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Nonlinear dynamics of three solvable aggregation models

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Abstract.

For three interesting kinetic models of clustering, we review results on dynamical phenomena related to the approach to self-similar form and their close connections to probability theory. For Smoluchowski's coagulation equation with additive rate kernel, we describe the scaling attractor and show how dynamics on it is trivialized in terms of Bertoin's Lévy–Khintchine-like representation. For a model motivated by domain coarsening dynamics in the Allen–Cahn equation, we describe the remarkable solution procedure found by Gallay and Mielke, and the ensuing classification of domains of attraction for self-similar solutions. And we describe the rigorous connection between Smoluchowski's equation and random shock coalescence in the inviscid Burgers equation. Recent work of Menon indicates that the latter problem is completely integrable for initial data comprising a spatial Markov process.

§1. Introduction

An important goal in nonlinear science is to understand something about systems that appear to behave *predictably*, but whose complexity precludes detailed analysis. One of the challenges in dealing with such systems, in fact, is to identify good *statistics*—some properties of the system—about which something can be said.

In this article we deal with models for the time evolution of a rather simple statistic in certain systems: the *size distribution of clusters*. In particular, we shall review results concerning three kinds of kinetic models of coagulation and clustering:

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- Smoluchowski's coagulation equation with additive rate kernel
- Min-driven clustering model of domain coarsening
- Random shock coalescence in the inviscid Burgers equation

While infinite-dimensional dynamical systems may exhibit a rich variety of phenomena in general, we will focus on reviewing some recent work on *dynamic scaling limits* in these models. Our study will point out some close analogies with classical limit theorems in probability theory. We also encounter some beautiful solution procedures which hint at deeper results, including connections to completely integrable systems theory.

The models mentioned above are dealt with successively in the subsequent sections of this paper. In Section 2, our main aim is to describe the notion of the *scaling attractor* introduced in [31], and the trivialization of dynamics on the attractor which is obtained from Bertoin's Lévy-Khintchine representation of eternal solutions of Smoluchowski's equation. The min-driven clustering model that we study in Section 3 was originally motivated by the problem of domain coarsening in the 1D Allen–Cahn PDE. We will describe the remarkable solution procedure found by Gallay and Mielke [20] for this model, and discuss the classification of domains of attraction for self-similar limits found in [28], and the dynamics of eternal solutions. In Section 4 we describe the direct connection found by Carraro and Duchon [8], [9] and Bertoin [3] between Smoluchowski's equation and the problem of coalescence of shock waves for the inviscid Burgers equation with initial data given by a random walk (a Lévy process with no upward jumps). For scalar conservation laws with more general kinds of Markov-process initial data with no upward jumps, recent work of Menon and Srinivasan [32] and Menon [27] formally indicates the complete integrability of the dynamics of entropy solutions.

§2. Scaling dynamics for a solvable coagulation equation

The main themes of this section will involve dynamic scaling limits and their connections to dynamical systems concepts and limit theorems in probability theory. These concepts can be compared to renormalization group methods, for example, but the fundamental ideas really originated with the pioneers of probability theory in the 1920s and 1930s, workers such as Lévy, Khintchine and Doeblin. I believe that the power of these ideas has not been fully appreciated in PDE theory, and there is considerable promise for extending their reach to address many problems beyond the ones we consider here.

2.1. A basic model for clustering.

Smoluchowski's coagulation equation is an oversimplified model for the aggregation or clustering of matter. It describes the evolution of a simple statistic, the *distribution of cluster size x at time t.* One describes this using a cumulative distribution function (CDF): Let

$$\nu_t(x) = \text{(expected) number of clusters of size} \leq x \text{ at time } t$$

and let $n(x,t)$ denote a density for this distribution (presuming it exists), so that

$$\nu_t(x) = \int_0^x n(y,t)\,dy.$$

The mechanism of clustering can be indicated schematically as follows:

$$\boxed{x} + \boxed{y} \Longrightarrow \boxed{x+y}$$

$$\boxed{x-y} + \boxed{y} \Longrightarrow \boxed{x}$$

Clusters of size x and y join to form one of size $x+y$ at presumed rate $K(x,y)n(x,t)n(y,t)$, separately proportional to the populations of 'incoming' clusters. We assume $K(x,y) = K(y,x) \geq 0$. Integrating to account for all events involving loss or gain of size-x clusters (and avoiding double-counting) yields the rate equation

$$(1) \quad \partial_t n(x,t) = -\int_0^\infty K(x,y)n(x,t)n(y,t)\,dy \qquad \text{(loss)}$$
$$+\frac{1}{2}\int_0^x K(x-y,y)n(x-y,t)n(y,t)\,dy \qquad \text{(gain)}$$

This is Smoluchowski's coagulation equation, first derived in size-discrete form in 1916 [35] to model Brownian particles, with the rate kernel

$$(2) \qquad K(x,y) = (x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3}).$$

Many other forms of rates appear in an extensive physical literature, in a wide range of fields: astrophysics, chemistry of colloids, polymers, aerosols (fog & smog), lines of descent in population biology, and also in probability theory (renewal processes), and random graph theory. See the surveys by Drake [12] and Aldous [1].

2.2. Dynamic scaling behavior

In a pure coagulation process, cluster sizes simply grow in time. Claimed in many physical papers and seen in numerics is *dynamic scaling* behavior (see the survey by Leyvraz [25]): As time increases, the size distribution approaches a *universal self-similar form*,

$$n(x,t) \sim a(t)f(b(t)x),$$

with some scaling profile f. This raises compelling mathematical questions: *Is this true? When? And why?* Naturally one should expect such behavior only if the system is (at least asymptotically) scale-free, meaning the kernel K is *homogeneous*, meaning that for any $c > 0$,

$$K(cx, cy) = c^\lambda K(x,y).$$

There are two first remarks to make about this:

1. Conservation of mass imposes the constraint $a = b^2$ because

$$\int_0^\infty xn(x,t)\,dx = \int_0^\infty axf(bx)\,dx = \frac{a}{b^2}\int_0^\infty yf(y)\,dy.$$

2. Explicit self-similar solutions are known for special rate kernels:

$$\begin{aligned} K = 2: \quad & n(x,t) = \frac{1}{t^2}\exp(-x/t), \\ K = x+y: \quad & n(x,t) = \frac{1}{\sqrt{2\pi}}x^{-3/2}e^{-t}\exp(-xe^{-2t}/2), \\ K = xy: \quad & n(x,t) = \frac{1}{\sqrt{2\pi}}x^{-5/2}\exp(-t^2x/2). \end{aligned} \tag{3}$$

The existence of self-similar solutions in general was a long-standing open problem; the first results were achieved in 2005 by Fournier and Laurençot [17] and Escobedo *et al.* [14]. But some kernels important in applications are not covered by these results, such as that in (2).

