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Results and conjectures in profinite Teichmüller theory

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Abstract.

We discuss a number of statements in profinite Teichmüller theory, usually in parallel with their discrete counterparts. Some of these can be shown to hold true, others can be brought back to a few conjectures which we state explicitly and are mostly related to the homotopy type of certain simplicial profinite complexes.

> J'ai pu constater en d'autres occasions que lorsque des augures (ici moi-même!) déclarent d'un air entendu (ou dubitatif) que tel problème est "hors de portée", c'est là au fond une affirmation entièrement subjective. Elle signifie simplement, à part le fait que le problème est censé ne pas être résolu encore, que celui qui parle est à court d'idées sur la question [...]
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> A. Grothendieck, *Esquisse d'un programme*, Note 1; a translation can be found at the end of the paper.

§1. Introduction

Profinite Teichmüller theory is a recent topic stemming from several sources, among which Grothendieck's *Esquisse d'un programme* which in part inspired what is now known as Grothendieck–Teichmüller theory. Still more recently, Marco Boggi introduced (in [B1]) natural and seemingly important objects in this context, namely the profinite completions of the curve complexes. Unfortunately a flaw was subsequently

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discovered in that paper (see in §5 below) thus temporarily invalidating some of its most significant results. Since then the gap in [B1] has been reorganized into a clearcut conjecture (see §4 below) which does seem central in the subject. The main goal of the present text is to present this hopefully emerging theme in a way which would make part of the structure more visible and perhaps enticing to the reader. Of course it may be that one part or another of this partly conjectural picture will turn out to look different than predicted here but even if this were to happen, we hope that the present paper may still be useful as a guideline and perhaps a source of inspiration for some interested and adventurous readers. Now, because this is a slightly unusual format and also because we will cover a lot of mathematical ground, we will have to be quite sketchy in terms of recalling most of the notions and results we make use of, including at times in terms of attributions. The necessary material, references and background can be gathered from the articles we quote and from their reference lists. Technically speaking however we have also partly shifted the emphasis from profinite *curve* complexes to *arc* complexes and since the completed versions of the latter are making their first appearance in the present note, we have worked out some of their basic properties in more detail.

A short reminder of a particular historical sequence may help put this note in perspective. In the mid-seventies D. Quillen explained a strategy aiming at proving the stability of the cohomology of $GL_n(\mathbb{Z})$ w.r.t. n, which was a necessary ingredient of his foundational work on K-theory. It was quickly implemented, proving the stability of various series $G_n(R)$ of linear reductive groups G_n over rings R. The main points can be tersely summarized by saying that one can study the cohomology of a group by letting it act faithfully on a *highly connected* space (in essence that part was known to A. Weil and others) and that in the case at hand buildings provide such spaces. Another essential requirement is that the stabilizers of the action possess some hierarchical structure enabling one to trigger some sort of induction. Concretely, when studying $G_n(R)$ they should be connected e.g. to $G_{n-1}(R)$, modulo tractable pieces. Technical (often cumbersome and not so easy) work using equivariant spectral sequences then completes the proof of stability. Almost at the same time (late seventies) and after conversations with J.-P. Serre about his work on symmetric spaces in collaboration with A. Borel, W. J. Harvey introduced curve complexes as a means to compactify (or bordify) Teichmüller spaces. Curve complexes can be seen as analogs of buildings, triggering the possibility to transfer Quillen's strategy to that situation. This was implemented in the early eighties by J. Harer and N. Ivanov; J. Harer's original paper ([H1]) on the subject indeed follows the pattern very roughly outlined above. Determining the homotopy type of curve and arc complexes proved difficult however; to that end J. Harer made use of W. Thurston's theory of laminations (Cf. [H1], Section 2) whereas N. Ivanov used—following A. Hatcher and W. Thurston—J. Cerf's theory of global Morse functions on differentiable surfaces (cf. [Iv1]). It is all the more remarkable that A. Hatcher subsequently found a truly astonishing shortcut (in [Hatcher]) reducing the main result, namely the contractibility of the arc complex (under quite general assumptions) to a mere two-page long elementary proof. Of course curve complexes subsequently found a hoard of other uses, just like buildings, and in fact have become an object of study in their own right. As an important example, one of the applications we will be interested in (in a profinite setting) is to the study of automorphisms of the Teichmüller groups and their open subgroups, via automorphisms of the complexes themselves (see [Iv3,4] for the discrete case).

So here we would like to add one episode (unfortunately at present still conditional) to the story sketched above, of a rather different nature since it has to do with completions with respect to the natural action of the Teichmüller group on most of the objects, something which comes in naturally when investigating more algebraic or arithmetic phenomena. A central conjecture states that the *profinite* arc complexes are (should be) acyclic, a statement which closely parallels the discrete situation. A. Hatcher's nice 'combing argument' alluded to above does not apply of course to the profinite case (it breaks down at line 1) but who can tell how difficult the proof of that statement, if true, can be? At this point we can only refer to the epigraph. It may however be also fit to add a few words about the overall parallel between the discrete and profinite situations. Practitioners will probably agree that this parallel fortunately has its limits, both in terms of statements and proofs. Statements can sometimes be essentially identical but discrete proofs just do not go through. For instance proving that a free *profinite* group on two generators is centerfree is another matter than showing the elementary discrete statement obtained by erasing the word 'profinite'. To-date we conjecture but do not know (in genus > 2) how to prove that the *profinite* Teichmüller groups are centerfree, whereas the discrete statement has been known for decades. But here, there are also surprises in store as far as *statements* are concerned, and these can in some sense be summarized under the heading 'Grothendieck–Teichmüller theory', which will be touched upon especially in the last two sections of the text. So we should insist on the fact that the conjectures are not made here by routinely continuing the discrete situation to the profinite one. Instead, their (purported) truth would in fact rely on very specific principles, prominent among which is Grothendieck's 2-level principle which lies at the very foundation of Grothendieck–Teichmüller theory and of which we will present a rather novel incarnation (see in particular §9.5).

Acknowledgments. It will be plain, in particular from the references, that this text constitutes a kind of prospective survey based on ideas which for a substantial part were introduced by Marco Boggi. I have tried my best to do justice to these ideas and organize the material into a hopefully coherent text, adding other ingredients on the way. To what extent this attempt has been successful, I leave it to the well-intentioned reader to decide.

It would be too long to enumerate the people to whom I am indebted in one way or another in this subject, but I certainly wish to thankfully acknowledge the help of R. Kent, G. Quick and P. Symonds, specifically in connection with the present text. Finally I would like to extend particularly warm thanks to the organizers of the Kyoto conferences on Galois–Teichmüller and Arithmetic Geometry who made these events so wonderfully pleasant and productive.

§2. Discrete and profinite curve and arc complexes

This section serves to fix notation and define the relevant completed objects. Profinite *arc* complexes make their first appearance here and we develop in somewhat more detail their definition and basic properties.

2.1. Discrete complexes

This paragraph does not go beyond a short reminder so that we remain sketchy even as far as the well-known definitions are concerned. We also refrain from giving references except concerning a few specific notions, for which [B1] and references therein contains more than we need.

A finite *type* is a pair (g, n) of non negative integers; it is *hyperbolic* if $2g - 2 + n > 0$. Given such a type, we let $S = S_{g,n}$ denote the —unique up to diffeomorphism—differentiable surface of genus g with n deleted points. We occasionally write $g(S)$ for the genus of S. The points are usually considered setwise, i.e. are not labeled; they can also be seen as 'holes' provided isotopies do not fix the boundary. Attached to a hyperbolic surface S of type (g, n) are the *Teichmüller space* $\mathfrak{T}(S)$ and *moduli space* $\mathfrak{M}(S)$. The Teichmüller space $\mathfrak{T}(S)$ is noncanonically identified with the standard Teichmüller space $\mathfrak{T}_{g,n}$ associated with the given type. It has dimension $d(S) = d_{g,n} = 3g - 3 + n$, which we call the *modular dimension* of S or of the given type—we will often drop the adjective 'modular'. In turn $\mathfrak{M}(S)$ is—again noncanonically—identified with $\mathfrak{M}_{g,[n]}$, the moduli space of curves of the given type, with unlabeled marked points. Here we use brackets ($[n]$) when the points are unlabeled, that is are considered setwise (points are always labeled on Teichmüller space). The moduli space of curves of genus g with n labeled (i.e. ordered) points is denoted $\mathfrak{M}_{g,n}$. The cover $\mathfrak{M}_{g,n}/\mathfrak{M}_{g,[n]}$ is finite, orbifold unramified (stack étale) and Galois with group $\mathcal{S}_n$, the permutation group on n symbols.

We let $Mod(S) = \pi_0(Diff(S))$ denote the (extended) *mapping class group* of S, i.e. the group of isotopy classes of diffeomorphisms of S. The index 2 subgroup of orientation preserving isotopy classes is denoted $Mod^+(S)$. More generally an upper $+$ will mean *orientation preservating.* We usually write $\Gamma(S) = Mod^+(S)$ and call it the *Teichmüller (modular) group.* It can be seen as the orbifold fundamental group of $\mathfrak{M}(S)$ and as the Galois group of the orbifold unramified cover $\mathfrak{T}(S)/\mathfrak{M}(S)$. The group $\Gamma(S)$ is (noncanonically) isomorphic to $\Gamma_{g,[n]}$, defined as the fundamental group of the complex orbifold $\mathfrak{M}_{g,[n]}$. The group $\Gamma_{g,[n]}$ is centerfree, except for 4 low-dimensional exceptions, i.e. types $(0,4)$, $(1,1)$, $(1,2)$ and $(2,0)$.

We now briefly summarize the definitions pertaining to various *arc* and *curve complexes* where 'complex' will always stand for 'simplicial set'. Except when this makes a real difference, we will not notationally record the simplicial character of the objects, nor distinguish between a complex and its geometric realization i.e. we often write X instead of $X_\bullet$ or $|X_\bullet|$. There exists at present a maze of such complexes (see in [H1] and more recent papers) and we are not going to enter it; variants will be left to the reader and are well documented in the literature. So to start with, given again a surface S hyperbolic and of finite type we define the set of isotopy classes of simple closed curves on S not isotopic to boundary curves, for short simply *curves.* A *multicurve* is a set of non intersecting elements of $\mathcal{S}$ where 'nonintersecting' means that there is a set of mutually disjoint representatives. The original *curve complex* $\mathscr{C}(S)$ introduced by W. J. Harvey has a k-simplex for each multicurve $\underline{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_k)$, so that vertices of $\mathscr{C}(S)$ correspond to curves. Boundary and face operators are defined in the usual way, which amounts to deletion and inclusion of curves. This makes $\mathscr{C}(S)$ into a (non locally finite) simplicial set of dimension $d(S) - 1$ where $d(S)$ is the modular dimension of S. We will occasionally write $\mathscr{C}^{(k)}(S)$ for the k-dimensional skeleton of $\mathscr{C}(S)$ and use a similar notation for the other complexes; curves themselves span the 0-skeleton or vertex set of $\mathscr{C}(S)$.

Another important and related object is the *pants complex* $\mathscr{C}_P(S)$, which was briefly mentioned in the appendix of the classical 1980 paper by A. Hatcher and W. Thurston (see [HLS] or [M] for the reference) and first studied in [HLS] where it is shown to be connected and simply connected. It is a two dimensional, not locally finite complex whose vertices are given by the pants decomposition (i.e. maximal multicurves) of S; these correspond to the simplices of highest dimension ($= d(S)-1$) of $\mathscr{C}(S)$. Given two vertices $\underline{s}, \underline{s}' \in \mathscr{C}_P(S)$, they are connected by an edge if and only if $\underline{s}$ and $\underline{s}'$ have $d(S)-1$ curves in common so that up to relabeling (and of course isotopy) $s_i = s_i'$, $i = 1, \ldots, d(S)-1$, whereas s_0 and s_0' differ by an *elementary move*, which means the following. Cutting S along the s_i's, $i > 0$, there remains a surface Σ of modular dimension 1, so Σ is of type $(1,1)$ or $(0,4)$. Then s_0 and s_0', which are supported on Σ, should intersect in a minimal way, that is they should have geometric intersection number 1 in the first case, and 2 in the second case (in the latter case their algebraic intersection number is 0). In the first case (genus 1), the edge (and move) is said to be of type S (for 'simple', see [HLS]); in the second case (genus 0) of type A (for 'associativity', see [HLS]). For $d(S) = 1$, the 1-skeleton of $\mathscr{C}_P(S)$ is the Farey graph F. We have so far defined only the 1-skeleton $\mathscr{C}_P^{(1)}(S)$ of $\mathscr{C}_P(S)$ which, following [M], we call the *pants graph* of S. We will not give here the definition of the 2-cells of $\mathscr{C}_P(S)$ (see [HLS] or [M]), and we will not make use of it. They describe certain relations between elementary moves, that is they can be considered as elementary homotopies; as mentioned above, pasting them in makes $\mathscr{C}_P(S)$ simply connected (cf. [HLS]). Furthermore, it is shown in [M] and [BL] how to recover the full 2-dimensional pants complex from the pants graph, quite a nontrivial fact.

