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Some recent progress on standing waves of FitzHugh–Nagumo system

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Abstract.

The FitzHugh–Nagumo system is a well-known reaction-diffusion model for exhibiting self-organized patterns. Besides regular patterns found in a neighborhood of Turing's instability, localized structures are also observed in experiment and numerical simulation. In particular, fronts and pulses are the most well-known localized structures in reaction-diffusion systems. This article is aimed at some recent results on the variational approach for studying standing waves of FitzHugh–Nagumo system.

§1. Introduction

In recent years pattern formation became an important research field, in which the traditional disciplines of physics, chemistry, biology and mathematics interact and significant progress [2], [24] has been made through the exchange of ideas. Following from Turing [37], reaction-diffusion systems serve as relevant models [5], [24], [35] for studying complex patterns [15], [22], [27] in several fields of sciences; not only these regular patterns found in a neighborhood of Turing's instability, localized structures [5], [24] also are observed in experiment and numerical simulation.

The existence of wavefronts and pulses is one of the central issues in understanding dynamics of reaction-diffusion system. Planar wavefronts are generic structures connecting two different homogeneous states of a system with bi-stable nonlinearity. A well-known model is the Allen–Cahn equation, in which a standing wavefront can easily be found from

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phase plane analysis; as matter of fact the explicit form of this solution is known. In the Allen-Cahn model, wavefront played a crucial role in the generation of interface in earlier stage of phase transition. Moreover, in the situation of slow diffusion, the motion of interface [12] is driven by its mean curvature. For the Nagumo equation, phase plane analysis shows that there is a standing pulse: nevertheless it is not stable.

In the study of diffusion-induced instability, the FitzHugh–Nagumo model received a great deal of attention:

$$u_t = d_1 \Delta u + f(u) - v, \tag{1}$$

$$\tau v_t = d_2 \Delta v + u - \gamma v. \tag{2}$$

Here u can be viewed as an activator while v acts as an inhibitor. Variants of (1)–(2) also appeared in neural net models for short-term memory [26] and in studying nerve cells of heart muscle [30]. The aim of this article is to report some recent results by using variational methods to study standing waves of FitzHugh–Nagumo system.

§2. Planar standing wavefront

In seek of standing wavefronts of (1)–(2), a natural question is to clarify the influence of diffusivity [11] to the existence of such waves. This amounts to studying the heteroclinic solutions of a second order Hamiltonian system

$$-du'' = f(u) - v, \tag{3}$$

$$-v'' = u - \gamma v. \tag{4}$$

The ratio of diffusivities is denoted by d in the non-dimensional form of (3)–(4).

Theorem 1. *Let $a = 2(\beta + 1)/3$. If $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$ and $d > \gamma^{-2}$, there exists a standing wave solution $(u(x), v(x))$ of (1)–(2) such that $(u, v) \to (0, 0)$ as $x \to -\infty$ and $(u, v) \to (a, a/\gamma)$ as $x \to \infty$.*

Let us remark that in system (1)–(2), two homogeneous states $(0, 0)$ and $(a, a/\gamma)$ are in the same energy levels only if $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$. As in the Allen–Cahn equation, standing wavefront appears in reaction-diffusion system with balanced potential wells.

In the proof of Theorem 1, a variational functional with a nonlocal term will be studied. We briefly sketch the main idea of this approach as follows. Pick a function $u_0 \in C^\infty(\mathbb{R})$ with the property

$$u_0(x) = \begin{cases} a & \text{if} \quad x \geq 1, \\ 0 & \text{if} \quad x \leq -1. \end{cases} \tag{5}$$

Let D_* be the differential operator $\gamma - \partial^2/\partial x^2$. For a given $\psi \in H^1(\mathbb{R})$, there is a unique solution satisfying

$$D_* w = \psi, \; w \in H^1(\mathbb{R}), \tag{6}$$

which will be denoted by $L\psi$. In the way to set up a variational functional for solving a heteroclinic solution of (3)–(4), a simple choice is $u_0 = D_* v_0$ with v_0 being a C^∞-function satisfying

$$v_0(x) = \begin{cases} a/\gamma & \text{for} \quad x \geq 1, \\ 0 & \text{for} \quad x \leq -1. \end{cases} \tag{7}$$

Let $F(\xi) = -\int_0^\xi f(s)ds$. For $\psi \in H^1(\mathbb{R})$, define

$$J_1(\psi) = \int_{-\infty}^{\infty} \frac{1}{2}[d(u_0' + \psi')^2 + (u_0 + \psi)(v_0 + L\psi)] + F(u_0 + \psi)dx. \tag{8}$$

If

$$J_1(\hat{\psi}) = \inf_{\psi \in H^1(\mathbb{R})} J_1(\psi)$$

and $(u, v) = (u_0 + \hat{\psi}, v_0 + L\hat{\psi})$, then (u, v) is a heteroclinic solution of (3)–(4).