The question of universal approach to the self-similar forms above is analogous to the *central limit theorem* in probability theory. Using the language of dynamical systems theory, one can list a number of further questions about scaling limits for Smoluchowski's equation with solvable kernel, that one can address rather completely by analogy to the more general limit theorems for heavy-tailed distributions found in probability:

(1) What are all the scaling solutions that exist?
(2) What are their domains of attraction? (Can one characterize which initial data converge to self-similar form as $t \to \infty$?)
(3) What other scaling limit points can exist, along subsequences $t_n \to \infty$? (We call the set of such points the *scaling attractor* for the system.)

(4) Can one describe precisely the "ultimate dynamics" on the scaling attractor?

In probability theory one considers scaled sums $S_n = (X_1 + \ldots + X_n)/c_n$ of iid random variables. The scaling limits as $n \to \infty$ are the *Lévy stable laws.* Limits along arbitrary subsequences $n_j \to \infty$ are the *infinitely divisible laws*, and are represented by the famous Lévy–Khintchine formula in terms of a class of measures. Our essential aim here is to give an account of 'infinite divisibility' for Smoluchowski's equation with $K = x + y$, following the works [29, 31].

2.3. Weak form of Smoluchowski's equation

To model in a unified way cluster size distributions that may either be discrete or have continuous densities, it is desirable to have a theory of measure solutions which satisfy a weak form of the equation. This is based on a *generalized moment identity* for Smoluchowski's coagulation equation. Multiplying (1) by a test function $a(x)$ and integrating we formally get

$$\begin{aligned}(4)\quad & \partial_t \int_0^\infty a(x) n(x,t)\,dx = \\ & \frac{1}{2}\int_0^\infty \int_0^\infty \big(a(x+y) - a(x) - a(y)\big) K(x,y) n(x,t) n(y,t)\,dx\,dy.\end{aligned}$$

Integrating in t yields the weak form for measure solutions $t \mapsto \nu_t(dx)$:

$$\begin{aligned}(5)\quad & \int_0^\infty a(x)\nu_t(dx) = \int_0^\infty a(x)\nu_{t_0}(dx) + \\ & \frac{1}{2}\int_{t_0}^t \int_0^\infty \int_0^\infty \big(a(x+y) - a(x) - a(y)\big) K(x,y)\, \nu_s(dx)\nu_s(dy)\,ds,\end{aligned}$$

for a suitable class of test functions $a(x)$ that depends on K and we will not specify here.

Formally, total mass is always conserved: taking $a(x) = x$ we find $\int_0^\infty x n(x,t)\,dx$ =const. This can *fail* however for fast-growing rate kernels K homogeneous of degree $\lambda > 1$: Solutions in this case can start to lose mass in finite time. This phenomenon is called *gelation* and has attracted much interest. There are basic math papers on it by Jeon [23] and by Escobedo *et al.* [13], and many physical papers. Much remains to be understood, however; see [26], [18]. The term 'gelation' suggests the loss of mass to an 'infinite cluster', but the model as presented does not contain such a cluster.

2.4. Solution procedure for $K = x + y$

For definiteness, and due to its special relation to the theory of shock clustering described below in Section 3, we restrict attention to the additive rate kernel $K(x, y) = x + y$. Consider moments $m_p = \int_0^\infty x^p n(x,t)\,dx$. By conservation of mass, $m_1(t)$ is constant—Often we normalize so $m_1 = 1$. For $p = 0$, using $a(x) = 1$ in the weak form yields

$$\begin{aligned} \dot{m}_0 &= -\frac{1}{2}\int_0^\infty \int_0^\infty (x+y)n(x)n(y)\,dx\,dy = -m_1 m_0, \\ m_0(t) &= m_0(0)\exp(-tm_1). \end{aligned}$$

Then the expected cluster size grows exponentially: $m_1/m_0 = Ce^{m_1 t}$.

Weak solutions generally for $K = x + y$ can be characterized using a variant of the Laplace transform: Note

$$a(x) = 1 - e^{-qx} \quad \Rightarrow \quad a(x+y) - a(x) - a(y) = -a(x)a(y).$$

Then the function

$$\varphi(t,q) := \int_0^\infty (1 - e^{-qx})\nu_t(dx) \tag{6}$$

satisfies $\partial_q \varphi = \int_0^\infty e^{-qx} x \nu_t(dx)$ and the evolution equation (after normalizing so $m_1 = 1$)

$$\boxed{\partial_t \varphi - \varphi \partial_q \varphi = -\varphi.} \tag{7}$$

This is a damped inviscid Burgers equation, and here solutions should be analytic. For use below, we recall how solutions can be found from an implicit equation determined by the method of characteristics.

Solution via characteristics. Along a characteristic curve $q = q(t, \alpha)$ with $q(0, \alpha) = \alpha$ we have (with $\dot{q} = \partial_t q$)

$$\dot{q} = -\varphi, \quad \frac{d}{dt}\varphi(t, q(t,\alpha)) = \partial_t \varphi + \partial_q \varphi\, \dot{q} \quad = -\varphi,$$

Hence

$$\varphi = e^{-t}\varphi_0(\alpha), \quad e^t \varphi = \varphi_0(\alpha), \quad \frac{d}{dt}(q - \varphi) = 0,$$

so

$$q - \varphi(t,q) = \alpha - \varphi_0(\alpha) = \int_0^\infty (e^{-\alpha x} - 1 + \alpha x)\,\nu_0(dx) =: \psi_0(\alpha).$$

Now

$$\alpha = q + (e^t - 1)\varphi,$$

so to find φ from given (t, q) we can solve the implicit equation

$$q = \varphi + \psi_0(q + (e^t - 1)\varphi). \tag{8}$$

Or, to find $\alpha = \alpha(t, q)$ from given (t, q), note

$$(e^t - 1)q = (e^t - 1)\varphi + (e^t - 1)\psi_0(\alpha),$$

so

$$e^t q = \alpha + (e^t - 1)\psi_0(\alpha). \tag{9}$$

The solution formula (8) was used in [29] to prove a well-posedness theorem for measure solutions to Smoluchowski's equation with $K = x + y$, which requires the initial data to have only finite mass.