We turn last to the *arc complex*, referring especially to [H1,2] and [Hatcher] for much more detail. In a nutshell, S is first equipped with a nonempty finite set P of marked points (or basepoints) $P = \{p_1, p_2, \ldots, p_n\}$. An *arc* is a simple curve up to isotopy joining two points in P; a *multiarc* is defined from this just as a multicurve above. Elements of a multiarc meet only at P and the arc complex $\mathscr{A}(S)$ has a k-simplex for every multiarc $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_k)$. Boundary and face operators are defined by deletion and inclusion of arcs respectively. We will often denote the arc complex by $\mathscr{A}(S,P)$; this is not quite consistent with the convention that P is part of the definition of S but it is also not likely to cause confusion. Here we can regard S as a closed (i.e. compact without boundary) surface of genus g with g large enough so that $S \setminus P$ is hyperbolic. There are several variants worth mentioning here. First S may actually have nonempty boundary ∂S and some points of P may lie on ∂S. Arcs then join these points of P on S and touch ∂S only at P. One can also define another set of points Q in the interior of S, which arcs are supposed to avoid. One could include curves in the definition of the complex, that is apart from arcs it could contain simple closed curves which avoid the boundary and the marked points etc.

Here for simplicity, we will (as in [H2]) often stick to the simplest variant, namely the case when P is reduced to a point (which we denote P again) and S is closed of genus $g > 0$. So arcs start from and end at the point P; elements of a multiarc meet only at that point. Most of the interest and difficulty of the situation is contained in this case. Note however that when it comes to proofs, one often needs to start cutting and pasting surfaces and such operations immediately make it necessary to include more general cases.

Before moving to completions we add just one remark. The complexes $\mathscr{C}(S)$, $\mathscr{C}_P(S)$, $\mathscr{A}(S,P)$ (and others) have been defined in a purely topological way. However the curve complex $\mathscr{C}(S)$ actually affords an algebro-geometric interpretation (see [B1]) which was indeed the main incentive for introducing its completion. In turn the pants complex $\mathscr{C}_P(S)$ has a modular interpretation (see [BL] and below) in terms of triangulations of Riemann surfaces. Finally the arc complex $\mathscr{A}(S,P)$ essentially arose from analytic considerations involving Strebel differentials (see [H2] for a summary and references). These analytic and algebraic interpretations are of course of vital importance for the relevance of these objects to the problems we are ultimately interested in.

2.2. Group actions and completions of complexes

Teichmüller groups act naturally on the objects defined above, by letting diffeomorphisms act on (actual) curves and quotienting by the appropriate equivalence relations, i.e. regarding curves and diffeomorphisms up to isotopies. More precisely $\Gamma(S)$ itself acts naturally on $\mathscr{C}(S)$ and $\mathscr{C}_P(S)$. Here S is of type (g,n) with a fixed orientation and $\Gamma(S) \simeq \Gamma_{g,[n]}$. The action is very close to being faithful: only the center of $\Gamma(S)$ acts trivially and that center is trivial except in the well-known low dimensional cases mentioned above. This is an immediate consequence of the following elementary formula, which we take this opportunity of recalling. Namely, given an orientation of S which we fix once and for all, to a curve γ there is associated the (Dehn) twist t_γ along γ and then for $g \in \Gamma(S)$ one has: $g t_\gamma g^{-1} = t_{g(\gamma)}$.

In the case of the arc complex $\mathscr{A}(S,P)$ with P a finite set of points, we need to consider $\Gamma(S,P)$, the group fixing P setwise. If S is of genus g (type $(g,0)$) and P has $n \geq 1$ elements, $\Gamma(S,P) \simeq \Gamma_{g,[n]}$. Referring to possible variants, one could take S of type (g,m) with a set Q of marked points as mentioned above, and then $\Gamma(S,P)$ would be the subgroup $\Gamma_{g,[m]+[n]}$ of $\Gamma_{g,[m+n]}$ preserving the sets P and Q. In our model case, P

consists of just one point and S is closed so that $\Gamma(S,P) \simeq \Gamma_{g,1}$. In all cases one gets a natural action of $\Gamma(S,P)$ on the arc complex $\mathscr{A}(S,P)$.

We can now introduce, essentially following [B1], the completions of the complexes with respect to the natural Γ-action described above. Let us start slightly more abstractly, using a small part of the notation and setting of [Q1] (see also below). In particular $\mathcal{S}$ will denote the category of simplicial sets and $\hat{\mathcal{S}}$ the category of simplicial profinite sets with continuous morphisms. An object of $\hat{\mathcal{S}}$ is thus given as $X = X_\bullet$ where each X_n is a profinite set. As usual a group G acts on a (not necessarily profinite) simplicial set X if there is a morphism $G \times X \to X$ and the usual diagrams commute. Moreover the defining morphism should respect the topology, i.e. be continuous in the profinite case.

In order to define completions, we add some restrictions on the actions we consider. So let G be a group acting on a simplicial set X. The action is said to be *simplicial* if it commutes with the face and degeneracy maps. More generally, it is called *geometric* if it satisfies some natural compatibility conditions which are detailed in [B1], §5. We do not state them here in order to save space and because they are obviously met in all the cases we will consider. Next the action is *open* if the set of G-orbits of X_n is finite for all $n \geq 0$ and the reader can check that this is also obviously satisfied in all the examples we will be interested in. The three properties above can be defined in the profinite case by adding in the usual continuity requirement, that is compatibility with the topology.

Let now G be a *discrete* group and G' a profinite completion of G, that is a profinite group equipped with a morphism $j : G \to G'$ with dense image. We denote by $(G^\lambda)_{\lambda\in\Lambda}$ the tower of subgroups of G which are inverse images of open subgroups of G'. In the cases we are interested in, j is injective, so Λ is a sub-inverse system of the one indexing the cofinite (i.e. finite index) subgroups of G and G' is a quotient of $\hat{G}$, the full profinite completion of G, which will be our main but not only example. We will usually ignore j and write simply: $G' = \varprojlim_{\lambda\in\Lambda} G/G^\lambda$.

Let G and a completion G' be as above, and suppose G acts on a simplicial set X. We will call this action *admissible* if it is geometric, open and virtually simplicial. This last requirement means that there exists a cofinite subgroup G^μ ($\mu \in \Lambda$) which acts simplicially on X. It is the only property which is not trivially satisfied in our examples and we will check its validity in the next subsection. Granted the admissibility of the action, the G'-completion X' of X is defined as the simplicial profinite set (i.e. the element of $\hat{\mathcal{S}}$):

$$X' = \varprojlim{}_{\lambda\in\Lambda} X/G^\lambda = \varprojlim{}_{\lambda\in\Lambda} X^\lambda.$$

There is of course a natural completion map $X \to X'$ which we again denote by j for simplicity, and X' is equipped with a natural (admissible) action of G' which moreover satisfies the following universal property: Given a G-equivariant map $f : X \to Y$ where $Y \in \hat{\mathcal{S}}$ is equipped with an admissible action of G', f factors uniquely into $f = f' \circ j$ for a unique G'-equivariant map $f' : X' \to Y$. We will write $\hat{X}$ rather than X' in the case of the full profinite completion, that is when $G' = \hat{G}$. In general G' (resp. X') is a quotient of $\hat{G}$ (resp. $\hat{X}$).

One should note right away that X' is an essentially asymptotic object in the sense that given G and G', X' is invariant under the replacement of G by any of the G^λ's. By definition again, the quotients X^λ have finite n-skeleta for all $n \geq 0$ and moreover, in the context we will be dealing with, i.e. curve and arc complexes, X is finite dimensional so that the quotients X^λ are simplicial finite sets. In particular the (geometric realization of) X' is compact, in sharp contrast with X. So for curve and arc complexes (or indeed as soon as X is finite dimensional) X' is literally a profinite complex, that is a limit of simplicial finite sets.

As could be predicted, we now let X be $\mathscr{C}(S)$, $\mathscr{C}_P(S)$ or $\mathscr{A}(S,P)$ equipped with the natural action of $\Gamma(S)$ for the first two and of $\Gamma(S,P)$ for the last of these complexes. The action of these Teichmüller groups is obviously geometric (from the definition in [B1], §5) and open. Indeed it has finite quotients as mentioned above. So again as mentioned above, in order to check that the action is admissible, it is enough in each case to check that it is virtually simplicial. For the curve complex $\mathscr{C}(S)$, we refer to [B1], §5 which also settles the case of the pants complex $\mathscr{C}_P(S)$. Given a completion $\Gamma(S)'$, we thus get the attending completed complexes $\mathscr{C}(S)'$ and $\mathscr{C}_P(S)'$, which we denote $\hat{\mathscr{C}}(S)$ and $\hat{\mathscr{C}}_P(S)$ when $\Gamma(S)' = \hat{\Gamma}(S)$. In the next subsection we will show that the action of $\Gamma(S,P)$ on $\mathscr{A}(S,P)$ is virtually simplicial as well, which will vindicate the introduction of the completions $\mathscr{A}(S,P)'$ and in particular of the full completion $\hat{\mathscr{A}}(S,P)$.

It is important to stress that the profinite curve and pants complexes have nice modular interpretations coming from their respective discrete counterparts. To be more precise and fix notation, let S be a surface as above, $\mathfrak{M} = \mathfrak{M}(S)$ the attending moduli stack (viewed here as a complex orbifold) and $\Gamma = \Gamma(S)$ the corresponding Teichmüller group. To a finite index subgroup Γ^λ there corresponds a finite orbifold unramified cover $\mathfrak{M}^\lambda/\mathfrak{M}$. We call the subgroups Γ^λ *levels* and the corresponding covers $\mathfrak{M}^\lambda$ *level structures*, a traditional terminology in this context. For $\lambda, \mu \in \Lambda$ we write $\mu \geq \lambda$ if $\Gamma^\mu \subseteq \Gamma^\lambda$ i.e. if $\mathfrak{M}^\mu$ is a covering of $\mathfrak{M}^\lambda$, and we say that $\mathfrak{M}^\mu$ dominates $\mathfrak{M}^\lambda$. The study of these levels and their stable compactifications has been an object of intense study in the late nineties (see [B1] for a summary and references). One way to envision $\hat{\mathscr{C}}(S)$ is to interpret its finite quotients $\mathscr{C}^\lambda(S)$ as follows. Namely for λ large enough, let $\overline{\mathfrak{M}}^\lambda$ be the stable completion of $\mathfrak{M}^\lambda$ and let $\partial\mathfrak{M}^\lambda$ be the divisor at infinity. This is a a normal crossing divisor (strictly so for λ large enough) and $\mathscr{C}^\lambda(S)$ is nothing but the nerve of its covering by its irreducible components. For a detailed study, see [B1], §3. As for the modular interpretation of $\mathscr{C}_P^\lambda(S)$, we refer to [BL], §4 or §9.2 below for a brief account.

Remark 2.1. The above constructions represent a rather direct an comparatively elementary way of entering the profinite world. However below we will have to deal with profinite *homotopy* theory and we are fortunate enough that the recent work of G. Quick ([Q1,2,3]) provides us with a modern framework which is taylored to our needs. In the present paper we will use only a very small part of these results but it is to be hoped that they will prove useful and indeed unavoidable in further investigations.

Here are a few first inputs. Let again $\mathcal{S}$ denote the category of simplicial sets, $\hat{\mathcal{S}}$ that of simplicial profinite sets. As has become fairly standard, elements of $\mathcal{S}$ (resp. $\hat{\mathcal{S}}$) are called *spaces* (resp. *profinite spaces*). We also let $\mathcal{E}$ denote the category of sets, $\mathcal{F} \subset \mathcal{E}$ that of finite sets and $\hat{\mathcal{E}}$ that of profinite sets. Given a category $\mathscr{C}$, $s(\mathscr{C})$ denotes the associated simplicial category and $pro - \mathscr{C}$ that of the pro-objects of $\mathscr{C}$. As is well-known, taking limits induces an equivalence between $pro - \mathcal{F}$ and $\hat{\mathcal{E}}$. But s and $pro-$ do not commute in general. In particular although taking limits provides a natural functor between $pro - s(\mathcal{F})$ and $\hat{\mathcal{S}} = s(pro - \mathcal{F}) = s(\hat{\mathcal{E}})$, it is not an equivalence. We defined the G'-completion of X as an element of $\hat{\mathcal{S}} = s(\hat{\mathcal{E}})$ by taking limits (dimensionwise). In our examples X/G and thus all the X/G^λ are finite, so we can build the system $(X^\lambda)_{\lambda\in\Lambda}$ of the finite quotients and take the limit levelwise to get the G'-completion $X' \in \hat{\mathcal{S}} = s(pro - \mathcal{F})$.

