinearities such as $a(\nabla u)^2 + b(\partial_t u)^2$ if we assume that $u_0 \in \mathbf{H}^{4,2}$.

Define the operator $\mathcal{J} = \langle i\nabla\rangle e^{-i\langle i\nabla\rangle t} x e^{i\langle i\nabla\rangle t} = \langle i\nabla\rangle x + it\nabla$, which is analogous to the operator $x + it\nabla = e^{-\frac{it}{2}\Delta} x e^{\frac{it}{2}\Delta}$ in the case of the nonlinear Schrödinger equation (see [3]) and commutes with $\mathcal{L}: [\mathcal{L}, \mathcal{J}] = \mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L} = 0$. However $\mathcal{J}$ is not a purely differential operator, so it is apparently difficult to calculate its action on the nonlinearities. We use also the first order differential operator $\mathcal{P} = t\nabla + x\partial_t$ which is closely related to $\mathcal{J}$ by the identity $\mathcal{P} = \mathcal{L}x - i\mathcal{J}$, acts easily on the nonlinearities and it almost commutes with $\mathcal{L}: [\mathcal{L}, \mathcal{P}] = -i\langle i\nabla\rangle^{-1}\nabla\mathcal{L}$, where we used the commutator $\left[x, \langle i\nabla\rangle^\beta\right] = \beta\langle i\nabla\rangle^{\beta-2}\nabla$.

We denote the Lebesgue space by $\mathbf{L}^p$ with $1 \leq p \leq \infty$. The weighted Sobolev space is $\mathbf{H}_p^{m,s} = \{\phi \in \mathbf{L}^p; \|\langle x\rangle^s \langle i\nabla\rangle^m \phi\|_{\mathbf{L}^p} < \infty\}$, for $m, s \in \mathbf{R}$, $1 \leq p \leq \infty$, where $\langle x\rangle = \sqrt{1 + |x|^2}$. For simplicity we write $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$ and $\mathbf{H}^m = \mathbf{H}^{m,0}$. We denote the Fourier transform of the function ϕ by

$$\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{-ix\cdot\xi}\phi(x)\,dx.$$

Define the function space $\mathbf{X}_T = \left\{\phi \in \mathbf{C}\left([0,T];\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_T} < \infty\right\}$, where the norm

$$\begin{aligned} \|\phi\|_{\mathbf{X}_T} &= \sup_{t\in[0,T]} \left(\|\phi(t)\|_{\mathbf{H}^\alpha} + \|\phi_t(t)\|_{\mathbf{H}^{\alpha-1}}\right. \\ &\quad \left. + \|\mathcal{P}\phi(t)\|_{\mathbf{H}^{\alpha-1}} + \|\mathcal{J}\phi(t)\|_{\mathbf{H}^{\alpha-1}}\right), \end{aligned}$$

with $\alpha > 1$. The local existence of solutions in $\mathbf{X}_T$ can be easily proved.

Proposition 1. *Assume that $u_0 \in \mathbf{H}^{\alpha,1}$, where $\varepsilon > 0$ is small, $\alpha > 1$. Then there exists $T = O\left(\|u_0\|_{\mathbf{H}^{\alpha,1}}^{-1}\right)$ such that (2) has a unique solution $u \in \mathbf{C}\left([-T,T];\mathbf{H}^{\alpha,1}\right)$ satisfying $\|u\|_{\mathbf{X}_T} \leq C\|u_0\|_{\mathbf{H}^{\alpha,1}}$.*

§2. Preliminary estimates

We first state a time decay estimate from paper [4].

Lemma 1. *The estimate is valid*

$$\|\phi\|_{\mathbf{L}^q} \leq C t^{\frac{2}{q}-1} \left(\|\phi\|_{\mathbf{H}^{2-\frac{4}{q}}} + \|\mathcal{J}\phi\|_{\mathbf{H}^{1-\frac{4}{q}}} \right)$$

for all $t > 0$, *where* $2 < q < \infty$, *provided that the right-hand side is finite.*

The following estimates for the bilinear operators $\mathcal{T}_j$ were proved in paper [5].