Theorem 1. (*Well-posedness of the initial-value problem*) *Let* ν_0 *be any measure on* $(0, \infty)$ *such that* $m_1 = \int_0^\infty x\,\nu_0(dx) < \infty$. *Then there exists a unique weakly continuous map* $t \mapsto x\nu_t(dx)$ *such that* $\int_0^\infty x\,\nu_t(dx) = m_1$ *for all* $t \geq 0$ *and* ν_t *is a weak solution to Smoluchowski's coagulation equation.*

A corresponding result for a general class of homogeneous kernels was subsequently obtained by Fournier and Laurençot [16] using a Gronwall inequality for a Wasserstein-type distance. (The required finite moment corresponds to the degree of homogeneity of the kernel.)

2.5. Self-similar limits

To study self-similar limits it is convenient to consider the normalized mass distribution, which corresponds to a probability distribution function:

$$F_t(x) = \int_{[0,x]} y\,\nu_t(dy) \Big/ \int_0^\infty y\,\nu_t(dy).$$

The following result is a dynamic version of the *central limit theorem* for this system. Assuming finiteness of one more moment, there is a universal scaling behavior.

Theorem 2. [25], [29] *Suppose* $\int_0^\infty x^2 \nu_0(dx) < \infty$. (*We assume* $= 1$ *without loss of generality.*) *Let* $\lambda(t) = e^{2t}$. *Then as* $t \to \infty$, *in the weak sense of probability measures*

$$F_t(\lambda(t)\,dx) \to F_*(dx) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x}\,dx.$$

With finiteness of further moments, rates of convergence have been obtained by Cañizo *et al.* [6] for $K = 2$ and Srinivasan [34] for $K = x+y$ and xy.

For initial data without a finite 2nd moment, *necessary and sufficient* conditions for convergence to self-similar form can be given that involve *regular variation* [5]—'almost power-law' tail behavior:

Theorem 3. [29] *Let $F_0(dx) = x\,\nu_0(dx)$ be a probability measure on $(0,\infty)$ and let $F_t(dx) = x\,\nu_t(dx)$ $(0 \le t < \infty)$ correspond to the weak solution of Smoluchowski's equation with $K = x+y$. Then the following are equivalent:*

$$\text{(1):}\qquad F_t(\lambda(t)x) \to F_*(x) \quad \text{as } t\to\infty \text{ for a.e. } x>0,$$

where $\lambda(t)\to\infty$ and $F_*(dx)$ is a (proper) probability measure on $(0,\infty)$.

$$\text{(2):}\qquad \int_{[0,x]} y^2\,\nu_0(dy) = x^\rho L(x),$$

where $0 \le \rho < 1$ and L is slowly varying at ∞ $(L(cx)/L(x) \to 1$ as $x\to\infty$, for all $c>0)$.

2.6. The scaling attractor, and eternal solutions

In this part our aim is to describe some of the main results of [31], concerning (a) the scaling attractor, (b) its measure representation analogous to the Lévy–Khintchine formula for infinitely divisible laws (based on a result of Bertoin [4]), and (c) how the dynamics on the attractor is made purely dilational in terms of this representation.

Although the proofs available now certainly rely on the Laplace transform, it seems plausible that only the scaling properties of Smoluchowski's equation should be important. Exactly how this should work for a general class of kernels is a mystery at present.

Scaling symmetries. Let ν_t be a measure solution of Smoluchowski's equation with $K = x+y$ (for $t\ge 0$, $m_1 = 1$). Then for all $T\in\mathbb{R}$ and $\lambda>0$ the quantity

$$\tilde\nu_t(dx) = \lambda\nu_{t+T}(\lambda dx)$$

is a measure solution also (for $t \ge -T$, $\tilde m_1 = 1$). Correspondingly,

$$\tilde\varphi(t,q) = \int_0^\infty (1-e^{-qx})\tilde\nu_t(dx) = \lambda\varphi(t+T, q/\lambda), \quad \tilde F_t(dx) = F_{t+T}(\lambda dx).$$

Consider any sequence of solutions $\nu_t^{(n)}(dx)$ (defined for $t\ge 0$, with $m_1 = 1$) and suppose sequences T_n, $\lambda_n\to\infty$ exist such that

$$F_{T_n}^{(n)}(\lambda_n dx) \underset{n\to\infty}{\longrightarrow} \hat F_0(dx) \tag{10}$$

where $\hat{F}_0(dx)$ is a probability distribution on $[0,\infty]$. Note that one can always pass to a subsequence to ensure that such a $\hat{F}_0$ exists—with the usual weak topology, the space of probability distributions on $[0,\infty]$ is compact. Of course, defective limits in $(0,\infty)$ are possible; probability can concentrate at 0 or leak off to ∞.

The set of $\hat{F}_0$ that arise in this way comprise all the cluster points of the set of solutions up to an arbitrary rescaling of cluster size.

Definition 4. The *scaling attractor* $\mathcal{A}$ is the set of probability measures $\hat{F}_0$ on $[0,\infty]$ such that sequences exist yielding (10).

We aim to characterize elements of this scaling attractor and their dynamics. In terms of corresponding 'Laplace exponents',

$$\begin{aligned}\tilde{\varphi}^{(n)}(0,q) = \lambda_n \varphi^{(n)}\left(T_n, \frac{q}{\lambda_n}\right) &= \int_0^\infty \frac{1-e^{-qx}}{x} F_{T_n}^{(n)}(\lambda_n dx) \\ &\underset{n\to\infty}{\longrightarrow} \int_0^\infty \frac{1-e^{-qx}}{x} \hat{F}_0(dx) =: \hat{\varphi}_0(q).\end{aligned}$$

Since $\tilde{\varphi}^{(n)}(t,q)$ is defined for $t > -T_n$, one can show without much difficultly that if we allow a finite advance or delay in time, convergence still holds. This means that *for each fixed* $t \in (-\infty,\infty)$ and all $q > 0$,

$$\tilde{\varphi}^{(n)}(t,q) \underset{n\to\infty}{\longrightarrow} \hat{\varphi}(t,q),$$

and then one can deduce (details omitted) that the corresponding probability measures converge weakly:

$$F_{t+T_n}^{(n)}(\lambda_n dx) \underset{n\to\infty}{\longrightarrow} \hat{F}_t(dx), \qquad -\infty < t < \infty, \tag{11}$$

where $\hat{F}_t$ is a probability measure on $[0,\infty]$, with

$$\hat{\varphi}(t,q) = \int_{[0,\infty]} \frac{1-e^{-qx}}{x} \hat{F}_t(dx), \qquad q > 0.$$