§3. Planar standing pulse

Fronts and pulses are the most well-known one-dimensional waves in reaction-diffusion systems. The profile of a pulse may stay in close proximity to a trivial background state except in one localized spatial region where change is substantial. In Hamiltonian systems pulses often result from balance between dispersion and nonlinearity [2], and they usually represent states which are far away from the homogeneous equilibrium.

When there is no diffusion term in (2), the system

$$u_t = du_{xx} + f(u) - v, \tag{9}$$

$$v_t = \varepsilon(u - \gamma v), \tag{10}$$

has been considered as a model for the Hodgkin–Huxley system [15], [27], [35] to describe the behavior of electrical impulses in the axon of the squid. The existence of traveling pulses of (9)–(10) has been established [6], [13], [18], [23] for $\varepsilon << 1$; in their works the system was treated as a singular perturbation problem in which the pulse is constructed by piecing together solutions of certain reduced systems. Related stability questions have been studied in [20], [42]. Whether such a pulse exists for ε not necessarily small is left open.

The absence of diffusion for v in (10) simplifies the existence analysis of standing pulse. If (u, v) is a standing pulse of (9)–(10), it is clear that $v = u/\gamma$ and u satisfies

$$du_{xx} + f(u) - u/\gamma = 0. \tag{11}$$

When $\gamma << 1$, (11) can be expressed as $du_{xx} - p(x)u = 0$ for some positive function $p \equiv u^2 - (\beta+1)u + (\beta+1/\gamma)$. An immediate consequence of the maximum principle excludes the possibility of having a standing pulse. On the other hand if $\gamma >> 1$, (11) can be cast in the form $du_{xx} - u(u-a_1)(u-a_2) = 0$ for some $\beta < a_1 < a_2 < 1$. Phase plane analysis indicates that (11) possesses a positive solution u which is homoclinic to 0 and $a_1 < \max u < a_2$.

Reinecke and Sweers [31] used finite domain approximation to establish a positive standing pulse solution for (1)–(2). They treated the case $\gamma >> 1$ and sufficiently large d. The presence of three distinct homogeneous solutions is essential in their construction of certain supersolutions.

In contrast with the known results of positive standing pulses, when $\gamma << 1$, a standing pulse (u, v) constructed in [7] is of different shape: u actually changes sign while v stays positive.

Theorem 2. *Let $\beta \in (0, 1/2)$ be given.*

(i) *There exist $\hat{\gamma} > 0$ and $\hat{d} = \hat{d}(\gamma) > 0$ such that if $\gamma < \hat{\gamma}$ and $d < \hat{d}$, then there is a standing pulse solution (u, v) of (1)–(2).*

(ii) *Both u and v are even functions on $(-\infty, \infty)$ and satisfy $(u, v) \to (0, 0)$ as $x \to \infty$.*

(iii) *u changes signs exactly once on $(0, \infty)$ while $v > 0$ and $v' < 0$ on $(0, \infty)$.*

By experimenting numerical calculation [36] with various initial data, we find a stable standing pulse solution of (1)–(2) with profile as shown in Fig. 1.

§4. Standing wavefront joining with Turing patterns

An interesting article by Kondo and Asai [22] demonstrated that the pattern formation and change on the skin of tropical fishes can be predicted well by reaction-diffusion models of Turing type. A common pattern structure in fish skin is the rearrangement of stripe pattern; the

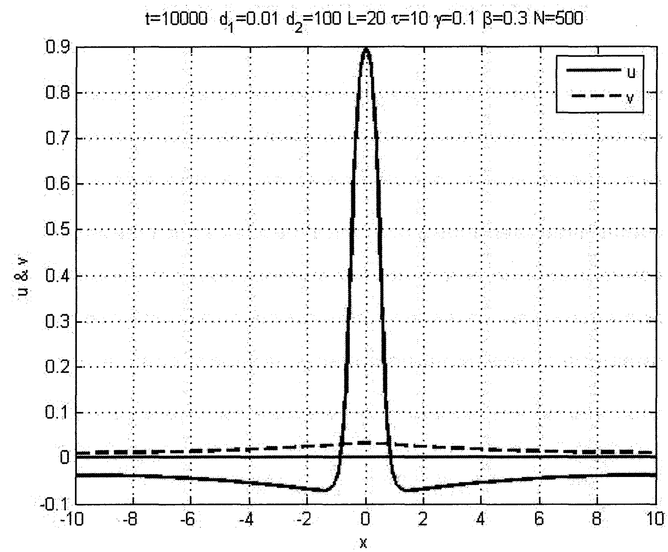

Fig. 1. Standing pulse solution u changes sign and v stays positive. Here $\gamma = 0.1$, $\beta = 0.3$ and $d = 10^{-4}$