Proposition 2. *The bilinear operators* $\mathcal{T}_j$, $j = 1, 2, 3$, *are bounded from* $\mathbf{H}^{\beta}_s(\mathbf{R}^2) \times \mathbf{H}^{\gamma}_r(\mathbf{R}^2)$ *to* $\mathbf{H}^{-\sigma}_p(\mathbf{R}^2)$, *i.e.*

$$\|\mathcal{T}_j(f, g)\|_{\mathbf{H}^{-\sigma}_p} \leq C \|f\|_{\mathbf{H}^{\beta}_s} \|g\|_{\mathbf{H}^{\gamma}_r}$$

where $1 \leq p \leq r \leq \infty$, $\frac{1}{s} + \frac{1}{r} = 1 + \frac{1}{p} - \frac{1}{l}$, $1 \leq l \leq 2$, $\sigma, \beta, \gamma \geq 0$ *are such that* $\sigma + \beta > 1, \gamma > 1$, *or* $\beta > 1$, $\sigma + \gamma > 1$.

From Proposition 2 we obtain

Lemma 2. *Let* $\sigma + \beta > 1, \gamma > 1$, *or* $\beta > 1$, $\sigma + \gamma > 1$, *and* $\sigma + \beta_1 > 1, \gamma_1 > 1$, *or* $\beta_1 > 1$, $\sigma + \gamma_1 > 1$, *and* $\frac{1}{s} + \frac{1}{r} = \frac{3}{2} - \frac{1}{l} = \frac{1}{p} + \frac{1}{q}$, $1 \leq l \leq 2$. *Then the following estimates are valid*

$$\|x\mathcal{T}_j(\phi, \psi)\|_{\mathbf{H}^{-\sigma}} \leq C \|\phi\|_{\mathbf{H}^{\beta,1}_s} \|\psi\|_{\mathbf{H}^{\gamma}_r}$$

and

$$\|\mathcal{P}\mathcal{T}_j(\phi, \psi)\|_{\mathbf{H}^{-\sigma}} \leq C \left(\|\mathcal{P}\phi\|_{\mathbf{H}^{\beta}_s} + \|\partial_t \phi\|_{\mathbf{H}^{\beta}_s} \right) \|\psi\|_{\mathbf{H}^{\gamma}_r} + C \|\phi\|_{\mathbf{H}^{\gamma_1}_p} \left(\|\mathcal{P}\psi\|_{\mathbf{H}^{\beta_1}_q} + \|\partial_t \psi\|_{\mathbf{H}^{\beta_1}_q} \right),$$

for $1 \leq j \leq 3$, *provided that the right-hand sides are finite.*

Proof. Integrating by parts we obtain $x\mathcal{T}_j(\phi, \psi) = \mathcal{T}_j(x\phi, \psi) + \mathcal{T}_{j(\xi)}(\phi, \psi)$, where the operator

$$\mathcal{T}_{j(\xi)}(\phi, \psi)(x) = \frac{1}{4\pi^2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} e^{ix\cdot(\xi+\eta)} \left(i\nabla_\xi S_j^{-1}(\xi, \eta) \right) \widehat{\phi}(\xi) \widehat{\psi}(\eta) \, d\xi d\eta$$

is estimated as in Proposition 2. Also integrating by parts we find

$$\begin{aligned}
&\mathcal{P}\mathcal{T}_j(\phi,\psi)(x)\\
=&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} e^{ix\cdot(\xi+\eta)}\left(\widehat{t\nabla\phi}(\xi)\,\widehat{\psi}(\eta)+\widehat{\phi}(\xi)\,\widehat{t\nabla\psi}(\eta)\right)\frac{d\xi d\eta}{4\pi^2 S_j(\xi,\eta)}\\
&+\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\nabla_\xi e^{ix\cdot(\xi+\eta)}\left(\partial_t\widehat{\phi}(\xi)\right)\widehat{\psi}(\eta)\frac{d\xi d\eta}{4i\pi^2 S_j(\xi,\eta)}\\
&+\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\nabla_\eta e^{ix\cdot(\xi+\eta)}\widehat{\phi}(\xi)\,\partial_t\widehat{\psi}(\eta)\frac{d\xi d\eta}{4i\pi^2 S_j(\xi,\eta)}\\
=&\mathcal{T}_j(\mathcal{P}\phi,\psi)(x)+\mathcal{T}_j(\phi,\mathcal{P}\psi)(x)\\
&+\mathcal{T}_{j(\xi)}(\partial_t\phi,\psi)(x)+\mathcal{T}_{j(\eta)}(\phi,\partial_t\psi)(x).
\end{aligned}$$

Therefore the second estimate of the lemma follows from Proposition 2. Q.E.D.