(This has to be interpreted properly to account for possible atoms at 0 and ∞.) Naturally, by forward-backward continuity of the solution along characteristics for the PDE, the limit $\hat{\varphi}(t,q)$ satisfies the same PDE:

$$\partial_t \hat{\varphi} - \hat{\varphi}\partial_q \hat{\varphi} = -\hat{\varphi}, \qquad \hat{\varphi}(0,q) = \hat{\varphi}_0(q), \quad q > 0. \tag{12}$$

It is striking that rescaled limits of solutions that may concentrate probability at 0 or ∞ have Laplace exponents $\hat{\varphi}$ that solve the same damped Burgers equation for all $t \in (-\infty, \infty)$. We take this to mean that F_t corresponds to an *eternal* solution of Smoluchowski's equation in an extended sense that allows for the presence of 'dust' and 'gel'—here, a positive probability that mass distribution includes clusters of size 0 or ∞. (Practically, this means sizes that are either negligible or too huge to account for precisely.) But it turns out that 'dust' and 'gel' cannot appear or disappear in finite time: For all t,

- $\hat{F}_0(0) > 0 \quad \Leftrightarrow \quad \mathcal{L}F_t(0) = \partial_q \hat{\varphi}(t, \infty) > 0 \qquad \text{(dust)}$
- $\hat{F}_0(\infty^-) < 1 \quad \Leftrightarrow \quad \hat{F}_t(\infty^-) < 1 \quad \Leftrightarrow \quad \partial_q \varphi(t, 0^+) < 1 \qquad \text{(gel)}$

Definition 5. The *proper scaling attractor* is the set

$$\mathcal{A}_{\mathrm{p}} := \{\hat{F}_0 \in \mathcal{A} : \hat{F}_0 \text{ is a proper probability distribution on } (0, \infty)\}.$$

We remark that the (proper) scaling attractor is *invariant*: $\hat{F}_0 \in \mathcal{A}_{\mathrm{p}}$ if and only if $\hat{F}_t \in \mathcal{A}_{\mathrm{p}}$ for all t. More precisely:

Theorem 6. $\hat{F}_0 \in \mathcal{A}_{\mathrm{p}} \quad \Leftrightarrow \quad \hat{F}_0 = \hat{F}_t|_{t=0}$, *where* $\hat{F}_t$ *is a (proper) eternal solution, defined for* $-\infty < t < \infty$.

2.7. Characterizing eternal solutions by taking $t \to -\infty$

The key result is due to Bertoin [4]. My main aim here is to give some idea how one might discover such a thing, and hint at the consequences for dynamics that were described in [31]. We first scale to simplify the dynamics of the damped inviscid Burgers equation:

1. Let $\nu_t(dx)$ be a (proper) eternal solution for $K = x + y$, with $\int_0^\infty x\, \nu_t(dx) \equiv 1$. Recall the function $\varphi(t, q)$ from (6) satisfies (7). Along characteristics we have $\varphi(t, q(\alpha, t)) = e^{-t}\varphi(\alpha)$,

$$q - \varphi(t, q) = \alpha - \varphi_0(\alpha) = \psi_0(\alpha),$$
$$\alpha = q + (e^t - 1)\varphi = e^t q - (e^t - 1)(q - \varphi).$$

2. We rescale via $x = s(t)\hat{x}$ with $s(t) = e^t$. (An important point is that *it is not well-understood why this choice of rescaling works!* There is a loose similarity to the computation of the generator of a convolution semigroup as in [15, ch. IX.2]. Perhaps the notion of nonlinear Markov process is relevant [24].) Then

$$\int_0^\infty (1 - e^{-q\hat{x}})\nu_t(s\, d\hat{x}) = \varphi\left(t, \frac{q}{s}\right),$$

and replacing q by q/s yields

$$\psi(s,q) := \frac{q}{s} - \varphi\left(t, \frac{q}{s}\right) = \psi_0(\hat{\alpha}),$$
$$\hat{\alpha} = e^t \frac{q}{s} - (e^t - 1)\left(\frac{q}{s} - \varphi\right) = q - (s-1)\psi.$$

Thus the implicit solution formula transforms to the form

(13) $$\psi(s,q) = \psi_0(q - (s-1)\psi), \quad \forall s = e^t > 0, \; q > 0.$$

This is just the implicit formula for the *usual inviscid Burgers equation*:

(14) $$\partial_s \psi + \psi \partial_q \psi = 0, \qquad \psi(1,q) = \psi_0(q).$$

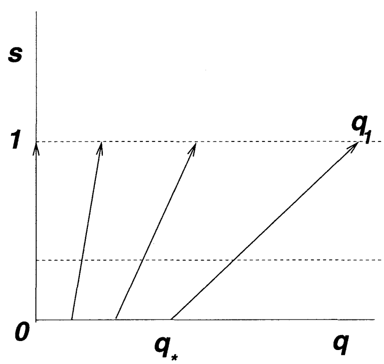

Fig. 1. Geometry of characteristics

3. As usual, ψ is *constant along straight-line characteristics*, whose speed increases with increasing q:

$$\frac{dq}{ds} = \psi = \psi_0(\alpha), \qquad q(1,\alpha) = \alpha, \qquad q = \alpha + (s-1)\psi.$$

At this point it seems natural to wonder what happens as $s \downarrow 0$. It is not hard to see there is a limit (see [31] for the proof):

Proposition 7. *For each $q > 0$, $\Psi(q) := \lim_{s\to 0^+} \psi(s,q)$ exists and determines $\psi(s,q)$ via $\psi = \Psi(q - s\psi)$ for all $s, q > 0$.*

Now there arises the question: *What does this proposition mean in terms of the measure solution $\nu_t(dx)$?* An answer is suggested by the formula

$$\psi(s,q) = \int_0^\infty \left(\frac{e^{-qx} - 1 + qx}{x^2} \right) x^2 \nu_t(s\,dx). \tag{15}$$

Define the measure

$$G_t(dx) = x^2 \nu_t(e^t\,dx). \tag{16}$$

Then the convergence $\psi(s,q) \to \Psi(q)$ suggests (and it is true) that

$$G_t(dx) \to H(dx) \quad \text{weakly as } t \to -\infty,$$

where H is some measure. (One should note $\int_0^\infty x^{-1} G_t(dx) = 1/s = e^{-t} \to \infty$, and possibly $\int_0^\infty G_t(dx) = \infty$.)