The paper [Q2] gives a careful treatment of group actions on profinite spaces and constructs a model category for such objects. It is important to note that our direct constructions do fit into that framework as detailed in [Q3], §4. The main point is that the profinite groups we are considering here, i.e. the profinite completions of the Teichmüller groups are strongly stable, as is any completion of a discrete group. We refer to [Q3], §4 for a careful discussion. It may be interesting to note that the profinite spaces we are considering turn out to be naturally equipped with an action of the absolute Galois group of $\mathbb{Q}$ (cf. §9.3 below) which is *not* strongly stable but which does not enter into building up the completion.

2.3. Teichmüller groups act virtually simplicially

In this paragraph we sketch a proof of the fact that the action of the Teichmüller group $\Gamma(S)$ (resp. $\Gamma(S,P)$) on the curve and pants complexes $\mathscr{C} = \mathscr{C}(S)$ and $\mathscr{C}_P = \mathscr{C}_P(S)$ (resp. the arc complex $\mathscr{A} = \mathscr{A}(S,P)$) is virtually simplicial. This may serve to make things more concrete but some readers may also prefer to skip this short paragraph altogether, which will not impair the understanding of later sections.

For simplicity we assume that S has no boundary and write Γ for the appropriate group i.e. $\Gamma(S)$ or $\Gamma(S,P)$ as the case may be. Also the result for $\mathscr{C}_P$ being an immediate consequence of the one for $\mathscr{C}$, we will hardly mention the former in the sequel. Now the action of Γ on the complexes $\mathscr{C}$ and $\mathscr{A}$ being geometric, what we have to show is that in each case there exists a finite index subgroup $G \subset \Gamma$ such that if an element of G fixes a simplex of $\mathscr{C}$ (resp. $\mathscr{A}$) setwise, it actually fixes it pointwise (for graphs, this is called acting without inversion). Note that one could take $G = \Gamma$ and replace $\mathscr{C}$ and $\mathscr{A}$ by their first barycentric subdivisions (which does not alter the geometric realizations) but this operation of subdivision does not behave well under completion.

First we recall the classical definition of the *abelian levels*: An element of Γ induces an automorphism of the homology group $H_1(S,\mathbb{Z}/m)$ for any integer $m > 0$. The subgroup $\Gamma^{(m)} = \Gamma^{(m)}(S)$ of level m is defined as the kernel of this map, i.e. the subgroup (clearly normal and of finite index) of those elements of Γ which act trivially on the homology of S modulo m.

With this definition we can state:

Proposition 2.2. *The abelian level $\Gamma^{(m)}$ acts simplicially on $\mathscr{C}(S)$ (and $\mathscr{C}_P(S)$) for $m > 2$. Given a finite set of points P on S, denote $\Gamma(S,P) \subset \Gamma(S)$ the subgroup fixing P* pointwise. *For $m > 2$, $\Gamma^{(m)}(S,P) = \Gamma^{(m)}(S) \cap \Gamma(S,P)$ acts simplicially on $\mathscr{A}(S,P)$.*

Proof. For curves this is actually done in [B1]: see §5 just above Proposition 5.1; one appeals to the useful Proposition 3.1. The point is that if two curves (again this means simple closed curves up to isotopy) on S, say γ_1 and γ_2, do not intersect and $f \in \Gamma^{(m)}$ (for some $m > 2$) maps γ_1 to γ_2, then $f = 1$ (and $\gamma_1 = \gamma_2$). One can also consult e.g. [Iv2].

As for the arc complex $\mathscr{A}(S,P)$, let us first assume that P is reduced to a single point (our model case). Let then α_1 and α_2 be two arcs i.e. simple closed curves based at the point P up to an isotopy fixing P, and assume that they are part of a simplex, that is they intersect only at P. Considering sufficiently thin tubular neighborhoods of α_1 and α_2 their boundaries determine two pairs of curves and out of these one can choose a curve γ_1 (resp. γ_2) with γ_1 (resp. γ_2) freely isotopic to α_1 (resp. α_2) such that γ_1 and γ_2 do not intersect (and per force are not isotopic). If $f \in \Gamma$ and $f(\alpha_1) = \alpha_2$, then $f(\gamma_1) = \gamma_2$; then apply the above result for curves to get that $\Gamma^{(m)}(S,P)$ acts simplicially on $\mathscr{A}$ for any $m > 2$.

Finally let P be an arbitrary finite set of points and note that for our present purpose we defined $\Gamma(S,P)$ as fixing P pointwise, not merely setwise. Let again α_1 and α_2 be two arcs of $\mathscr{A}(S,P)$ in the same simplex, i.e. they intersect only at points in P. Let $f \in \Gamma(S,P)$ such that $f(\alpha_1) = \alpha_2$. Since we have already dealt with the case of loops, we may assume that both α_1 and α_2 join p and p', which are two distinct elements of P. Let us orient both arcs from p to p' and call the oriented versions $\vec{\alpha}_1$ and $\vec{\alpha}_2$.

Fix an integer $m > 2$; an element of $\Gamma(S,P)$ induces an automorphisms of the relative homology group $H_1(S,P;\mathbb{Z}/m)$ and we let $\Gamma^{(m)}(S,P)$ denote the kernel of that map, that is the subgroup of $\Gamma(S,P)$ fixing the relative homology modulo m. Now the relative homology long exact sequence shows that the natural map $H_1(S;\mathbb{Z}/m) \to H_1(S,P;\mathbb{Z}/m)$ is injective with cokernel $H_0(P;\mathbb{Z}/m) \simeq \mathbb{Z}^{\sharp P}$ ($\sharp P$ denotes the cardinal of the set P). Since here, by definition the elements of $\Gamma(S,P)$ fix P pointwise, they act trivially on $H_0(P;\mathbb{Z}/m)$ and we conclude that actually: $\Gamma^{(m)}(S,P) = \Gamma^{(m)}(S) \cap \Gamma(S,P)$.

Let $f \in \Gamma^{(m)}(S,P)$ be such that $f(\alpha_1) = \alpha_2$; since f fixes p and p', we find that $f(\vec{\alpha}_1) = \vec{\alpha}_2$. So f maps the oriented curve $\vec{\alpha}_1 - \vec{\alpha}_2$ to its opposite and since f fixes the relative homology modulo m we find that $\vec{\alpha}_1 - \vec{\alpha}_2 = 0$ in $H_1(S,P;\mathbb{Z}/m)$, so also in $H_1(S;\mathbb{Z}/m)$. This implies that the curve $\alpha_1 \cup \alpha_2$ separates the surface S into two subsurfaces, say $S^{\pm}$. Now f is orientable and it induces an orientation reversing diffeomorphism of $\alpha_1 \cup \alpha_2$, so it has to permute the subsurfaces S^+ and S^-. But it also fixes $H_1(S,P;\mathbb{Z}/m)$. This is impossible if α_1 and α_2 are distinct, which we assume; indeed one must have that S has genus 0 and $P = \{p,p'\}$, in which case $\alpha_1 = \alpha_2$ anyway. In conclusion the group $\Gamma^{(m)}(S,P) = \Gamma^{(m)}(S) \cap \Gamma(S,P)$ ($m > 2$) does act simplicially on $\mathscr{A}(S,P)$. Q.E.D.

We briefly return to the general situation of a complex X with an admissible action of a group G and let $H \subset G$ be a cofinite subgroup acting simplicially. Given a profinite completion $j : G \to G'$ we get as above the corresponding system $(G^\lambda)_{\lambda \in \Lambda}$ of the inverse images of the open subgroups of G'. If $H \subset G^\mu$ for some $\mu \in \Lambda$, e.g. if $G' = \hat{G}$, we define X' directly as above. If however this is not the case, we have to replace the system $(G^\lambda)_{\lambda \in \Lambda}$ by its trace on H, namely $(H \cap G^\lambda)_{\lambda \in \Lambda}$, which we will do implicitly.

§3. Stabilizers in completed complexes

In this section we will review the situation regarding the stabilizers of simplexes in completed complexes under the action of the Teichmüller groups. We will also use this problem as an opportunity of getting acquainted with basic features of these complexes, sometimes in contrast with their discrete counterparts. In accordance with the goal of this paper we will not give all the details and will at times even use a slightly abbreviated notation favoring legibility over complete precision. A main point here is to take stock of certain important phenomena pertaining to the profinite situation. On the other hand the question of determining the stabilizers is a crucial one because, referring to our sketch in the introduction, it provides in principle the basis for a form of induction on the modular dimensions of the surfaces in play. This section has a somewhat preliminary character and the material will be elaborated further below, in particular in §8.4.

Let us start with a basic feature of completions, namely residual finiteness. Given G acting on X, $j : G \to G'$ a completion of G, one says it is residually finite if j is injective and we extend this terminology to the group completion, i.e. X', the G'-completion of X is called residually finite if the natural map $X \to X'$ is injective. Fortunately, for arc and curve complexes it holds:

Proposition 3.1. *The Γ'-completion of $\mathscr{C}(S)$ (resp. $\mathscr{C}_P(S)$, $\mathscr{A}(S)$) is residually finite if and only if this is the case for the completion $j : \Gamma \to \Gamma'$.*

Proof. For the curve complex $\mathscr{C}$, this is Proposition 5.1 of [B1], which also takes care of the pants complex $\mathscr{C}_P$. For the arc complex $\mathscr{A}$, one direction goes through without change. In the other direction, assume Γ' is residually finite. Let α, β be two arcs of $\mathscr{A} = \mathscr{A}(S, P)$ whose images coincide in $\mathscr{A}'$ (i.e. $j(\alpha) = j(\beta)$); we need to show that they actually coincide in $\mathscr{A}$, which is done basically as in [B1]. Namely let α and thus β connect p and p', two points of P. If $p \neq p'$, consider the curves a and b obtained as the respective boundaries of sufficiently thin tubular neighborhoods of α and β. We also get the associated twists t_a and t_b. Proceeding as in [B1] we then find that $j(\alpha) = j(\beta)$ implies that $j(t_a) = j(t_b)$, so $a = b$ because Γ' is residually finite, and $\alpha = \beta$. The case $p = p'$ is analogous. Q.E.D.

Next we recall an important and classical notion which will play a prominent role in the sequel:

Definition 3.2. Let $K \subset \pi = \pi_1(S)$ be an invariant (i.e. characteristic) cofinite group. It determines a map $\phi_K : \Gamma \to Aut(\pi/K)$ ($\Gamma = \Gamma(S)$ or $\Gamma(S,P)$) and we let $\Gamma^K = Ker(\phi_K)$ denote the corresponding *principal congruence subgroup.* A congruence subgroup is a subgroup of Γ containing a principal congruence subgroup.

The congruence or *geometric completion* of Γ, denoted $\check{\Gamma}$, is obtained by completing along the inverse system of the $(\Gamma^K)_K$.

It is of course enough to complete along *principal* congruence subgroups. There is a canonical surjection $\hat{\Gamma} \to \check{\Gamma}$ and the fundamental congruence subgroup conjecture originally proposed by N. Ivanov and to be discussed below (see §6), purports that this is actually an isomorphism. The geometric completion $\check{\Gamma}$ is known to be residually finite (see e.g. [Iv2], Exercises), just as the full profinite completion $\hat{\Gamma}$. Actually, proofs of the residual finiteness of $\hat{\Gamma}$ usually show that property for $\check{\Gamma}$, which is a stronger statement.

We now come to stabilizers. In the discrete case it is an elementary problem and provides the basis of a discrete Teichmüller Lego. It has been used (see [HLS], [NS]) in order to explore profinite Grothendieck–Teichmüller theory to which we will turn later. One can also remark that even the discrete case deserves a more detailed exploration; one for instance would like to *explicitly* write down Γ as an amalgamation of simpler groups. More generally, in this context inductions are performed in terms of amalgamations, a nonlinear operation which is incredibly sensitive to profinite completion, rather than (linear) extensions, which are (sometimes) tractable by means of classical comparison of spectral sequences arguments (see below for more).