number of stripes tends to increase with body size and defect like heteroclinic solution appeared between the patterns with different number of stripes. According to the observation [22], defect made change time to time during the growth of skin. The reaction-diffusion wave in generating stripe pattern is a kind of standing wave. Our particular interest is to seek standing wavefront joining with Turing patterns.

To investigate the spatially heterogeneous steady states of (1)–(2), we study the following system of elliptic equations:

$$-d_1 \Delta u = f(u) - v, \tag{12}$$

$$-d_2 \Delta v = u - \gamma v, \tag{13}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\omega} = \left.\frac{\partial v}{\partial \nu}\right|_{\partial\omega} = 0, \tag{14}$$

where ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\omega$. It is easily seen that a solution of (12)–(14) is a critical point of $\Phi(u, v)$ defined by

$$\Phi(u, v) = \int_\omega \frac{1}{2}(d_1|\nabla u|^2 - d_2|\nabla v|^2) + F(u, v), \tag{15}$$

where $F(u,v) = uv - \frac{\gamma}{2}v^2 - \int_0^u f(\xi)d\xi$. In fact, with the reaction terms being in coupled with a skew-gradient structure [40], (1)–(2) can be expressed as

$$Mw_t = Dw_{xx} + Q\nabla F(w) \tag{16}$$

if we set $w = (u,v)$; the general situation is to consider a system in which $w(x,t)$ is an n-dimensional vector function, M and D are $n \times n$ diagonal matrices with positive entries,

$$Q = \begin{pmatrix} I_j & 0 \\ 0 & -I_{n-j} \end{pmatrix},$$

and I_j is the $j \times j$ identity matrix. Following the work [40] of Yanagida, (16) will be referred to as a skew-gradient system.

In dealing with a strongly indefinite functional Φ, a critical point theorem established by Benci and Rabinowitz [4] can be employed to study the existence of solutions of (12)–(14). Let $\Phi''(\bar{u},\bar{v})$ be the second Frechet derivative of Φ at $(\bar{u},\bar{v})$. A critical point $(\bar{u},\bar{v})$ is said to be non-degenerate if the null space of $\Phi''(\bar{u},\bar{v})$ is trivial. Concerning the stability of steady states, Yanagida [40] introduced the notion of mini-maximizer of (15) as follows: A steady state $(\bar{u},\bar{v})$ is called a mini-maximizer of Φ if $\bar{u}$ is a local minimizer of $\Phi(\cdot,\bar{v})$ and $\bar{v}$ is a local maximizer of $\Phi(\bar{u},\cdot)$. Yanagida showed that non-degenerate mini-maximizers of Φ are linearly stable. This result gives a natural generalization of a stability criterion for the gradient system in which all the non-degenerate local minimizers are stable steady states.

More recently the stability of steady states of (16) was studied [10] in conjunction with a relative Morse index associated with the critical point of (15). Let $E = H^1(\omega) \oplus H^1(\omega)$. If G is a self-adjoint Fredholm opeartor on E, there is a unique G-invariant orthogonal splitting

$$E = E_+(G) \oplus E_-(G) \oplus E_0(G)$$

with $E_+(G), E_-(G)$ and $E_0(G)$ being respectively the subspaces on which G is positive definite, negative definite and null. In the way of investigating the stability of steady states of (12)–(14), Q is extended to an operator mapped from $H^1(\omega) \oplus H^1(\omega)$ onto itself. Suppose $(\bar{u},\bar{v})$ is a critical point of Φ. For the pair of Fredholm operators Q and $\Phi''(\bar{u},\bar{v})$, we define a relative Morse index $i(Q,\Phi''(\bar{u},\bar{v}))$ to be the relative dimension of $E_-(Q)$ with respect to $E_-(\Phi''(\bar{u},\bar{v}))$. For a gradient system, a non-degenerate critical point with non-zero Morse index is an unstable steady state. The next theorem [10] gives a parallel result for skew-gradient system.

Theorem 3. *Suppose* $i(Q, \Phi''(\bar{u}, \bar{v})) \neq 0$ *and* $dim E_0(\Phi''(\bar{u}, \bar{v})) = 0$, *then for any* $\tau > 0$, $(\bar{u}, \bar{v})$ *is an unstable steady state of* (12)–(14).