Now one can ask: *What measures H are possible limits?* The answer formulated in [31] goes as follows. We write $a \wedge b = \min(a,b)$.

Definition 8. A measure G on $[0,\infty)$ is a *g-measure* if

$$\int_{[0,\infty)} (1 \wedge y^{-1}) G(dy) < \infty.$$

We say a sequence $G^{(n)} \xrightarrow{g} G$ as $n \to \infty$ if $x(G^{(n)}) \to x(G)$ for a.e. $x > 0$, where

$$x(G) := \int_{[0,\infty)} (x^{-1} \wedge y^{-1}) G(dy) = \frac{1}{x} G(x) + \int_x^\infty y^{-1} G(dy).$$

We say G is a *divergent* g-measure if $x(G) \to \infty$ as $x \to 0^+$.

Theorem 9. [4], [31] *1. If $t \mapsto \nu_t$ is a (proper) eternal solution for $K = x + y$, then as $t \to -\infty$ we have $G_t \xrightarrow{g} H$ for some divergent g-measure H.*

2. For each divergent g-measure H, there is a unique proper eternal solution ν_t such that $G_t \xrightarrow{g} H$ as $t \to -\infty$. The solution ν_t is determined from

$$\Psi(q) := \int_{[0,\infty)} \frac{e^{-qx} - 1 + qx}{x^2} H(dx)$$

through the implicit relation $\psi(s,q) = \Psi(q - s\psi)$ and (15).

3. The correspondence

$$\hat{F}_0(dx) \leftrightarrow H(dx),$$

from the proper scaling attractor to divergent g-measures, is a bicontinuous bijection.

This result is the analog of the *Lévy–Khintchine representation formula* for infinitely divisible laws in probability theory. The proof in [31] relies on a (rather easy) extended continuity theorem for "2nd order Laplace exponents" of g-measures. Roughly, a sequence $G^{(n)} \xrightarrow{g} G$ if and only if $\psi^{(n)}(q) \to \psi(q)$ for all $q > 0$, where

$$\psi^{(n)}(q) = \int_{[0,\infty)} \frac{e^{-qx} - 1 + qx}{x^2} G^{(n)}(dx).$$

Remark. This representation of eternal solutions by divergent g-measures provides a strange kind of *nonlinear superposition rule* for eternal solutions. Namely, by adding together positive multiples of the representing divergent g-measures for two eternal solutions, one obtains a divergent g measure corresponding to a third eternal solution. How this corresponds to a well-defined operation acting directly on the measures $\hat{F}_0$ that lie on the scaling attractor is not clear.

2.8. Ultimate dynamics on the scaling attractor

The limit theorem above gives a remarkable simple representation of nonlinear clustering dynamics on the scaling attractor, as follows.

Let $T \in \mathbb{R}$. Then we map eternal solutions to eternal solutions by the time-T map, via

$$\nu_t(dx) \quad \to \quad \tilde{\nu}_t(dx) = \nu_{t+T}(dx).$$

Correspondingly, $F_0(dx) = x\,\nu_0(dx) \to \tilde{F}_0(dx) = F_T(dx)$ and

$$G_t(dx) = x^2 \nu_t(e^t dx) \quad \to \quad \tilde{G}_t(dx) = e^{2T} G_{t+T}(e^{-T} dx), \tag{17}$$

because

$$x^2 \tilde{\nu}_t(e^t dx) = x^2 \nu_{t+T}(e^t\, dx) = \nu_{t+T}(e^{t+T} e^{-T} dx)(e^{-T} x)^2 e^{2T}.$$

Now, just using the previous theorem to take the limit $t \to -\infty$ in the scaling relation (17) yields the following result!!

Theorem 10. *To the time-T map $\nu_t \mapsto \nu_{t+T}$ on the scaling attractor $\mathcal{A}_{\mathrm{p}}$ corresponds the pure scaling map $H \mapsto H_T$ of divergent g-measures given by*

$$\boxed{H_T(dx) = e^{2T} H(e^{-T} dx).}$$

To summarize: In the Lévy–Khintchine representation of an eternal solution ν_t by divergent g-measure H, the time-T map corresponding to nonlinear coagulation dynamics is represented by a pure dilational scaling as in the theorem. This is analogous to Bernoulli shift dynamics on sequences (after a log change of variables), and suggests a form of sensitive dependence upon initial conditions, in particular upon the shape of the tail in the distribution of large clusters. For further developments of this idea see [31].

§3. Min-driven domain coarsening models

3.1. One-dimensional bubble bath

We turn to consider a rather charming model that arises in describing the coarsening of domain-wall patterns in the one-dimensional Allen–Cahn PDE $\partial_t u = \partial_{xx} u + u - u^3$ [33], [7]. The model is simple to describe:

- The *smallest* domain collapses and joins with its two neighbors to form a single new domain with the same total size.
- Repeat.

I call this "one-dimensional bubble bath"—here, it is always the smallest bubble that 'pops' next. For this 'min-driven' coarsening process, numerical simulations initiated with domain sizes chosen randomly from various initial size distributions show the domain-size distribution approaching self-similar form.

To describe the coagulation rate equation used to model this process, let $\ell(t)$ denote the size of the smallest domain remaining at time t. The size density function $n(x,t)$ then vanishes for $x < \ell(t)$. In a time interval $(t, t+dt)$, the total number of coalescence events (per unit interval) is expected to be the total number of domains with size in $[\ell(t), \ell(t+dt)]$, which is approximately

$$n(\ell(t), t) \cdot \dot{\ell}(t)\, dt.$$

Now, domains of size x are

(i) *destroyed* when domains with sizes x, ℓ, y (or y, ℓ, x) combine,
(ii) *produced* when domains with sizes $y, \ell, x-y-\ell$ combine.