So the first discrete example is of course that of a curve γ and its stabilizer $\Gamma_\gamma \in \Gamma$. It is easily described via almost tautological short exact sequences (see e.g. [B1], §3). More generally if σ is a multicurve, determining a simplex of $\mathscr{C}(S)$, its stabilizer Γ_σ is again easily determined (cf. ibidem and §8.4 below). Write $\Gamma_{\vec{\sigma}} \subset \Gamma$ for the oriented pointwise stabilizer, comprising the elements of Γ which fix the curves of σ and preserve their orientation. Then $\Gamma_{\vec{\sigma}}$ has finite index in Γ_σ and there is a natural surjection $p : \Gamma_{\vec{\sigma}} \to \Gamma(S \setminus \sigma)$. Here $S \setminus \sigma$ denotes the surface S slit along σ and $\Gamma(S \setminus \sigma)$ denotes its Teichmüller group, which by definition is the product of the group associated to its connected components, as it may not be connected (here components cannot be permuted). Finally $Ker(p) = \langle t_\sigma \rangle \simeq \mathbb{Z}^k$ is the free abelian group generated by the twists along the components (vertices) of σ. A detailed and general description, including surfaces with boundaries etc. would be notationally messy but nothing more. A major point is that by cutting along a curve the modular dimension always decreases and strictly so, so it can be used as an induction parameter. We will not go into the description for arcs, which again is elementary in the discrete case.

What happens and where do we stand in the completed case? We will as usual review the case of curves and add a few details for arcs. Pants complexes will not appear because again they reduce to curves. So we start with a residually finite completion Γ' ($\Gamma' = \Gamma(S)'$ for curves and $= \Gamma(S, P)'$ for arcs) and the Γ'-completions $\mathscr{C}'$ and $\mathscr{A}'$. The first important remark is that by Γ'-equivariance, we can confine ourselves to investigating stabilizers Γ'_σ of *discrete* simplices. Here a discrete simplex of course means that it is in the image of the natural map j and since the completion is residual we always identify $\mathscr{C}$ (resp. $\mathscr{A}$) with its image in $\mathscr{C}'$ (resp. $\mathscr{A}'$).

Let us sketch that reduction argument because it is very simple and general, yet important. Return to the general setting of a G-complex X with completion G' (not even necessarily residual) and attending G'-completion X'. Assume that X/G is finite, as is the case for our geometric complexes. For any $k \geq 0$, let F_k be a finite set of representatives in X_k of the simplices of X_k/G; we assume as we may that the action is simplicial (as usual there is a hoard of minor details which we just skip). So X_k is the disjoint union $\coprod_{s \in F_k} G \cdot s$ of finitely many orbits. The point now is that in the same way X'_k is the union (now possibly *not* disjoint) of the orbits $G' \cdot s$ ($s \in F_k$). Indeed this finite union is dense in X' because it contains the image of X and it is closed as a finite union of compact sets, so it coincides with X'. This also provides the useful though somewhat deceptively simple image of X' as covered by finitely many G'-orbits.

Returning to our problem, any simplex of the completed complex lies in the Γ'-orbit of a discrete simplex and by Γ'-equivariance, we may indeed confine ourselves to the stabilizers Γ'_σ of discrete simplices σ of $\mathscr{C}'$ and $\mathscr{A}'$. Let Γ_σ be the stabilizer in the corresponding discrete complex and $\bar{\Gamma}_\sigma$ its closure in the completed complex. Since the Γ'-action on that complex is continuous, $\bar{\Gamma}_\sigma \subset \Gamma'_\sigma$, that is the closure of the discrete stabilizer is clearly contained in the stabilizer. It turns out that in our cases these two groups coincide:

Proposition 3.3. $\bar{\Gamma}_\sigma = \Gamma'_\sigma$ *for a simplex* σ *of* $\mathscr{C}$ *or* $\mathscr{A}$*, that is the stabilizer of a discrete simplex in the completed group coincides with the closure of its stabilizer in the discrete group.*

Proof. For the curve complex $\mathscr{C}$, this is Proposition 6.5 of [B1], which is not affected by the gap in §5. For the arc complex we can first reduce ourselves to vertices i.e. to arcs (as opposed to multiarcs) since the stabilizers are virtually (i.e. up to passing to an open subgroup) *pointwise* stabilizers. The reasoning then goes very much as in the previous proposition. Let α be an arc joining p and p' and assume $p \neq p'$, leaving the case $p = p'$ to the reader. This determines a curve γ, obtained as the boundary of a sufficiently thin tubular neighborhood of α. Let $f = (f_\lambda) \in \Gamma'$ fixing α, which means that $f_\lambda(\alpha) = \alpha$ modulo Γ^λ for all $\lambda \in \Lambda$; clearly we get the same congruence with γ instead of α, so f fixes γ and by the results for curves we are done.

Perhaps it is worth rephrasing this a little more generally. Consider a disk with two marked points p and p' in its interior. Observe that there is one and only one arc joining p to p' and in fact only one arc joining p (resp. p') to itself. That describes the arc complex of such an object—call it a trinion. In particular we get a bijection between curves encircling two points, trinions and arcs joining two distinct points; it extends to the corresponding pro-objects and the attending invariants like stabilizers. This viewpoint may not be disjoint from the notion of 'cuspidalization' (see [HM1] and references therein). Q.E.D.

Our last piece of information about stabilizers appears in Proposition 6.6 of [B1] (again independent of the gap in §5 of that paper) to which we refer for details. First say that a completion Γ' is finer than the geometric completion if there is a natural epimorphism $\Gamma' \twoheadrightarrow \check{\Gamma}$ i.e. if the subsystem defining Γ' is finer than the system of the congruence subgroups. Note that if a completion Γ' is finer than the geometric one, it is residually finite and in the system defining Γ' there exists a group which acts simplicially, namely take an abelian level. Most completions considered thereafter will be finer than geometric.

Let now $\sigma \in \mathscr{C}$ be a simplex of the curve complex (we concentrate on curves for definiteness). Let as above $\Gamma'_\sigma \subset \Gamma'$ be its stabilizer and $\Gamma'_{\vec{\sigma}}$ the pointwise stabilizer. Again as mentioned above, in the discrete case there is a natural onto map $\Gamma_{\vec{\sigma}} \twoheadrightarrow \Gamma(S \setminus \sigma)$ whose kernel is generated by the twists along the curves of σ. Given a completion Γ', we get a completed map $\Gamma'_{\vec{\sigma}} \to \Gamma(S \setminus \sigma)''$ where $\Gamma(S \setminus \sigma)''$ is some completion of $\Gamma(S \setminus \sigma)$. The same construction works for arcs as well. Then we have:

Proposition 3.4. *Assume Γ' is a completion which is finer than the geometric one. If σ is a simplex of $\mathscr{C}$ or $\mathscr{A}$, its pointwise stabilizer in $\mathscr{C}'$ or $\mathscr{A}'$ induces on $\Gamma(S \setminus \sigma)$ a completion which is finer than the geometric one.*

For the proof for curves, see [B1]. The statement for arcs should be clear by now, e.g. the definition of $\mathscr{A}(S \setminus \sigma, P)$ which actually means that the surface S is slit along the arcs of σ with the marked points appearing on the emerging boundary components. We will leave the proof to the untiring reader.

We stress that the problem of determining the stabilizer Γ'_σ is a very important one, technical as it may appear at first encounter. The problem is equivalent to describing the completion which we denoted $\Gamma(S \setminus \sigma)''$ above. The rest is dealt with by natural short exact sequences (see again [B1]; one does use some local monodromy computations for certain level structures, but these are well-known). Of course if the congruence subgroup property for Teichmüller groups obtains, that is if $\check{\Gamma} = \hat{\Gamma}$ (see below, §6), the above proposition simply solves the problem. This notwithstanding we can ask our first question, to which we will return in §8.4:

Question 3.1. Does the full profinite completion $\hat{\Gamma}$ induce the full profinite completion on the stabilizer? In other words, given a simplex σ in $\mathscr{C}$ or $\mathscr{A}$, is the closure $\bar{\Gamma}_\sigma \subset \hat{\Gamma}$ of its stabilizer Γ_σ isomorphic to the full profinite completion $\widehat{\Gamma_\sigma}$?

Again this would be (modulo Proposition 3.4) a consequence of the congruence conjecture to be discussed later. In particular it holds true whenever the congruence conjecture has been established already, that is in genus 0, 1 and 2 (see again §6).

It may be useful to briefly indicate the geometric version of this question for curve complexes. Consider $\mathfrak{M} = \mathfrak{M}(S)$, its completion $\overline{\mathfrak{M}}$ and an irreducible component $\mathfrak{N}$ of the divisor at infinity $\partial\mathfrak{M}$. In the case of a curve (that is σ is a vertex of $\mathscr{C}$) to which we can reduce ourselves, the question asks whether, given an étale cover V of a formal neighborhood U of $\mathfrak{N}$ one can find an étale cover of $\mathfrak{M}$ such that its trace on $U \setminus \mathfrak{N}$ dominates (the trace of) V.

We insist in closing that most questions and conjectures in this paper are intimately related and that we are trying to capture a very small number of mathematical phenomena. For instance, as the above rephrasing should make clear, Question 3.1 also features what is needed in order to be able to start unraveling the profinite Teichmüller groups in terms of amalgamations of similar groups with strictly lower modular dimensions.

§4. The contractibility conjecture for profinite arc complexes

In this section we state and comment on what can be termed the main conjecture of (part of) the subject. It reads simply as follows:

Conjecture 4.1. Let S be a closed surface (of genus $g \geq 0$), P a nonempty finite set of points of S, $\mathscr{A} = \mathscr{A}(S, P)$ the corresponding arc complex; let also Γ' be a completion of $\Gamma = \Gamma(S, P)$ which is finer than the geometric completion $\check{\Gamma}$. Then the Γ'-completion $\mathscr{A}'$ of $\mathscr{A}$ is weakly contractible.

Remark 4.2. It is actually enough to prove the conjecture in the particular case of the geometric completion $\check{\mathscr{A}}$ itself (see below §6). So the conjecture is *equivalent* to that seemingly special case. The above formulation, although unnecessarily general underscores the fact that partial results can already be significant. For instance proving that the full *profinite* completion $\hat{\mathscr{A}}$ is contractible would have very interesting consequences (see §7).

This is of course the direct analogue of what is known to happen in the discrete case. Namely $\mathscr{A}$ itself is indeed contractible. One can also accommodate surfaces with boundary and extend the conjecture to them. There are only a few cases, listed e.g. in [Hatcher], in which the discrete arc complex is not contractible. Even then, its homotopy type is known and the conjecture can at least be meaningfully extended. Perhaps we should insist that although it may look like we are simply mimicking the discrete case, there are in fact surprises in store, a remarkable one being the eventual emergence of the Grothendieck–Teichmüller groups, to be discussed toward the end of this paper.

In this paper we will need only a very minimal dose of (profinite) homotopy theory. We will thus content ourselves with some 'elementary' reminders and remarks, hoping however that the fairly concrete problems we are facing may attract the attention of some experts. So let us turn to an elucidation of the contractibility conjecture. It states in effect that $\pi_k(\mathscr{A}')$ is (should be) trivial for all $k \geq 0$, where these homotopy groups for profinite spaces (i.e. objects of $\hat{\mathcal{S}}$) are defined in [Q1]. As usual, if $X \in \hat{\mathcal{S}}$, $\pi_0(X)$ is a profinite set, $\pi_1(X)$ is a profinite group, and $\pi_k(X)$ an abelian profinite group for $k > 1$. As for the the connection between homotopy, cohomology and homology, it is essentially built into the homotopical formalism and in our case can be summarized by the following equivalences:

Proposition 4.3. *Let $X \in \hat{\mathcal{S}}$ be a profinite space. The following conditions are equivalent:*

i) X is weakly contractible, i.e. its homotopy groups are trivial;
ii) X is connected, simply connected ($\pi_1(X) = \{1\}$) and $H^k(X, \mathbb{Z}/p) = \{0\}$ for every $k > 1$ and every prime p;
iii) X is connected, simply connected and $H_k(X, \hat{\mathbb{Z}}) = \{0\}$ for $k > 1$.

Proof. i) and ii) are equivalent as in [AM] (see e.g. [F], Theorem 6.2). We call ii) an acyclicity condition. In the definition of a weak equivalence to a point as in [AM] (see [Q1], Definition 2.6 in our case) one should have that $H^k(X, M)$ is trivial for any $k > 1$ and any finite abelian local system M. But over a simply connected space such a local system is nothing but a constant (finite abelian) module and by dévissage one is reduced to the case $M = \mathbb{Z}/p$. Finally i) and iii) are equivalent by the classical Hurewicz theorem, which extends to our case (see [Q1], §2.5). Q.E.D.

Before proceeding, we stress that we are dealing with objects, like the arc and curve complexes, which even in the discrete case are *not* fibrant. So the contractibility conjecture is indeed a statement about their *weak* contractibility. In the sequel we will drop the adjective for simplicity, bearing however the above in mind.

We have at present almost no significant information about Conjecture 4.1 although after reading this and the next few sections it will hopefully appear more natural and believable, apart from the obvious parallel with the discrete case. Let $\mathscr{A}'$ be as in the conjecture, i.e. a completion of $\mathscr{A}$ which is finer than the geometric one. Then it is connected (i.e. $\pi_0(\mathscr{A}') = \{*\}$) because the finite quotients $\mathscr{A}^\lambda$ are connected and π_0 commutes with inverse limits. Here $\lambda \in \Lambda'$, which is a subsystem of Λ containing $\check{\Lambda}$, where the latter defines the geometric completion.