§3. Proof of Theorem 1

We consider the a priori estimate of the solution in the norm

$$\begin{aligned}
\|u\|_{\mathbf{X}_T} =& \sup_{t\in[0,T]}\left(\|u(t)\|_{\mathbf{H}^\alpha}+\|u_t(t)\|_{\mathbf{H}^{\alpha-1}}\right.\\
&\left.+\|\mathcal{P}u(t)\|_{\mathbf{H}^{\alpha-1}}+\|\mathcal{J}u(t)\|_{\mathbf{H}^{\alpha-1}}\right),
\end{aligned}$$

where $\alpha>1$. Also we denote $\mathbf{X}_{T,\rho}=\left\{\phi\in\mathbf{X}_T;\|\phi\|_{\mathbf{X}_T}\le\rho\right\}$, where $\rho=\varepsilon^{\frac{2}{3}}$, $\varepsilon>0$. By the Sobolev embedding theorem with $\frac{1}{r}=\frac{1}{q}-\frac{\chi}{2}$, $\chi\ge0$ and by Lemma 1 we get the estimate

$$\begin{aligned}
\|u\|_{\mathbf{H}^\gamma_r} \le& C\|u\|_{\mathbf{H}^{\chi+\gamma}_q}\le Ct^{\frac{2}{q}-1}\left(\|u\|_{\mathbf{H}^{2-\frac{4}{q}+\chi+\gamma}}+\|\mathcal{J}u\|_{\mathbf{H}^{1-\frac{4}{q}+\chi+\gamma}}\right)\\
\le& Ct^{\frac{2}{r}+\chi-1}\left(\|u\|_{\mathbf{H}^\alpha}+\|\mathcal{J}u\|_{\mathbf{H}^{\alpha-1}}\right)\le C\rho t^{\frac{2}{r}+\chi-1}, \qquad (5)
\end{aligned}$$

where $2<r<\infty$, $0\le\gamma<\alpha$ and $\chi=\max\left(0,2+\gamma-\frac{4}{r}-\alpha\right)$.

First we estimate the norm $\|u(t)\|_{\mathbf{H}^\alpha}$. We choose $\nu=\alpha-1>0$ sufficiently small. Also we denote $\gamma=1+\mu$, $\mu=\nu^2$. By equation (4) we get

$$\begin{aligned}
&\|u(t)+\mathcal{N}_1(t)\|_{\mathbf{H}^\alpha}\le\|u(0)+\mathcal{N}_1(0)\|_{\mathbf{H}^\alpha}\\
&+C\int_0^t\|\mathcal{N}_2(\tau)+\mathcal{N}_3(\tau)+\mathcal{N}_4(\tau)\|_{\mathbf{H}^\alpha}d\tau.
\end{aligned}$$

Applying Proposition 2 with $s = 2$, $r = \frac{4}{2+\gamma-\alpha}$ and estimate (5) we obtain

$$\begin{aligned} &\|\mathcal{N}_1\|_{\mathbf{H}^\alpha} \le Ct^\mu \|\mathcal{T}_1(\overline{u},\overline{u}) + \mathcal{T}_2(u,u) + \mathcal{T}_3(u,\overline{u})\|_{\mathbf{H}^{\alpha-\gamma}} \\ \le\ & Ct^\mu \|u\|_{\mathbf{H}^\alpha} \|u\|_{\mathbf{H}^\gamma_r} \le C\rho^2 t^{\mu-\frac{\alpha-\gamma}{2}} \le C\rho^2 t^{-\mu}. \end{aligned}$$

Next applying the Hölder inequality, the Sobolev embedding theorem and estimate (5) with $q = \frac{2}{\nu}$ we get

$$\begin{aligned} &\|\mathcal{N}_2\|_{\mathbf{H}^\alpha} \le Ct^{\alpha-2} \|(u+\overline{u})\nabla(u+\overline{u})\|_{\mathbf{L}^2} \\ \le\ & Ct^{\alpha-2} \|u\|_{\mathbf{L}^{\frac{2}{\nu}}} \|u\|_{\mathbf{H}^\alpha} \le C\rho^2 t^{-\alpha}. \end{aligned}$$