When we make the mean-field assumption that domains undergoing coalescence have sizes taken randomly and independently from the current overall size distribution, the probability density for events of these types is respectively

$$\rho_t(x)\rho_t(y), \quad \rho_t(y)\rho_t(x), \quad \rho_t(y)\rho_t(x-y-\ell),$$

where

$$\rho_t(x) = \frac{n(x,t)}{m_0(t)}, \qquad m_0(t) = \int_\ell^\infty n(y,t)\,dy.$$

Summing over all loss and gain terms one obtains the rate equation

$$\partial_t n(x,t) = n(\ell,t)\dot{\ell}\left(\int_\ell^{x-\ell} \rho_t(y)\rho_t(x-y-\ell)\,dy - 2\rho_t(x)\right), \quad x > \ell.$$

A significant feature of this model is that it is invariant under an arbitrary reparametrization of time: changing variables via $t = T(\tilde{t})$, $\tilde{n}(x,\tilde{t}) = n(x,t)$, $\tilde{\ell}(\tilde{t}) = \ell(t)$ leaves the equation in the same form. A remarkable solution procedure was found by Gallay and Mielke [20], who used the choice $\ell(t) = t$. Writing the Fourier transform as

$$\mathcal{F}\rho_t = \int_{\mathbb{R}} e^{-i\xi x}\rho_t(x)\,dx,$$

let $\Phi(z) = \frac{1}{2}\ln\frac{1-z}{1+z}$ and make the change of variable

$$v_t(x) = \mathcal{F}^{-1} \circ \Phi \circ \mathcal{F}\rho_t.$$

Then in terms of v_t the solution is given by the simple formula

$$v_t(x) = \begin{cases} v_1(x), & x \geq t, \\ 0, & x < t. \end{cases}$$

3.2. Solvable min-driven multiple collision models

The solution procedure of Gallay and Mielke adapts also to a more general family of min-driven clustering models that involve multiple cluster mergers. With probability p_k, each clustering event involves the merger of the smallest cluster of size $\ell(t)$ with k other randomly chosen clusters. The rate equation for number density in this situation takes the form

$$\partial_t n(x,t) = n(\ell,t)\dot{\ell}\sum_{k=1}^{\infty} p_k\left(\rho_t^{*k}(x-\ell) - k\rho_t(x)\right), \quad x > \ell.$$

Here ρ_t^{*k} indicates k-fold self-convolution. The 'linear' case $p_1 = 1$ corresponds to a "paste-all" model of coarsening that was studied by Derrida *et al.* [11]. Rather similar solution formulas appear also in a model for social conflicts studied in [22]. (The appendix of [11] points out the relevance of self-similar solutions to the diffusion equation with heavy tails and anomalous scaling.)

For this model with $p_k = 0$ for large k, Gallay and Mielke proved a number of strong results, concerning, for example: well-posedness of the initial value problem for densities ρ_t in L^1; the existence of self-similar solutions with heavy tails; sufficient conditions for solutions to approach self-similar form; and rates of approach to self-similar form that vary depending on the rate of decay of initial data.

A more recent analysis by Menon *et al.* [28] makes use of a different time scale, one chosen so that

$$m_0(t) = 1/t.$$

Forcing total number to be continuous in this way allows one to establish well-posedness for arbitrary size distributions which are arbitrary finite measures with support away from 0. Regarding the probability coefficients p_k we presume only

$$\sum_{k=1}^{\infty} p_k = 1, \quad Q_1 = Q'(1) = \sum_{k=1}^{\infty} k p_k < \infty, \quad Q(z) = \sum_{k=1}^{\infty} p_k z^k.$$

Writing $F_t(dx) = \rho_t(x)\,dx$, the weak form of the equation analogous to (5) is written

$$\begin{aligned} \int_{\mathbb{R}_+} a(x) F_t(dx) &= \int_{\mathbb{R}_+} a(x) F_{t_0}(dx) + \\ &\int_{t_0}^{t} \sum_{k=1}^{\infty} p_k \int_{\mathbb{R}_+^k} \left[a\left(\ell(s) + \sum_{i=1}^{k} y_i \right) - a(\ell(s)) \right] \prod_{i=1}^{k} F_s(dy_i) \frac{ds}{Q_1 s}. \end{aligned} \tag{18}$$

Here $\ell(t)$ is what we call the *min* of F_t, meaning the minimum of the support of F_t.

3.3. Solution procedure

Let us sketch how the solution procedure works. Taking $a(x) = e^{-qx}$ and writing $\bar{F}_t(q) = \int_0^\infty e^{-qx} F_t(dx)$ for the Laplace transform, one finds

$$\frac{Q_1}{1 - Q(\bar{F}_t(q))} \partial_t \bar{F}_t(q) = \frac{e^{-q\ell(t)}}{t}.$$

By integration in time, using that $\bar{F}_t(q) \to 0$ as $t \to \infty$ one can show

$$\bar{F}_t(q) = \Phi^{-1}\left(\int_t^\infty \frac{e^{-q\ell(s)}}{s}\, ds \right), \qquad \Phi(x) = \int_0^x \frac{Q_1}{1 - Q(y)}\, dy. \tag{19}$$

Formula (19) provides the solution procedure: Given an initial size distribution F_{t_0}, taking $t = t_0$ in (19) determines the *min history* $\ell(s)$ for all $s \geq t_0$. Then by truncation of this min history, (19) determines F_t for all $t \geq t_0$.

In [28] this solution procedure is justified first for solutions having continuous densities, which may be constructed by a simple stepwise procedure on strips $j\ell(t_0) \leq x \leq (j+1)\ell(t_0)$, $j = 1, 2, \ldots$. Then solutions for arbitrary initial probability measures F_{t_0} with *positive min* are constructed by passing to the limit using the continuity theorem for Laplace transforms.

3.4. Approach to self-similarity

In the time scale where $m_0(t) = 1/t$, the self-similar solutions found by Gallay and Mielke take the form

$$(20) \qquad F_t(x) = F^{(\theta)}\left(\frac{x}{\ell^{(\theta)}(t)}\right), \quad \ell^{(\theta)}(t) = t^{1/\theta}, \quad \theta \in (0,1].$$

The profile $F^{(\theta)}(dx)$ is a probability measure supported on $[1, \infty)$, determined by its explicit Laplace transform

$$\bar{F}^{(\theta)}(q) = \Phi^{-1}(\theta \mathrm{Ei}(q)),$$

where $\mathrm{Ei}(q) = \int_q^\infty e^{-s}\, ds/s$ is the exponential integral. These measures have smooth densities which have heavy tails for $0 < \theta < 1$, satisfying

$$\partial_x F^{(\theta)}(x) \sim c_\theta x^{-(1+\theta)}, \quad x \to \infty.$$

The profile for $\theta = 1$ is the only one which may have a finite first moment. Curiously, there is a sharp condition for this to happen: The first moment

$$(21) \qquad \int_1^\infty x F^{(1)}(dx) < \infty \quad \Leftrightarrow \quad \sum_{k=1}^\infty p_k k \log k < \infty.$$

The following result from [28] establishes *necessary and sufficient* conditions on initial data for approach to self-similar form.