We can also show that: $\varprojlim_{\lambda \in \Lambda} \pi_1(\mathscr{A}^\lambda) = \{1\}$. Indeed for $\lambda \in \Lambda$ there is an epimorphism: $\Gamma^\lambda \twoheadrightarrow \pi_1^{top}(\mathscr{A}^\lambda)$ at the topological level, stemming from the fact that $\mathscr{A}^\lambda$ is obtained from $\mathscr{A}$ as a Γ^λ-covering, which however is not necessarily étale. The kernel is generated by the inertia elements, which here are the reducible elements in Γ^λ (when viewed as mapping classes). This map can be completed into: $\hat{\Gamma}^\lambda \twoheadrightarrow \pi_1(\mathscr{A}^\lambda)$; one can then take limits over $\lambda \in \Lambda$ on both sides. The inverse limit being exact on profinite groups, we obtain an epimorphism between the respective limits and the left-hand side converges to $\bigcap_\lambda \hat{\Gamma}^\lambda$, which is the trivial group, thus completing the proof. This is however quite a weak statement and it does not in particular imply the simple connectedness of $\hat{\mathscr{A}}$, which is unknown to-date.

We now compare the group completions we are using with the completions stemming from Artin-Mazur pioneering work. Again [Q1] contains the proper setting for our purpose and we refer mainly to that article for details and numerous references to relevant papers, as well as to [Q3] (esp. §3) for a detailed comparison between the set and group theoretic completions. To start with, let $X \in \mathcal{S}$ be a space (a simplicial set) and denote by $\mathcal{R} = \mathcal{R}(X)$ the set of simplicial open equivalence relations on X. An element $R \in \mathcal{R}$ is thus given as a simplicial subset of $X \times X$ such that in every dimension n, $R_n \subset X_n \times X_n$ defines an equivalence relation on X_n with finitely many classes. In particular each quotient X/R is a simplicial finite set. The elements of $\mathcal{R}$ are naturally ordered by inclusion and so one can define $\hat{\hat{X}}$ by taking the limit in $\hat{\mathcal{S}}$:

$$\hat{\hat{X}} = \varprojlim{}_{R \in \mathcal{R}} X/R.$$

(Remark: Here and below we will use indifferently *limit* and *inverse limit*, resp. *colimit* and *direct limit*.) The profinite space $\hat{\hat{X}} \in \hat{\mathcal{S}}$ is called the (set theoretic) profinite completion of X; it is usually denoted $\hat{X}$ but that piece of notation is not available here. For homotopical purposes, it is necessary to replace $\hat{\hat{X}}$ as defined above by a fibrant replacement (for the model structure defined in [Q1]) and this can be done functorially in X. In other words, for any $X \in \hat{\mathcal{S}}$ there is a functorially defined fibrant replacement $F(X)$ (up to weak equivalence) and [Q3] provides an explicit construction (inspired by prior work of several authors) of the functor F. Here we will hardly need to distinguish between X and $F(X)$ but recall that the homotopy groups of X are defined in terms of $F(X)$ and note again that it may be useful to keep in mind that the (discrete) spaces we are starting from, most notably arc and curve complexes, are *not* Kan complexes, that is are not fibrant objects of $\mathcal{S}$.

Assume now that a discrete group G acts on the (discrete) space X in an admissible way in the sense defined above (i.e. the action is geometric, virtually simplicial and open). To a completion G' of G defined by a system $(G^\lambda)_{\lambda \in \Lambda'}$ of finite index subgroups of G, we may associate the G'-completion $X' \in \hat{\mathcal{S}}$. As mentioned already, it is invariant if we replace G by one of the G^λ and for simplicity we assume that one of these act simplicially. Replacing G by that subgroup if need be, we may thus assume that G acts simplicially on X (this is really for notational convenience only). As we have seen above these hypotheses are fulfilled in most cases we are interested in, and in particular for any completion of a curve, arc or pants complex which is finer than the geometric one.

Now each G^λ defines an element $R^\lambda \in \mathcal{R}$ in the obvious way. Namely R^λ viewed in $X \times X$ comprises the pairs of simplices (x, y), $y = gx$, $g \in G^\lambda$. So the G'-completion $X' = \varprojlim_{\lambda \in \Lambda'} X/G^\lambda = \varprojlim_{\lambda \in \Lambda'} X/R^\lambda$ occurs naturally as a quotient of $\hat{\hat{X}}$. Let as usual Λ denote the cofiltering system of *all* finite index subgroups of G, $\hat{G}$ the resulting full profinite completion and $\hat{X}$ the attending $\hat{G}$-completion. For any G'-completion we thus get a sequence of natural *epimorphisms*: $\hat{\hat{X}} \twoheadrightarrow \hat{X} \twoheadrightarrow X'$. Finally, if X is one of our favourite complexes, namely one of $\mathscr{A}(S, P)$, $\mathscr{C}(S)$ or $\mathscr{C}_P(S)$, its geometric completion $\check{X}$ is defined and we are often interested in completions X' which are finer than $\check{X}$, i.e. that admit a map $X' \twoheadrightarrow \check{X}$ *onto* the geometric completion $\check{X}$.

How can we compare these various completions, especially $\hat{\hat{X}}$ and $\hat{X}$, and what for? There are at least two main incentives for trying to unravel this point. First the cohomologies of X and $\hat{\hat{X}}$ (with coefficients in a finite not necessarily constant module) are canonically isomorphic. Apply this to the arc complex $\mathscr{A}(S)$ with the assumptions of Conjecture 4.1 which ensure that $\mathscr{A} = \mathscr{A}(S)$ is contractible. One finds that the set theoretic profinite completion $\hat{\hat{\mathscr{A}}} \in \hat{\mathcal{S}}$ is contractible too. By contrast, even showing that $\hat{\mathscr{A}}$ is simply connected would be a significant step toward showing Conjecture 4.1. So in this respect $\hat{\hat{X}}$ is in some sense easier to handle. There is a payoff though. Namely for any completion G', the G'-completion X' is naturally equipped with a G'-action, whereas the action of G cannot in general be lifted to $\hat{\hat{X}}$. These two phenomena meet so to speak, and constrain the situation as follows:

Proposition 4.4. *Let $X \in \mathcal{S}$ be a space equipped with an admissible action of a discrete group G, let G' be a profinite completion of G and X' the corresponding G'-completion.*

Assume that the G-action on X extends to a G'-action on the profinite completion $\hat{\hat{X}}$. Then $\hat{\hat{X}}$ and X' coincides, that is the natural projection $p : \hat{\hat{X}} \twoheadrightarrow X'$ is an isomorphism.

Proof. Let us spell out the respective universal properties of the completions $\hat{\hat{X}}$ and X'. Namely, any simplicial map $X \to Y$ with $Y \in \hat{\mathcal{S}}$ a profinite space factors uniquely through $\hat{\hat{X}}$ (with $Y = X'$, this shows the existence of p). As for X', recall that by construction it is universal for maps into profinite G'-spaces. In other words, any simplicial G-equivariant map $f : X \to Y$ of X to a profinite space $Y \in \hat{\mathcal{S}}$ which is endowed with an admissible (continuous) G'-action factors uniquely as

$f = f' \circ j'$ where $j' : X \to X'$ is the completion map and $f' : X' \to Y$ is G'-equivariant.

Assume now as in the statement that $\hat{\hat{X}}$ is equipped with a G'-action extending the G-action on X. Then the natural G-equivariant map $X \to \hat{\hat{X}}$ factors through X' and we get a natural map $q : X' \to \hat{\hat{X}}$. Let $\hat{\hat{\jmath}}$ denote the completion map $X \to \hat{\hat{X}}$. Clearly $j' = p \circ \hat{\hat{\jmath}}$ and $\hat{\hat{\jmath}} = q \circ j'$. Hence $q \circ p \circ \hat{\hat{\jmath}} = q \circ j' = \hat{\hat{\jmath}}$. In other words $q \circ p$ induces the identity on $\hat{\hat{\jmath}}(X)$ which is dense in $\hat{\hat{X}}$, so by continuity $q \circ p = id$ and similarly $p \circ q = id$. Q.E.D.

It may look a little strange at first sight that above one can use *any* profinite completion G'. But the finer the completion the easier it is to extend the G-action on X to a G'-action on $\hat{\hat{X}}$. In particular any G'-action induces a $\hat{G}$-action by composing with the projection $\hat{G} \twoheadrightarrow G'$. So we actually get that $\hat{\hat{X}} = X' = \hat{X}$. For arc (and curve) complexes, one can also refer to Remark 4.2 above; any completion $\mathscr{A}'$ (finer than geometric) is so to speak 'sandwiched' between $\check{\mathscr{A}}$ and $\hat{\hat{\mathscr{A}}}$ where the latter is indeed contractible. Note that we did not require above that the completion X' be residual, i.e. that the map j' be injective (note that if j' is injective, $\hat{\hat{\jmath}}$ is too) but that will always be true in the cases of interest here and moreover, as mentioned already, the fact that the completion $G \to G'$ is residual will imply in these situations that the completion $X \to X'$ is residual as well.

Turning back to the case of the arc complex, the above proposition states in essence that the contractibility conjecture is equivalent to the fact that the set theoretic completion $\hat{\hat{\mathscr{A}}}$ of the arc complex should inherit a natural action of the geometric completion $\check{\Gamma}$.

Finally we refer again to [Q3] for an in-depth comparison of the two kinds of completions in a very general setting, recalling that here we will always be dealing with comparatively 'nice' groups, i.e. completions of Teichmüller groups, which among other things are topologically finitely generated, residually finite and strongly stable.

§5. On the homotopy type of the profinite curve complexes

We now come to the curve complex $\mathscr{C}(S)$. Although this is not essential for our purposes in this paper, there are are several reasons why one can be interested in its homotopy type in the profinite context, one of them being that at present its algebro-geometric description, namely its modular interpretation seems more natural than for the arc complex, which was first introduced in an analytic context (via Strebel differentials etc.).

Here we will first recall the main result in the discrete case. This leads to an obvious conjecture in the profinite case which we state explicitly. After explaining the technical nature of the gap in [B1], we summarize the information we have gathered on the problem as well as the connection with the contractibility conjecture for the profinite arc complex.

In the discrete case, we have the following well-known description:

Theorem 5.1. *([H1,2,3], [Iv1]) If S is hyperbolic of type (g,n), the curve complex $\mathscr{C}(S)$ is (homotopically equivalent to) a wedge of spheres of dimension $h(S) = h_{g,n}$ with $h_{g,0} = 2g-2$ for $g > 1$, $h_{0,n} = n-4$ for $n \geq 3$, $h_{g,n} = 2g-3+n$ for $g \geq 1$, $n \geq 1$.*

For simplicity, in this section we will work with the full profinite completion only. One could accommodate completions which are finer than the geometric one and of course under the congruence conjecture, which we discuss in the next section, that makes no difference. It would be interesting to analyse other types of completions, especially the *pro*$-p$ completions. We now make the obvious conjecture, mimicking the discrete situation:

Conjecture 5.2. If S is a hyperbolic surface of finite type, the completed curve complex $\hat{\mathscr{C}}(S)$ has the homotopy type of a pro-wedge of spheres of dimension $h = h(S)$.