The nonlinear term $\mathcal{N}_3$ is estimated similarly to $\mathcal{N}_1$

$$\|\mathcal{N}_3\|_{\mathbf{H}^\alpha} \le Ct^{\mu-1} \|\mathcal{T}_1(\overline{u},\overline{u}) + \mathcal{T}_2(u,u) + \mathcal{T}_3(u,\overline{u})\|_{\mathbf{H}^{\alpha-\gamma}} \le C\rho^2 t^{-\gamma}.$$

Finally we estimate $\mathcal{N}_4$ by Proposition 2 with $s = \frac{2}{2-\alpha}$, $r = \frac{4}{2+\gamma-\alpha}$ and estimate (5)

$$\begin{aligned} &\|\mathcal{N}_4\|_{\mathbf{H}^\alpha} \le Ct^{\alpha+\gamma-2} \sum_{j=1}^{3} \left\| \mathcal{T}_j\left(u, \langle i\nabla\rangle^{-1}(u+\overline{u})^2\right)\right\|_{\mathbf{H}^{1-\gamma}} \\ \le\ & Ct^{\alpha+\gamma-2} \|u\|_{\mathbf{H}^\gamma_r} \|u\|^2_{\mathbf{L}^{2s}} \le C\rho^3 t^{\frac{3}{2}\gamma-\frac{\alpha}{2}-2} \le C\rho^3 t^{-\gamma}. \end{aligned}$$

By equation (2) we find

$$\|u_t\|_{\mathbf{H}^{\alpha-1}} \le \|u\|_{\mathbf{H}^\alpha} + C\left\|(u+\overline{u})^2\right\|_{\mathbf{H}^{\alpha-2}} \le C\rho + C\rho^2.$$

Next by the identity $\mathcal{J} = i\mathcal{P} - i\mathcal{L}x$ we get

$$\|\mathcal{J}u\|_{\mathbf{H}^{\alpha-1}} \le \|\mathcal{P}u\|_{\mathbf{H}^{\alpha-1}} + \|\mathcal{L}xu\|_{\mathbf{H}^{\alpha-1}}. \tag{6}$$

Multiplying both sides of equation (2) by x, we obtain

$$\begin{aligned} \|\mathcal{L}xu\|_{\mathbf{H}^{\alpha-1}} \le\ & \|[x,\langle i\nabla\rangle]u\|_{\mathbf{H}^{\alpha-1}} + C\left\|\left[x,\langle i\nabla\rangle^{-1}\right](u+\overline{u})^2\right\|_{\mathbf{H}^{\alpha-1}} \\ & + C\left\|x(u+\overline{u})^2\right\|_{\mathbf{H}^{\alpha-2}}. \end{aligned} \tag{7}$$

Since $x = \langle i\nabla\rangle^{-1}\mathcal{J} - it\nabla\langle i\nabla\rangle^{-1}$, by estimate (5) we have

$$\begin{aligned} \left\|x(u+\overline{u})^2\right\|_{\mathbf{H}^{\alpha-2}} \le\ & C\left\|u\langle i\nabla\rangle^{-1}\mathcal{J}u\right\|_{\mathbf{L}^2} + Ct\left\|u\nabla\langle i\nabla\rangle^{-1}u\right\|_{\mathbf{L}^2} \\ \le\ & C\|u\|_{\mathbf{L}^4}\|\mathcal{J}u\|_{\mathbf{L}^2} + Ct\|u\|^2_{\mathbf{L}^4} \le C\rho^2. \end{aligned}$$

Therefore by (6) and (7) we obtain $\|\mathcal{J}u\|_{\mathbf{H}^{\alpha-1}} \leq C\rho + C\rho^2$. Thus we need to estimate the norm $\|\mathcal{P}u\|_{\mathbf{H}^{\alpha-1}}$. Since $[\mathcal{L},\mathcal{P}] = -i\langle i\nabla\rangle^{-1}\nabla\mathcal{L}$, the application of the operator $\mathcal{P}$ to equation (4) yields

$$\mathcal{L}\left(\mathcal{P}\left(u+\mathcal{N}_1\right)\right) = \left(\mathcal{P} - i\langle i\nabla\rangle^{-1}\nabla\right)\left(\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_4\right). \tag{8}$$

Hence integrating with respect to time and taking $\mathbf{H}^{\alpha-1}$-norm we get

$$\begin{aligned} &\|\mathcal{P}\left(u+\mathcal{N}_1\right)(t)\|_{\mathbf{H}^{\alpha-1}} \leq \|\mathcal{P}\left(u+\mathcal{N}_1\right)(0)\|_{\mathbf{H}^{\alpha-1}} \\ &+C\int_0^t \left\|\left(\mathcal{P} - i\langle i\nabla\rangle^{-1}\nabla\right)\left(\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_4\right)\right\|_{\mathbf{H}^{\alpha-1}} d\tau. \end{aligned} \tag{9}$$