Theorem 11. *Let $F_{t_0}(dx)$ be a probability measure on $(0, \infty)$ with positive min $\ell(t_0)$, and let $F_t(dx)$ $(t \geq t_0)$ be the weak solution of the min-driven clustering model, with min $\ell(t)$.*
(i) Suppose there is a probability measure F_* with positive min ℓ_* and a positive $\lambda(t) \to \infty$ as $t \to \infty$ such that

$$F_t(\lambda(t)x) \to F_*(x) \quad \textit{for a.e. } x > 0.$$

Then there exist $\theta \in (0,1]$ and slowly varying L, $\tilde{L}$ such that

$$\int_{[0,x]} y\, F_{t_0}(dy) \sim x^{1-\theta} L(x), \qquad \lambda(t)\ell_* \sim \ell(t) \sim t^{1/\theta}\tilde{L}(t). \tag{22}$$

(ii) Conversely, the condition on F_{t_0} in (22) is sufficient to ensure $\ell(t)$ is as described and $F_t(\ell(t)x) \to F^{(\theta)}(x)$, for a.e. $x \in (0,\infty)$.

The proof in [28] is not long, but (like many arguments involving regular variation) is subtle in some places. At one point use is made of the handy *exponential Tauberian theorem* of de Haan [10], which states that, given any function $A : \mathbb{R}_+ \to \mathbb{R}_+$ that is non-decreasing and right continuous, with Laplace transform $\bar{A}$, $\exp A(t)$ is regularly varying as $t \to \infty$ with exponent θ if and only if $\exp \bar{A}(1/t)$ is, and in this case,

$$e^{A(t)} \sim e^{\bar{A}(1/t)} e^{\gamma\theta},$$

where $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant.

3.5. Eternal solutions and their scaling dynamics

In general, the min-driven clustering model has scaling properties that are rather similar to those of Smoluchowski's coagulation equation with constant kernel $K = 2$. For example, the time dilation of a solution is a solution (see below). Also, an *eternal* solution is naturally defined as one that exists for all $t \in (0,\infty)$, since the total number is normalized so $m_0(t) = 1/t \to \infty$ as $t \to 0^+$.

An interesting difference occurs, however, in the case when

$$\sum_{k=1}^{\infty} p_k k \log k = \infty. \tag{23}$$

In this case, one needs an anomalous scaling of eternal solutions to obtain the Lévy–Khintchine representation. The analog of (16) is

$$G_t(dx) = \frac{xF_t(dx)}{t\kappa^{\#}(t)}, \tag{24}$$

where $\kappa^{\#}$ is a slowly varying function defined as follows. Define $\kappa(q)$ for $q \in (0,1)$ by

$$-\log \kappa(q) = \Phi(1-q) + \log q = \int_0^{1-q} \left(\frac{Q_1}{1-Q(z)} - \frac{1}{1-z} \right) dz. \tag{25}$$

It turns out $\kappa(q)$ is slowly varying as $q \to 0$, and $\kappa(q) \to 0$ if and only if (23) holds. The function $\kappa^\#$ is the de Bruijn conjugate of κ, satisfying

$$\kappa(q)\,\kappa^\#(q\kappa(q)) \sim 1 \quad \text{as } q \to 0.$$

The Lévy–Khintchine representation of eternal solutions is expressed in the following result from [28].

Theorem 12. 1. *Let F be an eternal solution of the min-driven clustering model. Then as $t \to 0^+$, $G_t \xrightarrow{g} H$ for some divergent g-measure H.*

2. *Conversely, for each divergent g-measure H, there is a unique eternal solution F such that $G_t \xrightarrow{g} H$ as $t \to 0^+$.*

3. *The Laplace exponent of H, given by*

$$\eta(q) = \int_0^\infty \frac{1 - e^{-qx}}{x} H(dx),$$

is related to the min history $\ell(t)$ of F by

$$\log \eta(q) = \int_0^1 \left(1 - e^{-q\ell(s)}\right) \frac{ds}{s} - \int_1^\infty e^{-q\ell(s)} \frac{ds}{s}.$$

This representation makes the dynamics of eternal solutions purely dilational in the following way, which was not made explicit in [28]. Any given $\tau > 0$ determines a *time-dilation* map on eternal solutions by

$$F_t(dx) \quad \to \quad \tilde{F}_t(dx) = F_{\tau t}(dx).$$

One easily checks that $\tilde{F}$ is a weak solution by verifying (18). Correspondingly,

$$G_t(dx) \quad \to \quad \tilde{G}_t(dx) = \frac{x\tilde{F}_t(dx)}{t\kappa^\#(t)} = \frac{\tau\kappa^\#(\tau t)}{\kappa^\#(t)} G_{\tau t}(dx). \tag{26}$$

By taking $t \to 0$, since $\kappa^\#$ is slowly varying we find the following.

Theorem 13. *To the time-dilation map $F_t \mapsto F_{\tau t}$ on eternal solutions corresponds the pure scaling map $H \mapsto H_\tau$ of divergent g-measures given by*

$$\boxed{H_\tau(dx) = \tau H(dx).} \tag{27}$$

§4. Random shock clustering and Smoluchowski's equation

For Smoluchowski's equation, the special rate kernel $K = x + y$ turns out to be very interesting, because it is rigorously connected to the theory of "Burgers turbulence," the study of how random initial data is propagated by nonlinear PDE dynamics. In this section we give a heuristic description of this surprising connection, which was established implicitly by Carraro & Duchon [8] and Bertoin [3], see [30]. This connection was combined with the classification of domains of attraction described in Theorem 3 above, to describe all domains of attraction of self-similar process solutions of the inviscid Burgers equation with initial data consisting of a one-sided Lévy process with no upward jumps [30]. Recently Menon & Srinivasan [32] and Menon [27] found formally that the problem of Burgers turbulence with Markov-process initial data is *completely integrable.* Much work remains to be done to develop the implications.

4.1. Heuristic derivation

We start by considering entropy solutions of

$$\partial_t u + u \partial_x u = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{28}$$

consisting of a random staircase of shocks (downward jumps):

$$u(x,t) = \sum_k -s_k H(x - x_k(t)). \tag{29}$$

(See Fig. 2.) We suppose this looks roughly linear on a very large scale.