The conjecture is meaningful only for $h(S) > 1$, which is equivalent to $d(S) > 2$ $(d(S_{g,n}) = 3g-3+n)$, which is again equivalent to the discrete complex $\mathscr{C}(S)$ being simply connected. It then states that: $\pi_1(\hat{\mathscr{C}}(S)) = \{1\}$ and $H_k(\hat{\mathscr{C}}(S), \hat{\mathbb{Z}}) = \{0\}$ for all positive integers $k \neq 0, h(S)$; equivalently $H^k(\hat{\mathscr{C}}(S), \mathbb{Z}/p) = \{0\}$ for all k in the same range and all primes p. In fact the difficulty is concentrated in lower ranks, at least for $n > 0$, because then the (co)homology of $\mathscr{C}(S)$ and of the quotients $\mathscr{C}^\lambda(S) = \mathscr{C}(S)/\Gamma^\lambda$ vanishes in ranks $> h(S)$, hence also for the completed complex $\hat{\mathscr{C}}(S)$. We can record this discussion as:

Proposition 5.3. *In Conjecture 5.2 one can assume that $d(S) > 2$; then if $n > 0$, the statement is equivalent to $\hat{\mathscr{C}}(S)$ being simply connected and the vanishing $H_k(\hat{\mathscr{C}}(S), \hat{\mathbb{Z}}) = \{0\}$ for all integers k, $1 < k < h(S)$ or else $H^k(\hat{\mathscr{C}}(S), \mathbb{Z}/p) = \{0\}$ for k in that range and every prime p.*

Conjecture 5.2 was purportedly proved in [B1] but a serious gap was subsequently discovered. It occurs in the proof of Lemma 5.5 there, invalidating that statement and thus also Theorem 5.4. More precisely the argument showing that one can do away with torsion is flawed; it is still unknown how to deal with possible nontrivial torsion phenomena at this point. However the Hodge theoretic argument taking care of the nontorsion part of the cohomology of the quotients, which stems from the classical Deligne's Hodge III paper and was used in a similar context by E. Arbarello and M. Cornalba, that argument does apply and it shows the following remarkable fact about the finite quotients of the curve complex:

Proposition 5.4. *Let S be hyperbolic of finite type and $\Gamma^\lambda \subset \Gamma = \Gamma(S)$ be a level which dominates an abelian level of level $m > 2$, i.e. $\Gamma^\lambda \subset \Gamma^{(m)}$ (cf. §2.3 above). Then the* rational *cohomology groups $H^k(\mathscr{C}^\lambda(S), \mathbb{Q})$ vanish for $1 \leq k < h(S)$.*

This is about the extent of our knowledge or our ignorance on that conjecture to-date. Let us now make the connection with the contractibility conjecture, which predicts the triviality of the homotopy type of the completed *arc* complex, in the form of the following:

Proposition 5.5. *The contractibility conjecture (Conjecture 4.1) implies Conjecture 5.2 for open surfaces (i.e. $n > 0$).*

Proof. For simplicity we restrict ourselves to the model situation, namely a closed surface $S \simeq S_g$ ($g > 1$) with just one marked point P, so that $\Gamma = \Gamma(S, P) \simeq \Gamma_{g,1}$ and $h(S) = h_{g,1} = 2g - 2 = -\chi(S)$. First let us recall the core of the reasoning in the discrete case, showing how to deduce that $\mathscr{C}$ is $(h(S) - 1)$-connected from the contractibility of $\mathscr{A}(S, P)$.

Let $\underline{\alpha} \in \mathscr{A} = \mathscr{A}(S, P)$ be a multiarc. It is said to *fill* the surface if every component of its complement is simply connected (i.e. is a topological disc). Then $\mathscr{A}_\infty$ is defined to be the subcomplex of $\mathscr{A}$ spanned by the multiarcs which do *not* fill S; it is obviously stable under the action of Γ. Moreover $\mathscr{A}$ has dimension $6g - 4$ because it takes $6g - 3$ arcs to triangulate the surface and the crucial point is that $\mathscr{A}_\infty$ contains the $(2g - 2)$-skeleton of $\mathscr{A}$ because it takes $2g$ curves to fill a surface of genus g.

The main property in the discrete setting says that $\mathscr{C}$ is actually Γ-equivariantly homotopically equivalent to $\mathscr{A}_\infty$. We refer to [H1,2,3], especially [H2], for a review of this material, including the connection with Strebel differentials. The homotopy equivalence of $\mathscr{C}$ with $\mathscr{A}_\infty$, which embodies the connection between curve and arc complexes, is proved (in the general case) in Theorem 3.4 of [H3] (but see [H2] for the context). We should also emphasize that the implied homotopies are also Γ-equivariant and linear on each simplex of the geometric realization of the complexes (see [H2], proof of Lemma 4.3 or [H3], proof of Theorem 3.4). We can now form the completion $\hat{\mathscr{A}}_\infty$ and the above implies that the profinite completions $\hat{\mathscr{C}}(S)$ and $\hat{\mathscr{A}}_\infty$ are also homotopically equivalent.

Concerning the respective fundamental groups, if we assume the validity of Conjecture 4.1, we certainly have that $\hat{\mathscr{A}}(S)$ is simply connected. Moreover $\mathscr{A}_\infty$ contains the $(2g-2)$-skeleton of $\mathscr{A}$ and since $2g-2 \geq 2$, $\hat{\mathscr{A}}_\infty$ will be simply connected too, along with $\hat{\mathscr{C}}(S)$.

To deal with higher dimensional invariants, let us first stick to the discrete setting for a short while. Given $X \in \mathcal{S}$, let $X^{(q)} = Sk^q(X)$ denote its q-th skeleton; then $H^k(X) = H^k(X^{(q)})$ for $k < q$ and any system of coefficients (which we do not indicate explicitly). For instance, in the discrete setting this immediately implies the vanishing properties in small ranks (as in Proposition 5.3) which is enough to conclude. Indeed $\mathscr{A}_\infty^{(q)} = \mathscr{A}^{(q)}$ for $q \leq 2g-2$, and so for $k < 2g-2$ we can write:

$$H^k(\mathscr{C}) = H^k(\mathscr{A}_\infty) = H^k(\mathscr{A}_\infty^{(k+1)}) = H^k(\mathscr{A}^{(k+1)}) = H^k(\mathscr{A}) = 0$$

as was to be shown. Turning to the profinite case and a G-space X with admissible action, we can still write, for $k < q$:

$$\begin{aligned} H^k(\hat{X}^{(q)}) &= \varinjlim {}_\lambda H^k(X^{(q)}/G^\lambda) = \varinjlim {}_\lambda H^k\big((X/G^\lambda)^{(q)}\big) \\ &= \varinjlim {}_\lambda H^k(X/G^\lambda) = H^k(\hat{X}), \end{aligned}$$

restricting if necessary to the $\lambda \in \Lambda$ large enough so that G^λ acts simplicially. This vindicates the same string of equalities as above in the profinite case; in other words, we can write, for $k < 2g-2$:

$$H^k(\hat{\mathscr{C}}) = H^k(\hat{\mathscr{A}}_\infty) = H^k(\hat{\mathscr{A}}_\infty^{(k+1)}) = H^k(\hat{\mathscr{A}}^{(k+1)}) = H^k(\hat{\mathscr{A}}) = 0,$$

assuming of course the validity of the contractibility conjecture. This completes the proof of the proposition, by Proposition 5.3. Q.E.D.

§6. The congruence subgroup property

We first briefly recall the setting and the state of the art on that question, first raised and emphasized by N. N. Ivanov and which remains one of the important unsolved problems about the deep structure of the Teichmüller groups. In group theoretic terms, the conjecture asks whether the full profinite and the geometric or congruence completions coincide; do we have $\check{\Gamma} = \hat{\Gamma}$? In other words, do the congruence subgroups form a cofinal sequence in Γ?

We remark from the start that it does not matter whether $\Gamma = \Gamma(S)$ denotes a colored group (preserving marked points or punctures pointwise), or allows for permutations. So for ease of notation we will sometimes be lead to innocuous ambiguities in what follows. Anyway we use essentially colored (or 'pure') groups and will emphasize the topological type of the surface $S = S_{g,n}$, which here has no boundary (one could accommodate surfaces with boundaries, again at the cost of a heavier notation).

Let us first recall the geometric counterpart of the conjecture. Given a hyperbolic type (g, n) $(2g-2+n > 0)$ and $S = S_{g,n}$ as usual (see §2.1), write $\pi_{g,n}$ for its topological fundamental group $\pi_1(S)$ with its classical presentation, which we omit here; we just mention that $\pi_{g,n}$ is free of rank $2g+n-1$ for $n > 0$ and is a so-called surface group (cohomological dimension 2) in the compact case $n = 0$. After taking proper care of basepoints, which we do not need to trace here, one gets the usual short exact sequence:

$$(1) \qquad 1 \to \pi_{g,n} \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1.$$

It gives rise rise to a representation

$$\rho_{g,n} : \Gamma_{g,n} \to Out(\pi_{g,n}),$$

which can be regarded as the monodromy representation arising from considering the moduli space $\mathfrak{M}_{g,n}$ as a complex orbifold equipped with the tautological fibration $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$. Here $\mathfrak{C}_{g,n}$ is the universal curve (or Riemann surface) of type (g, n) sitting over $\mathfrak{M}_{g,n}$ and $\mathfrak{C}_{g,n} \simeq \mathfrak{M}_{g,n+1}$; $\pi_{g,n}$ is the fundamental group of the fiber and the construction is (essentially) independent of the choice of a basepoint on $\mathfrak{C}_{g,n}$.

The *topological* monodromy representation ρ is well-known to be faithful with an image which can easily be described. Actually it provides an isomorphism $\Gamma_{g,n} \simeq Out^*(\pi_{g,n})$ (which can also be used as a definition of the left-hand side), where the right-hand side denotes, in algebraic parlance, the subgroup of *inertia preserving* outer automorphisms. We will not dwell on this here but will encounter these notions again below.

Now, in the algebraic setting, regard $\mathfrak{M}_{g,n}$ as a $\mathbb{C}$-stack with fundamental group $\hat{\Gamma}_{g,n}$. We now have an exact sequence:

$$(2) \qquad 1 \to \hat{\pi}_{g,n} \to \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \to 1.$$

Here exactness on the left is ensured by the fact that $\hat{\pi}_{g,n}$ has trivial center (including when $n = 0$), which is a highly nontrivial fact. This sequence gives rise to $\hat{\rho} : \hat{\Gamma}_{g,n} \to Out(\hat{\pi}_{g,n})$, the universal *geometric* monodromy map. A natural question consists in inquiring whether the representation $\hat{\rho}$ is faithful, which is the geometric form of the congruence subgroup conjecture. Indeed, it is easily seen that the geometric completion $\check{\Gamma}_{g,n}$ as defined and used above is nothing but the image of $\hat{\rho}$; so the congruence subgroup conjecture asks whether $\hat{\rho}$ induces an isomorphism from $\hat{\Gamma}_{g,n}$ onto $\check{\Gamma}_{g,n}$. Note that in this equivalence with the group theoretic definition, one has used the fact that $\pi_{g,n}$ is finitely generated, which implies that its finite index *invariant* subgroups form a cofinal sequence.

The exact sequence (2) gives rise to another (*bona fide*, not outer) representation $\tilde{\rho}_{g,n} : \hat{\Gamma}_{g,n+1} \to Aut(\hat{\pi}_{g,n})$. Now using yet again the center triviality of $\hat{\pi}_{g,n}$ (i.e. that $Inn(\hat{\pi}_{g,n}) \simeq \hat{\pi}_{g,n}$) we get, essentially by definition, the following short exact sequence:

$$1 \to \hat{\pi}_{g,n} \to \tilde{\Gamma}_{g,n+1} \to \check{\Gamma}_{g,n} \to 1, \tag{3}$$

which holds for any finite hyperbolic type (g, n) (i.e. $g \geq 0$, $n \geq 0$, $2g - 2 + n > 0$). The following lemma was noted in the course of the proof of Theorem 1 in [A] (see also [B1], Lemma 4.2):

Lemma 6.1. *For $n > 0$, there is a natural (continuous) epimorphism $\phi : \check{\Gamma}_{g,n} \twoheadrightarrow \tilde{\Gamma}_{g,n}$.*

Proof. We start from the tautological epimorphism $\tilde{\Gamma}_{g,n+1} \twoheadrightarrow \check{\Gamma}_{g,n}$; there is also a natural epimorphism $p : \tilde{\Gamma}_{g,n+1} \twoheadrightarrow \tilde{\Gamma}_{g,n}$ induced by filling in the last puncture. More explicitly p arises from the natural projection $\hat{\pi}_{g,n+1} \twoheadrightarrow \hat{\pi}_{g,n}$ obtained by quotienting by (the normal closure of the group generated by) the generating loop $u_{n+1} \in \pi_{g,n+1}$ around the last puncture.

One now remarks that $f \in \check{\Gamma}_{g,n}$ can be lifted to $\tilde{f} \in \tilde{\Gamma}_{g,n+1}$ preserving u_{n+1} (i.e. $\tilde{f}(u_{n+1}) = u_{n+1}$). Then set $\phi(f) = p(\tilde{f})$. To prove that this is well-defined, observe that if $\tilde{f}$ and $\tilde{f}'$ denote two lifts of f as above, $\tilde{f}^{-1}\tilde{f}'$ is an inner automorphism of $\hat{\pi}_{g,n+1}$ preserving u_{n+1}, in other words it is determined by an element of the centralizer of u_{n+1} in $\hat{\pi}_{g,n+1}$. But the latter is procyclic, generated by u_{n+1} (see e.g. [N], Lemma 2.1.2 or [B1]), which finishes the proof of the lemma. Q.E.D.