By the above estimates we have

$$\left\|\langle i\nabla\rangle^{-1}\nabla\left(\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_4\right)\right\|_{\mathbf{H}^{\alpha-1}} \leq C\rho^2 t^{-\gamma} + C\rho^3 t^{-\gamma}.$$

We use the commutators

$$\begin{aligned} \left[x,\langle i\nabla\rangle^{\beta}\right] &= \beta\langle i\nabla\rangle^{\beta-2}\nabla, \left[x,\left\langle\frac{i}{t}\nabla\right\rangle^{\beta}\right] = \beta t^{-2}\left\langle\frac{i}{t}\nabla\right\rangle^{\beta-2}\nabla, \\ \left[\partial_t,\left\langle\frac{i}{t}\nabla\right\rangle^{-\beta}\right] &= -\beta t^{-3}\Delta\left\langle\frac{i}{t}\nabla\right\rangle^{-\beta-2}, \left[\mathcal{P},\langle i\nabla\rangle^{\beta}\right] = \beta\langle i\nabla\rangle^{\beta-2}\nabla\partial_t, \end{aligned}$$

$[\mathcal{P}_k,\partial_l] = -\delta_{kl}\partial_t$, and

$$\left[\mathcal{P},\left\langle\frac{i}{t}\nabla\right\rangle^{-\beta}\right] = \beta t^{-2}\left\langle\frac{i}{t}\nabla\right\rangle^{-\beta-2}\nabla - \beta t^{-3}x\Delta\left\langle\frac{i}{t}\nabla\right\rangle^{-\beta-2}.$$

Therefore

$$\begin{aligned} &\left\|\mathcal{P}\left\langle\frac{i}{t}\nabla\right\rangle^{-2}\langle i\nabla\rangle^{-1}\phi\right\|_{\mathbf{H}^{\alpha-1}} \leq C\left\|\left\langle\frac{i}{t}\nabla\right\rangle^{-2}\mathcal{P}\phi\right\|_{\mathbf{H}^{\alpha-2}} \\ &+C\|\partial_t\phi\|_{\mathbf{H}^{\alpha-3}} + C\|\phi\|_{\mathbf{H}^{\alpha-3}} + Ct^{-1}\|x\phi\|_{\mathbf{H}^{\alpha-2}}. \end{aligned} \tag{10}$$

Taking $\phi = \mathcal{T}_1(\overline{u},\overline{u}) + \mathcal{T}_2(u,u) + 2\mathcal{T}_3(u,\overline{u})$ we get

$$\begin{aligned} &\|\mathcal{P}\mathcal{N}_1\|_{\mathbf{H}^{\alpha-1}} \leq Ct^{\mu}\|\mathcal{P}\phi\|_{\mathbf{H}^{\alpha-1-\gamma}} + C\|\partial_t\phi\|_{\mathbf{H}^{\alpha-3}} \\ &+C\|\phi\|_{\mathbf{H}^{\alpha-3}} + Ct^{-1}\|x\phi\|_{\mathbf{H}^{\alpha-2}}. \end{aligned}$$

Applying Lemma 2 with $s=2$, $r=\frac{4}{2+\gamma-\alpha}$ and estimate (5) we find

$$\begin{aligned} \|\mathcal{P}\phi\|_{\mathbf{H}^{\alpha-1-\gamma}} &\leq C\left(\|\mathcal{P}u\|_{\mathbf{H}^{\alpha-1}} + \|\partial_t u\|_{\mathbf{H}^{\alpha-1}}\right)\|u\|_{\mathbf{H}^{\gamma}_r} \\ &\leq C\rho^2 t^{\frac{\gamma-\alpha}{2}} \leq C\rho^2 t^{-\mu}. \end{aligned}$$

Since $x = \langle i\nabla\rangle^{-1}\mathcal{J} - it\nabla\langle i\nabla\rangle^{-1}$ we have $t^{-1}\|u\|_{\mathbf{H}^{\alpha,1}} \le t^{-1}\|\mathcal{J}u\|_{\mathbf{H}^{\alpha-1}} + \|u\|_{\mathbf{H}^{\alpha}}$. Therefore by Lemma 2 we find

$$t^{-1}\|x\phi\|_{\mathbf{H}^{\alpha-2}} \le Ct^{-1}\|u\|_{\mathbf{H}^{\alpha,1}}\|u\|_{\mathbf{H}^{\gamma}_{r}} \le C\rho^2 t^{-\mu}.$$