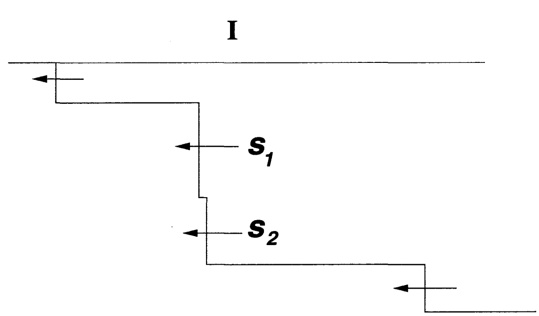

Fig. 2. Binary clustering of shocks

Each shock at position $x_j(t)$ has constant size s_j and constant speed $\dot{x}_j = \frac{1}{2}(u_- + u_+)$ between collisions. The shocks aggregate upon collision like ballistic particles, conserving total 'mass' (size) and 'momentum', due to the standard jump conditions.

We formally describe a mean-field statistical model for the shock-size distribution as follows: Let $n(s,t)\,ds$ be the expected number of shocks per unit length I (on the large scale) with size in $[s, s+ds]$, assuming this distribution is stationary in space. There are two mechanisms of evolution of the density n:

1. Net influx of shocks into I: The velocity difference across I is $|u_R - u_L| \approx \int_0^\infty s\,n(s,t)\,ds =: m_1(t)$. In time dt, the net influx of shocks with size in $[s, s+ds]$ then should be the product of the density of such shocks and the length swept into I in this time:

$$n(s,t)\,ds \cdot m_1(t)\,dt$$

2. *Shock coalescence* in I occurs at relative velocity

$$v = \dot{x}_2 - \dot{x}_1 = \frac{1}{2}(s_2 + s_1).$$

(The labels should be taken to indicate any two consecutive shocks.) In time dt, any given shock at x_1 with size in $[s_1, s_1 + ds_1]$ will collide with one with size in $[s_2, s_2 + ds_2]$ if there is such an s_2-shock in the interval of length roughly $2v\,dt$ centered at x_1. So the number of collisions of such shocks in time dt is expected to be

$$n(s_1,t)\,ds_1 \cdot n(s_2,t)\,ds_2 \cdot (s_1 + s_2)\,dt.$$

Considering all cases with $s = s_1 + s_2$ fixed and adding up over gain and loss terms, we obtain the density rate equation

$$\partial_t n(s,t) = m_1(t) n(s,t) + Q(n,n), \tag{30}$$

$$\begin{aligned} Q(n,n) = &- \int_0^\infty n(s,t) n(s_2,t)(s+s_2)\,ds_2 \\ &+ \frac{1}{2}\int_0^s n(s-s_1,t) n(s_1,t) s\,ds_1. \end{aligned} \tag{31}$$

Integration over s yields $\dot{m}_1 = \partial_t \int_0^\infty s n(s,t)\,ds = m_1^2$ (since by (4) we have $\int_0^\infty sQ(n,n)\,ds = 0$) whence one easily derives

$$\frac{1}{m_1}\partial_t\left(\frac{n}{m_1}\right) = Q\left(\frac{n}{m_1}, \frac{n}{m_1}\right). \tag{32}$$

Up to a change in time scale ($d\hat{t} = m_1\,dt$) and size distribution ($\hat{n} = n/m_1$), this is exactly Smoluchowski's coagulation equation with additive kernel $K(s_1, s_2) = s_1 + s_2$.

Remark. There is a very nice symmetry of the inviscid Burgers equation which allows one to add a constant slope to initial data, and express a solution $\tilde{u}(x,t)$ with initial data $\tilde{u}_0(x) = u_0(x) + ax$ in terms of the solution $u(x,t)$ with initial data $u_0(x)$. Namely,

$$\tilde{u}(x,t) = \frac{1}{1+at} u\left(\frac{x}{1+at}, \frac{t}{1+at}\right) + \frac{ax}{1+at}. \tag{33}$$

4.2. Solutions with Lévy process initial data

J. M. Burgers initiated the study of the solution of equation (28) with initial data $u_0(x)$ that is *random*, as a model for understanding how nonlinear dynamics propagates statistical uncertainty. He considered the case of white noise, for which the exact solution was found by Frachebourg and Martin [19] based on work of Groeneboom [21].

In the early 1990s the case when $x \mapsto u_0(x)$ is a (one-sided) Brownian motion began to be addressed. The work of Carraro & Duchon [8], [9] and Bertoin [3] showed that in this case, the solution increment process $x \mapsto u(x,t) - u(0,t)$ is a random walk with stationary, independent increments. In particular, it is a *Lévy process* that is spectrally negative (with no upward jumps).

Such a Lévy process $x \mapsto X_x$ is characterized by its Laplace exponent ψ determined from $\mathbb{E}(e^{qX_x}) = e^{x\psi(q)}$, and which admits the following Lévy–Khintchine representation [2]:

$$\psi(q) = bq + \frac{\sigma^2 q^2}{2} + \int_0^\infty (e^{-qs} - 1 + qs)\,\Lambda(ds), \quad q > 0.$$

Such a process is the superposition of three independent processes related to this formula: a Brownian motion with variance σ^2 and drift b, a compound Poisson process with jump measure $\Lambda \mathbf{1}_{|s|\ge 1}$, and a pure jump martingale with jump measure $\Lambda \mathbf{1}_{|s|<1}$. The *Lévy triple* (b, σ^2, Λ) characterizes the process.

Taking any such process as initial data for the entropy solution to (28), after normalizing net drift to zero the increment process at time $t > 0$ is again such a process, one characterized by a Lévy triple $(0, 0, \Lambda_t)$ with zero Brownian part. As found by Carraro & Duchon, the Laplace exponent $\psi(q,t)$ corresponding to this triple satisfies (again!) the inviscid Burgers equation

$$\partial_t \psi + \psi \partial_q \psi = 0.$$

And as explained explicitly in [30], *the jump measure Λ_t corresponds to a weak solution to Smoluchowski's equation with additive kernel*, according to the following natural scaling: Let

$$M(t) = \int_0^\infty s\Lambda_t(ds).$$

One knows $M(t) < \infty$ for $t > 0$, but $M(0) = \infty$ if u_0 is not of bounded variation (meaning $\sigma^2 > 0$). Set

$$\hat{t} = -\log M(t), \qquad \nu_{\hat{t}}(ds) = \Lambda_t(M(t)\, ds). \tag{34}$$

Then the map $\hat{t} \mapsto \nu_{\hat{t}}$ is a (unit mass) weak solution of Smoluchowski's equation for $\hat{t} > \hat{t}_0 = -\log M(0)$, in the sense made precise in [29].

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