Note that again the main technical ingredient here lies in the determination of the centralizers of the generators of a (free) surface group, which implies of course the center triviality of that group. We have included the proof of the lemma because it immediately implies:

Theorem 6.2. *The congruence subgroup conjecture holds true in genus 0, that is* $\Gamma_{0,n}$ $(n \geq 3)$ *has the congruence subgroup property.*

Proof. By induction, starting from $\Gamma_{0,3} = \{1\}$. Assume the conjecture holds for n and consider the sequence (3). One has $\check{\Gamma}_{0,n} = \hat{\Gamma}_{0,n}$ by the inductive assumption; hence $\tilde{\Gamma}_{0,n+1} = \hat{\Gamma}_{0,n+1}$ from sequence (2), and by the lemma we find that $\check{\Gamma}_{0,n+1} = \tilde{\Gamma}_{0,n+1} = \hat{\Gamma}_{0,n+1}$, which completes the proof. Q.E.D.

Remark 6.3. It seems a little difficult to attribute this result to a given author, as it may have appeared in print or have been known to a few individuals before it was formally stated and proved as such. Certainly [A] provides a proof which is basically the one we gave. Nowadays it can be reduced to just one page (see above or [B1], §4), the lemma above being the only technical ingredient. In other words, one needs to determine the centralizer of a topological generator in the free group $\pi_{0,n}$.

The epimorphism constructed in Lemma 6.1 is actually an isomorphism. In other words:

Theorem 6.4. *For any* $g > 0$, $n \geq 0$, *one has* $\tilde{\Gamma}_{g,n+1} \simeq \check{\Gamma}_{g,n+1}$.

We record this statement as a Theorem because it is in fact much more difficult than Theorem 6.2. Note that we have excluded the case $g = 0$ for no other reason than the fact that it has been covered already: $\check{\Gamma}_{0,n+1} = \tilde{\Gamma}_{0,n+1} = \hat{\Gamma}_{0,n+1}$ $(n > 2)$.

Before commenting on the result we write down the exact sequence that it immediately yields for the geometric completion; namely for any hyperbolic type (g, n), one has:

$$1 \to \hat{\pi}_{g,n} \to \check{\Gamma}_{g,n+1} \to \check{\Gamma}_{g,n} \to 1. \tag{4}$$

As for the proof of the theorem, in the open (or affine) case $n > 0$, it is a formal consequence of Theorem 2.2 in [Mat]. The closed (or projective) case $n = 0$ is still more involved and formally follows from Corollary 6.2 of [HM1]; a more elementary proof targeted to the present setting appears in [B3]. We also call the attention of the reader to a paper of R. P. Kent ([K]), which very roughly speaking places a new restriction on the congruence kernel in the affine case and as a technical by-product reproves Theorem 6.4 above in the affine case $(n > 0)$.

As an immediate corollary of the Theorem 6.4, or in fact of the exact sequence (4) above, we get:

Corollary 6.5. *For any any hyperbolic type (g,n), $\Gamma_{g,n}$ has the congruence subgroup property if and only if this is true for $\Gamma_{g,n+1}$.*

In other words, whether the congruence conjecture holds for $\Gamma_{g,n}$ actually depends only on g, so from now on we can talk of the congruence conjecture for a given genus g. Note that the *only if* part of the result, that is increasing the number n of punctures, is a consequence of the sole Lemma 6.1 above (and implies the conjecture in genus 0). The *if* part, that is decreasing the number of punctures, is more tricky, especially passing from the affine case $\Gamma_{g,1}$ to the projective case Γ_g ($g > 1$, $\Gamma_g = \Gamma_{g,0}$). This may prove very useful, for instance because arc complexes require at least one marked point. So the contractibility conjecture (and less; see below) would prove the congruence conjecture for $\Gamma_{g,1}$ but by the hardest case of Theorem 6.4 above, that would suffice.

Apart from the corollary above, the state of the art can be summarized in:

Theorem 6.6. *The congruence subgroup conjecture holds in genus $g = 0, 1, 2$.*

We have repeated the case $g = 0$ for convenience. The genus 1 case is contained in [A]. Perhaps at this point we should warn the newcomer that in the case of $\Gamma_{1,1} \simeq SL_2(\mathbb{Z})$ the congruence subgroup conjecture we are dealing with and which holds true in that case refers to a form of non abelian congruence subgroups, as opposed to the usual abelian congruences considered—say—in the theory of modular forms. Of course the usual (abelian) form of the congruence property does *not* hold for $SL_2(\mathbb{Z})$. Finally the hyperelliptic case $g = 2$ was settled in [B2] in the open ($n > 0$) case and [B3] provides an alternate proof of Theorem 6.4 for $n = 0$, thus settling in particular the case of Γ_2.

For any (g,n) we call $d(g,n) = 3g-3+n = dim(\mathfrak{M}(S_{g,n}))$ its associated modular dimension. So we see from the above that the congruence conjecture has been settled in particular in modular dimension ≤ 5. However... it seems fair to say that, as is often the case in Teichmüller theory, 'serious' problems really start in genus 3! So the case of Γ_3, or $\Gamma_{3,1}$ which may be easier to settle and yet is equivalent by the above, represents the real 'frontier'.

Let us pause a moment in order to note yet another easy consequence of Theorem 6.4. It is not hard to see that $\tilde{\Gamma}_{g,n+1}$ is centerfree by its very definition. From Theorem 6.4 we thus get:

Corollary 6.7. *For any any hyperbolic type (g,n) with $n > 0$, the geometric completion $\check{\Gamma}_{g,n}$ has trivial center.*

Note that this result does not cover the closed case ($n = 0$). One can however find a significantly stronger result, which does cover that last case, in [HM2], Theorem 6.13.

Let us now return to our favourite complexes (no pun intended) after stressing again that the above results afford a greater flexibility in terms of parameters: only the genus g is really at issue. The connection between the contractibility and the congruence conjectures goes as follows. Since everything depends on the genus only, it is enough to consider the model case of a closed surface S of genus $g > 1$ with a marked point P, the attending (modular) dimension being $d(S, P) = d(S \setminus P) = 3g - 2$. The notation below refers to that situation. Let $K = K(S, P)$ denote the congruence kernel, that is the kernel of the natural projection $p : \hat{\Gamma} \twoheadrightarrow \check{\Gamma}$. Then we have the following:

Proposition 6.8. *Let $\mathscr{A} = \mathscr{A}(S, P)$ and assume the validity of the congruence conjecture in dimensions $< d(S, P)$. Then there is an exact sequence:*

$$1 \to \pi_1(\hat{\mathscr{A}}) \to \pi_1(\check{\mathscr{A}}) \to K \to 1.$$

Proof. The first map in the sequence is of course $\pi_1(p)$, induced by the natural projection p. The group $\hat{\Gamma}$ acts on $\hat{\mathscr{A}}$ and the congruence kernel K acts transitively on the fibers of p. So the proposition is equivalent to the fact that p is unramified, i.e. K acts fixed point free. Let then $k \in K$ be an element of $\hat{\Gamma}$ fixing $\sigma \in \hat{\mathscr{A}}$; after conjugating by an element of $\hat{\Gamma}$, (recall that K is normal in $\hat{\Gamma}$), we may and do assume that $\sigma \in \mathscr{A}$ is discrete. Then k belongs to $\bar{\Gamma}(S \setminus \sigma, P) \subset \hat{\Gamma}$, the closure of $\Gamma(S \setminus \sigma, P)$. By Proposition 3.4, the induced topology on $\bar{\Gamma}(S \setminus \sigma, P) \subset \hat{\Gamma}$ is finer than the geometric topology and since we assume the congruence conjecture in strictly smaller dimensions, that last topology coincides with the profinite topology, that is $\bar{\Gamma}(S \setminus \sigma, P) \simeq \hat{\Gamma}(S \setminus \sigma, P) \subset \hat{\Gamma}$. So k belongs to the congruence kernel pertaining to $S \setminus \sigma$, which is trivial by assumption. In short the map p is unramified and that proves the validity of the exact sequence in the statement. Q.E.D.

We can collect some immediate consequences in the following statement, bearing in mind that we know the validity of the congruence conjecture in dimensions < 6 (the first unknown case being for closed surface of genus 3).

Corollary 6.9. *The congruence conjecture in genus $g > 2$ is equivalent to its validity in genus $< g$ plus the isomorphism $\pi_1(\check{\mathscr{A}}(S, P)) \simeq \pi_1(\hat{\mathscr{A}}(S, P))$ where S is closed of genus g and P is a basepoint on S.*

In particular if $\check{\mathscr{A}}(S,P)$ *is simply connected for* $g(S) \leq g$, *the congruence conjecture holds in genus* $\leq g$.

Also one can note again that this of course reduces the contractibility conjecture to the case of the geometric completion. The significance of this reduction derives from the fact that geometric completions, especially $\check{\mathscr{A}}(S)$ and $\check{\mathscr{C}}(S)$, can in principle be defined and investigated using objects which pertain directly to the surface S, without recourse to 'modular' objects, attached to the moduli space $\mathfrak{M}(S)$; see [B3] and especially [HM1,2] in this direction.

It is sometimes easier to work with the curve rather than the arc complex. As we have seen, the congruence conjecture depends only on the fundamental group of the geometric arc complex and this means that everything can also be translated in terms of the curve complex as in the last section, in contrast with the properties investigated in the next section.

We finally remark that as a (modest?) intermediate result, it would be interesting to show that the congruence kernel K is not only a normal but in fact an *invariant* subgroup of $\hat{\Gamma}$. In a later section we will indeed be interested in the automorphisms of $\hat{\Gamma}$ and this would ensure that every such automorphism descends to a (possibly trivial) automorphism of $\check{\Gamma}$.

§7. The étale type of the moduli stacks of curves

At variance with the last paragraph—and for no reason other than a need for diversity—we have emphasized in the title the geometric rather than the group theoretic version of the property we will explore in this section. Much as in the last section we will first tersely review three versions of 'goodness', mentioning a few references; then we summarize the state of the art as far as Teichmüller groups are concerned, and finally we make the connection with arc and curve complexes.

So first from a homotopical viewpoint: in a nutshell, if G is a group and $\hat{G}$ denotes its profinite completion, one has a natural map $j : G \to \hat{G}$, hence a map $BG \to B\hat{G}$ between the respective classifying spaces, giving rise in $\hat{\mathcal{S}}$ to a natural map $\widehat{BG} \to B\hat{G}$ between the completion of the classifying space of G and the classifying space of its profinite completion. The group G is called *good* if this map is a weak equivalence. We refer to [AM], §6 and especially to [Q3], §3 for this homotopical version of the story. Again in this section we will confine ourselves to the full profinite completion, but one can and does adapt this definition and the equivalent ones below to a more general setting, essentially by picking an admissible class of groups with respect to which the completion is effected (see [AM], §3 and especially [N], §1).

In terms of groups, that is the respective fundamental groups, we know that $\pi_1(\widehat{\widehat{BG}}) = \widehat{\pi_1(BG)}$, so $\pi_1(\widehat{\widehat{BG}}) = \hat{G}$ (see e.g. [Q1], Proposition 2.1). Since this coincides with $\pi_1(B\hat{G})$, the two spaces are weakly equivalent (i.e. coincide in $Ho(\hat{\mathcal{S}})$) if and only if their cohomologies with coefficients in any finite (equivalently, any torsion) local system coincide. Moreover $H^*(\widehat{\widehat{BG}}) = H^*(BG)$ so these are just the respective *group* cohomologies of G and $\hat{G}$. Summarizing, $\widehat{\widehat{BG}}$ and $B\hat{G}$ coincide in $Ho(\hat{\mathcal{S}})$ if and only if the natural map $j^* : H^*(\hat{G}, A) \to H^*(G, A)$ is an isomorphism for any finite $\hat{G}$-module. The latter condition is precisely the definition of G being *good* as introduced by J.-P. Serre; see [S], §I.2.6. We also refer to Lemma 8.2 in [B1] which refines the indications of [S] and to [N], §1.

Finally, the original motivation of J.-P. Serre for emphasizing this property stems from the geometric context and the so-called Artin good neighborhoods which help prove that under mild conditions the étale topology on a scheme is locally acyclic (see SGA 4, Exposé 11). These good neighborhoods are in particular algebraic $K(\pi, 1)$. In order to define the latter notion, start with a connected scheme X regarded with its étale topology (i.e. the associated site) and denote by $\pi = \pi_1(X)$ its fundamental group based at some geometric point which we omit in the notation. There is a natural map $H^*(\pi, A) \to H^*(X, A)$ where A is again a finite (or torsion) π-module and the coefficients on the right are defined by the associated local system. The scheme X is defined to be an algebraic $K(\pi, 1)$ if the latter map is an isomorphism for every A. We refer to [Sx], Appendix A, for a thorough discussion and several equivalent formulations. An important and suggestive characterization says that X is algebraically $K(\pi, 1)$ if and only if, for every prime p and every $k > 0$, the direct limit

$$\varinjlim{}_{Y/X} H^k(Y, \mathbb{Z}/p)$$

vanishes (cohomology with constant coefficients), where Y runs through the finite étale covers of X. One way of putting it is that it denotes an abundance of étale covers, enough so that every cohomology class as above 'dies' by restriction to a suitable such cover.