Hence we get

$$\|\mathcal{P}\mathcal{N}_1\|_{\mathbf{H}^{\alpha-1}} \le C\rho^2 t^{-\mu}.$$

Next we estimate $\mathcal{P}\mathcal{N}_2$. By (10), the Sobolev embedding theorem and estimate (5) with $q = \frac{2}{\nu}$ we get

$$\begin{aligned} &\|\mathcal{P}\mathcal{N}_2\|_{\mathbf{H}^{\alpha-1}} \\ \le\ & Ct^{\alpha-2}\left(\|u\mathcal{P}u\|_{\mathbf{L}^2} + \|u\partial_t u\|_{\mathbf{L}^2} + \|u\|^2_{\mathbf{L}^4} + t^{-1}\left\|xu^2\right\|_{\mathbf{L}^2}\right) \\ \le\ & Ct^{\alpha-2}\|u\|_{\mathbf{L}^{\frac{2}{\nu}}}(\|\mathcal{P}u\|_{\mathbf{H}^{\alpha-1}} + \|\partial_t u\|_{\mathbf{H}^{\alpha-1}} \\ &+ \|u\|_{\mathbf{H}^{\alpha-1}} + t^{-1}\|xu\|_{\mathbf{H}^{\alpha-1}}) \le C\rho^2 t^{-\alpha}. \end{aligned}$$

The term $\mathcal{P}\mathcal{N}_3$ is estimated in the same manner as $\mathcal{P}\mathcal{N}_1$

$$\begin{aligned} &\|\mathcal{P}\mathcal{N}_3\|_{\mathbf{H}^{\alpha-1}} \le Ct^{\mu-1}\|\mathcal{P}\phi\|_{\mathbf{H}^{\alpha-1-\gamma}} + Ct^{-1}\|\partial_t\phi\|_{\mathbf{H}^{\alpha-3}} \\ &+Ct^{-1}\|\phi\|_{\mathbf{H}^{\alpha-3}} + Ct^{-2}\|x\phi\|_{\mathbf{H}^{\alpha-2}} \le C\rho^2 t^{-\gamma}. \end{aligned}$$

Finally we estimate $\mathcal{P}\mathcal{N}_4$. By (10) with $\phi = \mathcal{T}_1\left(\overline{u}, \overline{\mathcal{L}u}\right) + \mathcal{T}_2(u, \mathcal{L}u) + \mathcal{T}_3\left(u, \overline{\mathcal{L}u}\right) + \mathcal{T}_3(\mathcal{L}u, \overline{u})$ we find

$$\begin{aligned} \|\mathcal{P}\mathcal{N}_4\|_{\mathbf{H}^{\alpha-1}} \ \le\ & Ct^{\mu}\|\mathcal{P}\phi\|_{\mathbf{H}^{\alpha-1-\gamma}} + C\|\partial_t\phi\|_{\mathbf{H}^{\alpha-3}} \\ &+C\|\phi\|_{\mathbf{H}^{\alpha-3}} + Ct^{-1}\|x\phi\|_{\mathbf{H}^{\alpha-2}}. \end{aligned}$$