Remark. For a critical point $(\bar{u}, \bar{v})$ of Φ, the work of Abbondandolo and Molina [1] provides a way to calculate $i(Q, \Phi''(\bar{u}, \bar{v}))$.

In [40] Yanagida showed that non-degenerate mini-maximers of Φ are always stable for any $\tau > 0$. He also pointed out that in convex domain a mini-maximizer of (12)–(14) must be spatially homogeneous. For a given system, the reaction rates in general should also play significant roles in generating a self-organized pattern. We next illustrate a criterion [10] for justifying a Turing pattern in skew-gradient system with its stability depending on the reaction rates. Let P^+ and P^- be the orthogonal projections from E to $E_+(Q)$ and $E_-(Q)$ respectively. Set $\Gamma = H^2(\omega) \oplus H^2(\omega)$,

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\tau}} \end{pmatrix} \text{ and } D = \begin{pmatrix} -d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Define $\hat{\Psi} = T(D\Delta - \nabla^2 F(\bar{u}, \bar{v}))T$, $\hat{\Psi}_+ = P^+ \hat{\Psi} P^+$, $\hat{\Psi}_- = P^- \hat{\Psi} P^-$,

$$\rho_i = \inf_{z \in \Gamma} \frac{\langle \hat{\Psi}_+ z, z \rangle_{L^2}}{\|P^+ z\|_{L^2}^2},$$

and

$$\rho_s = \sup_{z \in \Gamma} \frac{\langle \hat{\Psi}_- z, z \rangle_{L^2}}{\|P^- z\|_{L^2}^2}.$$

Theorem 4. *Assume that* $i(Q, \Phi''(\bar{u}, \bar{v})) = 0$ *and* $dim E_0(\Phi''(\bar{u}, \bar{v})) = 0$. *Then* $(\bar{u}, \bar{v})$ *is stable if* $\rho_i > \rho_s$.

Standing wavefront of scalar reaction-diffusion equation on cylinders has been studied in [3], [38], [39]. A typical example is

$$u_t = \Delta u + u_{yy} + h(u), \text{ for } (x, y) \in \omega \times \mathbb{R}, t > 0, \tag{17}$$

$$\frac{\partial u}{\partial \nu} = 0, \text{ on } \partial\omega \times \mathbb{R}, \tag{18}$$

$$u(x, y) \to u_\pm(x) \text{ as } y \to \pm\infty, \tag{19}$$

where u_+ and u_- are the solutions of

$$\Delta u + h(u) = 0, \qquad x \in \omega, \tag{20}$$

$$\frac{\partial u}{\partial \nu}|_{\partial\omega} = 0. \tag{21}$$

The case of homogeneous Dirichlet boundary conditions has been treated as well. In [38] Vega considered the wavefront solution $\bar{u}(x,y)$ with the property

$$u_+(x) > \bar{u}(x,y) > u_-(x) \text{ for all } (x,y) \in \omega \times \mathbb{R}.$$

Under certain stability assumptions on u_+ and u_-, he proved existence and uniqueness results [38], [39] for this type of standing waves. As a consequence of the maximum principle, such a wave is strictly increasing in the y-direction. If ω is convex all the stable solutions of (20)–(21) must be constant.

In many species of tropical fishes, the stripes run in parallel either to the anterior-posterior axis or to the dorso-ventral axis. It has been observed that on the two dimensional plane the stripe pattern generated by standard reaction-diffusion models of Turing type does not have a fixed direction. In the subsequent works of [22], the authors [33], [34] proposed that anisotropic diffusion might have an effect on the contrasting difference in the directionality of stripes on the fish skin, because most scales are arranged parallel to the anterior-posterior axis. This suggests that the substances (for example, activators and inhibitors) controlling the pattern formation may diffuse along the anterior-posterior axis at a speed different from that along the dorso-ventral axis. Motivated by [22], [33], [34], we consider a FitzHugh–Nagumo system [8], [9] with anisotropic diffusion:

$$\begin{aligned} (22) \qquad u_t &= d_1\Delta u + d_3 u_{yy} + f(u) - v, \\ (23) \qquad \tau v_t &= d_2\Delta v + d_4 v_{yy} + u - \gamma v, \ t > 0, \ (x,y) \in \omega \times \mathbb{R}. \end{aligned}$$

We look for a standing wavefront (u,v) with asymptotic properties $(u(x,y),\ v(x,y)) \to (u_1(x), v_1(x))$ as $y \to -\infty$ and $(u(x,y), v(x,y)) \to (u_2(x), v_2(x))$ as $y \to +\infty$. Here $(u_i(x), v_i(x))$, $i = 1, 2$, are the solutions of (12)–(14). The situation of (u_i, v_i) being a non-constant solution is of particular interest.

For a given $u \in H^1(\omega)$, we let $A_0 u$ denote the unique solution of

$$-d_2\Delta v + \gamma v = u, \frac{\partial v}{\partial \nu}|_{\partial\omega} = 0.$$

Then $(\bar{u}, A_0\bar{u})$ is a solution of (12)–(14) if and only if $\bar{u}$ is a critical point of J_0 defined by

$$J_0(u) = \int_\omega [\frac{d_1}{2}|\nabla u|^2 + \frac{1}{2} u A_0 u - \int_0^u f(\xi) d\xi] dx.$$

By making use of variational structure associated with J_0, many existence results [10], [14], [29], [32] for the non-constant steady states of (12)–(14) have been established.

Let u be a critical point of J_0. Straightforward calculation yields

$$J_0''(u) = -d_1\Delta + A_0 - f'(u).$$

Here J_0'' is the second Frechet derivative of J_0, and the Morse index of this critical point will be denoted by $i_*(J_0''(u))$. On the other hand, for any critical point u of J_0, we know that (u, A_0u) is a critical point of Φ. With the aid of the next proposition, we are able to justify the stability of (u, A_0u) if u is a critical point of J_0.

Proposition 1. *Suppose u is a critical point of J_0 and $v = A_0u$, then*

$$dimE_0(J_0''(u)) = dimE_0(\Phi''(u,v))$$

and

$$i_*(J_0''(u)) = i(Q, \Phi''(u,v)).$$

Since there exist $C_1 > 0$ and $C_2 > 0$ such that $F(\xi) \geq \frac{1}{2}C_1\xi^2 - C_2$, by adding a constant to F if necessary, we may assume that

$$\inf_{u \in H^1(\omega)} J_0(u) = 0.$$

Let $M_0 = \{u | u \in H^1(\omega) \text{ and } J_0(u) = 0\}$. A stable non-constant steady state of (12)–(14) is referred to as a Turing pattern.

Let $\Omega = \omega \times \mathbb{R}$ and $\hat{E} = H^1_{loc}(\Omega) \cap L^2(\Omega)$. For $i \neq j$, let u_i^*, $u_j^* \in M_0$ and $\hat{v}$ be a function in $C^\infty(\Omega, \mathbb{R})$ with the following properties:

(i) $\frac{\partial \hat{v}}{\partial \nu}(x,y) = 0$ if $(x,y) \in \partial\Omega$

(ii) $\hat{v}(x,y) = \begin{cases} A_0u_i^*(x) & \text{if} \quad y \leq -1, \\ A_0u_j^*(x) & \text{if} \quad y \geq 1. \end{cases}$

For a given $\psi \in \hat{E}$, we let $A\psi$ denote the unique solution of

$$-d_2\Delta v - d_4v_{yy} + \gamma v = \psi, \ v \in \hat{E}.$$

Define $\hat{u} = -d_2\Delta\hat{v} - d_4\hat{v}_{yy} + \gamma\hat{v}$ and

$$\begin{aligned} \Psi_{i,j}(\psi) \quad = \quad & \int_\Omega \frac{1}{2}[d_1|\nabla(\hat{u}+\psi)|^2 + d_3\left|\frac{\partial(\hat{u}+\psi)}{\partial y}\right|^2 + (\hat{u}+\psi)(\hat{v}+A\psi)] \\ & + F(\hat{u}+\psi)dxdy \end{aligned}$$

for $\psi \in \hat{E}$. Set

$$c_{i,j} = \inf_{\psi \in \hat{E}} \Psi_{i,j}(\psi)$$

and

$$c_i = \inf_k c_{i,k}.$$

Then $(\hat{u} + \psi, \hat{v} + A\psi)$ is a standing wave of (22)–(23) if $c_i = c_{i,j}$ and ψ is a minimizer of $\Psi_{i,j}$ over $\hat{E}$.

Theorem 5. *Assume that* $\gamma > \sqrt{\frac{d_4}{d_3}}$. *Then there exists a standing wave solution* $(u(x,y), v(x,y))$ *of* (22)–(23) *such that* $(u(x,y), v(x,y)) \to (u_i^*(x), A_0 u_i^*(x))$ *as* $y \to -\infty$ *and* $(u(x,y), v(x,y)) \to (u_j^*(x), A_0 u_j^*(x))$ *as* $y \to \infty$.

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