We use the equation (3) and Proposition 2 to get

$$\begin{aligned} &t^{\mu}\|\mathcal{P}\phi\|_{\mathbf{H}^{\alpha-1-\gamma}} \\ \le\ & Ct^{\mu}\sum_{j=1}^{3}\left\|\mathcal{P}\mathcal{T}_j\left(u, \left\langle\frac{i}{t}\nabla\right\rangle^{-2}\langle i\nabla\rangle^{-1}(u+\overline{u})^2\right)\right\|_{\mathbf{H}^{\alpha-1-\gamma}} \\ &+Ct^{\mu-2}\sum_{j=1}^{3}\left\|\mathcal{P}\mathcal{T}_j\left(u, \Delta\langle i\nabla\rangle^{-1}\left\langle\frac{i}{t}\nabla\right\rangle^{-2}(u+\overline{u})^2\right)\right\|_{\mathbf{H}^{\alpha-1-\gamma}} \\ \le\ & Ct^{\mu+\gamma-1}(\|\mathcal{P}u\|_{\mathbf{H}^{\alpha-1}} + \|u_t\|_{\mathbf{H}^{\alpha-1}})\left(\left\|u^2\right\|_{\mathbf{L}^{\frac{2}{2-\alpha}}} + t^{-1}\|u\nabla u\|_{\mathbf{L}^2}\right) \\ &+Ct^{\mu}\|u\|_{\mathbf{H}^{\alpha-1}_{4}}\left\|\mathcal{P}\left\langle\frac{i}{t}\nabla\right\rangle^{-2}\langle i\nabla\rangle^{-1}(u+\overline{u})^2\right\|_{\mathbf{H}^{\gamma}_{\frac{4}{4-\alpha}}} \\ &+Ct^{\mu-2}\|u\|_{\mathbf{H}^{\gamma}}\left\|\mathcal{P}\Delta\langle i\nabla\rangle^{-1}\left\langle\frac{i}{t}\nabla\right\rangle^{-2}(u+\overline{u})^2\right\|_{\mathbf{H}^{\alpha-1}}. \end{aligned}$$

Then as above in view of estimate (5) we obtain

$$t^{2\mu}\left\|u^2\right\|_{\mathbf{L}^{\frac{2}{2-\alpha}}}\le C\rho^2t^{2\gamma-2\alpha},$$

$$t^{2\mu-1}\left\|u\nabla u\right\|_{\mathbf{L}^2}\le Ct^{2\mu-1}\left\|u\right\|_{\mathbf{L}^{\frac{2}{\nu}}}\left\|u\right\|_{\mathbf{H}^{\alpha}}\le C\rho^2t^{2\gamma-2\alpha-1},$$

$$\begin{aligned}&t^{\mu}\left\|u\right\|_{\mathbf{H}^{\alpha-1}_4}\left\|\mathcal{P}\left\langle\frac{i}{t}\nabla\right\rangle^{-2}\left\langle i\nabla\right\rangle^{-1}\left(u+\overline{u}\right)^2\right\|_{\mathbf{H}^{\gamma}_{\frac{4}{4-\alpha}}}\\ \le\ & Ct^{2\mu-\frac{1}{2}}\left(\left\|u\mathcal{P}u\right\|_{\mathbf{L}^{\frac{4}{4-\alpha}}}+\left\|uu_t\right\|_{\mathbf{L}^{\frac{4}{4-\alpha}}}+t^{-1}\left\|xu^2\right\|_{\mathbf{L}^{\frac{4}{4-\alpha}}}\right)\\ \le\ & C\rho^3t^{-\gamma}\end{aligned}$$

and

$$\begin{aligned}&t^{\mu-2}\left\|u\right\|_{\mathbf{H}^{\gamma}}\left\|\mathcal{P}\Delta\left\langle i\nabla\right\rangle^{-1}\left\langle\frac{i}{t}\nabla\right\rangle^{-2}\left(u+\overline{u}\right)^2\right\|_{\mathbf{H}^{\alpha-1}}\\ \le\ & C\rho t^{\mu+\alpha-2}\left(\left\|u\mathcal{P}u\right\|_{\mathbf{L}^2}+\left\|uu_t\right\|_{\mathbf{L}^2}+t^{-1}\left\|xu^2\right\|_{\mathbf{L}^2}\right)\\ \le\ & C\rho^3t^{-\gamma}.\end{aligned}$$

Hence we find $t^{\mu}\left\|\mathcal{P}\phi\right\|_{\mathbf{H}^{\alpha-1-\gamma}}\le C\rho^3t^{-\gamma}$. In the same manner we estimate $\left\|\partial_t\phi\right\|_{\mathbf{H}^{\alpha-3}}$, $\left\|\phi\right\|_{\mathbf{H}^{\alpha-3}}$ and $t^{-1}\left\|x\phi\right\|_{\mathbf{H}^{\alpha-2}}$. Thus

$$\left\|\mathcal{P}\mathcal{N}_4\right\|_{\mathbf{H}^{\alpha-1}}\le C\rho^3t^{-\gamma}.$$

Collecting the above estimates we get

$$\left\|u\right\|_{\mathbf{X}_T}\le C\left\|u_0\right\|_{\mathbf{H}^{\alpha,1}}+C\rho^2. \tag{11}$$

The time decay estimate of Theorem 1 follows from (11). By the integral equation associated with (4)

$$w\left(t\right)=e^{-i\langle i\nabla\rangle t}u_0+\int_0^t e^{-i\langle i\nabla\rangle(t-\tau)}F\left(u\right)d\tau,$$

where $w=u+\mathcal{N}_1$, $F\left(u\right)=\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_4$, we obtain

$$e^{i\langle i\nabla\rangle t}w\left(t\right)-e^{i\langle i\nabla\rangle s}w\left(s\right)=\int_s^t e^{-i\langle i\nabla\rangle(t-\tau)}F\left(u\right)d\tau.$$

Therefore as above we get the estimate

$$\left\|e^{i\langle i\nabla\rangle t}w\left(t\right)-e^{i\langle i\nabla\rangle s}w\left(s\right)\right\|_{\mathbf{H}^{\alpha}}\le C\left\langle s\right\rangle^{-\mu}$$

for all $t > s > 0$. Since $w = u + \mathcal{N}_1$ and $\|\mathcal{N}_1(t)\|_{\mathbf{H}^\alpha} \le C\langle t\rangle^{-\mu}$ we find

$$\left\| e^{i\langle i\nabla\rangle t}u(t) - e^{i\langle i\nabla\rangle s}u(s)\right\|_{\mathbf{H}^\alpha} \le C\langle s\rangle^{-\mu}.$$

By the Cauchy–Schwarz inequality we find

$$\begin{aligned} &\left\| e^{i\langle i\nabla\rangle t}u(t) - e^{i\langle i\nabla\rangle s}u(s)\right\|_{\mathbf{H}^{\alpha.\theta}} \\ \le\ & C\left\| e^{i\langle i\nabla\rangle t}u(t) - e^{i\langle i\nabla\rangle s}u(s)\right\|_{\mathbf{H}^{\alpha,1}}^{\theta} \\ & \times \left\| e^{i\langle i\nabla\rangle t}u(t) - e^{i\langle i\nabla\rangle s}u(s)\right\|_{\mathbf{H}^{\alpha}}^{1-\theta} \le C\langle s\rangle^{-(1-\theta)} \end{aligned}$$

with $0 < \theta < 1$. This completes the proof of the theorem.

Remark 1. We briefly explain the reason why the proof given in [10] requires smooth data. By the energy estimate we get from (2)

$$\sum_{|\alpha|\le 2m} \|\Lambda^\alpha u\|_{\mathbf{L}^2} \le C\varepsilon + C\int_0^t \sum_{|\alpha|\le m} \|\Lambda^\alpha u\|_{\mathbf{L}^\infty} \sum_{|\alpha|\le 2m} \|\Lambda^\alpha u\|_{\mathbf{L}^2}\, dt$$

where $\Lambda = (\partial_t, \nabla, \mathcal{P}, \Omega)$. Then applying the time decay estimate of Georgiev to (4) we obtain a rough estimate

$$\begin{aligned} &\langle t\rangle \sum_{|\alpha|\le m} \|\Lambda^\alpha (u - \mathcal{T}_1(\overline{u},\overline{u}) - \mathcal{T}_2(u,u) - \mathcal{T}_3(u,\overline{u}))\|_{\mathbf{L}^\infty} \\ \le\ & C\varepsilon + C\int_0^t \langle t\rangle^{1-\varepsilon} \sum_{|\alpha|\le\left[\frac{m+l}{2}\right]+k} \|\Lambda^\alpha u\|_{\mathbf{L}^\infty}^2 \sum_{|\alpha|\le m+l+k} \|\Lambda^\alpha u\|_{\mathbf{L}^2}\, dt \\ \le\ & C\varepsilon + C\int_0^t \left(\langle t\rangle \sum_{|\alpha|\le m} \|\Lambda^\alpha u\|_{\mathbf{L}^\infty}\right)^2 \sum_{|\alpha|\le m+l+k} \|\Lambda^\alpha u\|_{\mathbf{L}^2} \frac{dt}{\langle t\rangle^{1+\varepsilon}}, \end{aligned}$$

if $\left[\frac{m+l}{2}\right] + k \le m$ (the correct estimate can be written by the Paley-Littlewood partition of unity). From which the desired estimate follows if we take $k = 4$ due to the derivative loss in the estimates for the bilinear operators and $l = 4$ due to the derivative loss in the time decay estimates by Georgiev. Hence we need $2m \ge 24$.

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