Now start from a complex variety $X_{\mathbb{C}}$ with (topological) fundamental group $G = \pi_1^{top}(X_{\mathbb{C}})$ and assume $X_{\mathbb{C}}$ is a (topological) $K(G, 1)$. Consider the associated $\mathbb{C}$-scheme X; one finds that X is an algebraic $K(\hat{G}, 1)$ if and only the group G is good. A last piece of information is that all these notions are invariant by passing to finite étale covers, i.e. finite index subgroups.

Let us specialize to the moduli spaces of curves $\mathfrak{M}_{g,[n]}$. We view them as Deligne–Mumford stacks over $\mathbb{Q}$, with associated geometric fundamental group $\hat{\Gamma}_{g,[n]}$; the analytification of $\mathfrak{M}_{g,[n]}(\mathbb{C})$ has a structure of complex orbifold with orbifold fundamental group $\Gamma_{g,[n]}$. The theory above can be elaborated for complex orbifolds on the analytic side and Deligne–Mumford stacks on the geometric side, but here the situation is actually is easier since $\mathfrak{M}_{g,[n]}$ is virtually a scheme, i.e. it has a finite étale cover (an abelian level structure) which is a scheme (idem with orbifold and manifold), so if need be the reader can pass to such a cover and this will not alter the statements below. In any case we will now discuss a well-known conjecture (T. Oda and others) stating that the Teichmüller groups $\Gamma_{g,n}$ should be good, that is the $\mathfrak{M}_{g,[n]}$ should be algebraic $K(\pi, 1)$.

First everything is up to a finite cover, so we can just as well use ordered points, that is the $\Gamma_{g,n}$ and $\mathfrak{M}_{g,n}$. It will emerge that one could add boundary components as well, but we refrain from doing so for (essentially notational) simplicity. The next step consists in showing that, as is the case for the congruence property (cf. Corollary 6.5), goodness depends only on the genus g.

To this end, let us explore more generally some stability properties of goodness under group extension. We start with $G \in Ext(K, H)$, i.e. an exact sequence of discrete groups:

$$1 \to H \to G \to K \to 1. \tag{5}$$

Recall that a group G is of type FP if $\mathbb{Z}$, considered as a trivial G-module, admits a resolution of finite length by projective G-modules of finite type; G is virtually FP if it has an open subgroup which is of type FP. In particular, a group of type FP (resp. virtually FP) is finitely generated and has finite (resp. virtually finite) cohomological dimension. With this standard terminology in mind we can state:

Proposition 7.1. *Let the discrete group G be an extension of K by H as in* (5)*; then:*
i) If H is finitely generated and K is good, the term-by-term completion of (5)*:*

$$1 \to \hat{H} \to \hat{G} \to \hat{K} \to 1 \tag{6}$$

is exact;

ii) If H is good, virtually FP and K is good, then G is good too;
iii) Conversely, assume that H is good and virtually FP, that G is finitely generated and good and that the completed sequence (6) is exact, then the quotient K is good.

Proof. i) Since completion is right exact, the point is to show that the map $\hat{H} \to \hat{G}$ is injective. This results from [N], Proposition 1.2.4 which describes more generally the situation for completion with respect to any admissible (or 'full') class of finite groups.

In short ii) states that goodness is stable for extensions *by* virtually FP groups. Note that by i) the completed sequence is then exact. By passing to open subgroups, one can assume that H itself is of type FP and then refer to [S] or [N] (§1.2.5). We sketch the proof nonetheless because it is typical and short. This stability of goodness by extension comes from a standard comparison of spectral sequences argument. For A a finite $\hat{G}$-module, there is a spectral sequence with E_2 term $E_2^{p,q} = H^p(K, H^q(H, A))$ (resp. $\hat{E}_2^{p,q} = H^p(\hat{K}, H^q(\hat{H}, A))$ and abutment $H^{p+q}(G, A)$ (resp. $H^{p+q}(\hat{G}, A)$). Since H and K are good, the E_2 terms of the two sequences coincide, noting that H of type FP ensures that $H^q(H, A)$ ($\simeq H^q(\hat{H}, A)$) is finite.

Item iii) is more complicated. First note that the assumptions imply that all three groups are finitely generated. We now need to show (see e.g. [N], Proposition 1.1.5 in the present group theoretic context) that for $k > 0$ and every prime ℓ, the direct limit $\varinjlim_V H^k(V, \mathbb{Z}/\ell)$ vanishes, where V varies over the finite index subgroups of K. Fix a prime ℓ and denote by p the projection $p : G \to H$. We first note that there exists a cofinal system $(G^\lambda)_{\lambda \in \Lambda}$ of finite index subgroups (or *cofinite* subgroups) of G with the following properties:
a) Every G^λ is normal in G;
b) The projections $K^\lambda = p(G^\lambda)$ form a cofinal system of cofinite subgroups of K;
c) The intersections $H^\lambda = G^\lambda \cap H$ form a cofinal system of cofinite subgroups of H;
d) For every $k \geq 0$ the action $G^\lambda \to Aut(H^k(H^\lambda, \mathbb{Z}/\ell))$ is trivial.

Indeed, start from the system of all cofinite subgroups of G. Since G is finitely generated, we can refine it to the *normal* subgroups and call it (G^λ) (we will refine it further but keep the notation). The projections $K^\lambda = p(G^\lambda)$ are indeed cofinal in K and b) is mentioned only for completeness. Next $H^\lambda = G^\lambda \cap H$ is normal in G^λ (indeed in G), so it acts on H^λ by conjugation and the monodromy action in d) is derived from this. Moreover, the fact that the completed sequence is exact, that is $\hat{H} \to \hat{G}$ is injective, precisely means that the H^λ's form a cofinal system in H. Returning to the monodromy action, since H is virtually FP, G^λ has finite cohomological dimension ($= vcd(H)$) for λ large enough (and it is independent of λ), so we can fix k. Again because H is virtually FP, $H^k(H^\lambda, \mathbb{Z}/\ell)$ is finite for λ large enough, so passing to a cofinite subgroup of G^λ trivializes it. This shows the existence of a system (G^λ) as above.

We now have a system of spectral sequences E_λ indexed by $\lambda \in \Lambda$ with E_2 terms:

$$E_{2,\lambda}^{p,q} = H^p(K^\lambda, H^q(H^\lambda, \mathbb{Z}/\ell)).$$

The sequence E_λ converges to $H^k(G^\lambda, \mathbb{Z}/\ell)$ with $k = p + q$. Since H is virtually FP, E_λ collapses at $E_{r(\lambda),\lambda}$ ($E_{r(\lambda),\lambda} = E_{\infty,\lambda}$) with $r(\lambda)$ bounded above uniformly in λ (indeed by $vcd(H) + 1$). This makes it possible to perform the limit over λ: we get a spectral sequence with E_2 term: $\varinjlim_\lambda E_{2,\lambda}^{p,q}$ converging to $\varinjlim_\lambda H^k(G^\lambda, \mathbb{Z}/\ell)$ and collapsing at E_r with $r \leq vcd(H) + 1$.

But $\varinjlim_\lambda H^k(G^\lambda, \mathbb{Z}/\ell)$ vanishes for $k > 0$ since G is good by assumption and G^λ is cofinal. By universal coefficients:

$$H^p(K^\lambda, H^q(H^\lambda, \mathbb{Z}/\ell)) = H^p(K^\lambda, \mathbb{Z}/\ell) \otimes H^q(H^\lambda, \mathbb{Z}/\ell)$$

(tensor product over $\mathbb{Z}/\ell$) since $H^q(H^\lambda, \mathbb{Z}/\ell)$ is a finite dimensional $\mathbb{Z}/\ell$-vector space (hence a *free* module). Since the second factor is finite we get, for the E_2 term of the limit sequence:

$$\varinjlim_\lambda E_{2,\lambda}^{p,q} = \varinjlim_\lambda H^p(K^\lambda, \mathbb{Z}/\ell) \hat{\otimes} \varinjlim_\lambda H^q(H^\lambda, \mathbb{Z}/\ell)$$

(completed tensor product over $\mathbb{Z}/\ell$). The second limit on the right-hand side is $\mathbb{Z}/\ell$ for $q = 0$ and vanishes for $q > 0$ since H is good by assumption and the H^λ's are cofinal. The limit sequence thus collapses at E_2 and since it converges to 0 for $p+q > 0$, this implies the vanishing of $\varinjlim_\lambda H^p(K^\lambda, \mathbb{Z}/\ell)$ for $p > 0$, i.e. the goodness of K, taking into account that the sequence K^λ is cofinal in K. Q.E.D.

We now apply this proposition to the exact sequences (1) and (2) of the previous section in order to show the analog of Corollary 6.5, namely:

Proposition 7.2. *For all finite hyperbolic types (g, n), $\Gamma_{g,n}$ is good if and only if this holds for $\Gamma_{g,n+1}$.*

Proof. As usual the *only if* part, or increasing the number of punctures, is the easier one. Indeed one simply applies ii) of the last proposition to sequence (1). The right-hand term ($\Gamma_{g,n}$) is good by the inductive assumption and one needs to know that $\pi_{g,n}$ is good too since it is certainly of FP type. For $n > 0$, this is a finitely generated free group and there is nothing to prove (because it has cohomological dimension 1). For $n = 0$, $\pi_g = \pi_{g,0}$ $(g > 1)$ is a surface group (cohomological dimension 2) with $H^2(\pi_g, \mathbb{Z}/\ell) = \mathbb{Z}/\ell$; restricting the fundamental class to a normal subgroup of index ℓ will 'kill' it.

In the other direction, one applies iii) of the last proposition. To check the assumptions, one needs only show that the completed sequence is exact, which results from the fact that the completion $\hat{\pi}_{g,n}$ has trivial center. This is well-known but nontrivial and we have used this property several times already (see e.g. [N], §1.3 for the proof). Note that again the compact case $n = 0$ is (relatively speaking) the hardest. It depends on the fact that π_g is good and that S_g is a $K(\pi, 1)$ with nonzero Euler characteristic. Q.E.D.

So goodness depends only on the genus g. The question is easy to settle in genus < 3, paralleling the situation for the congruence conjecture (cf. Theorem 6.6) and many other statements in Teichmüller theory. The following statement hardly deserves the title of 'theorem':

Proposition 7.3. *Teichmüller groups are good in genus* $g = 0, 1, 2$.

Equivalently, the moduli stacks $\mathfrak{M}_{g,n}$ are algebraically $K(\pi, 1)$ for $g \leq 2$, with the proviso that they are effectively stacks (and not schemes) for $g = 1, 2$ and n small so one should use the proper definition (or pass to a finite cover).

Proof. There does not remain much to prove really. If $g = 0$ one can simply observe that $\Gamma_{0,3}$ is trivial hence good and appeal to the last proposition. But in a nutshell, one is simply saying that the $\Gamma_{0,n}$ are iterated extensions by free groups. More geometrically, the affine schemes $\mathfrak{M}_{0,n}$ are $K(\pi, 1)$ obtained as iterated fibrations by hyperbolic curves (pointed spheres), i.e. they *are* Artin good neighborhoods.

In genus 1, $\Gamma_{1,1} \simeq SL_2(\mathbb{Z})$ contains the principal congruence subgroup of order 3 (the subgroup of matrices congruent to the identity mod. 3), which is cofinite and finitely generated, free, hence good. Alternatively, $\Gamma_{1,2}$ appears as an index 5 subgroup of $\Gamma_{0,5}$.

In genus 2, $\Gamma_2/Z \simeq \Gamma_{0,6}$ where $Z \simeq \mathbb{Z}/2$ is the center (generated by the hyperelliptic involution). So Γ_2/Z is good and one then appeals to (a very special case of) Proposition 7.1, iii). Q.E.D.

Just as in the last section, we now return to the completed arc complexes. It is enough to show the goodness of $\Gamma(S)$ for any surface of genus g, with any number of marked points, punctures and in fact also possibly boundary components although we are not considering these explicitly here. So the point is:

Proposition 7.4. *The contractibility conjecture (Conjecture 4.1) implies the goodness of the Teichmüller groups.*

Proof. Let $\mathscr{A} = \mathscr{A}(S, P)$ and $\Gamma = \Gamma(S, P)$ be as in Conjecture 4.1. Let M be a finite $\hat{\Gamma}$-module and consider the Γ-equivariant cohomology groups $H^k_\Gamma(\mathscr{A}, M)$ for $k \geq 0$; since $\mathscr{A}$ is contractible, they coincide with the group cohomology of Γ: $H^k_\Gamma(\mathscr{A}, M) \simeq H^k(\Gamma, M)$. Similarly, if we assume the contractibility of the completed complex $\hat{\mathscr{A}}$, we get a natural isomorphism: $H^k_{\hat{\Gamma}}(\hat{\mathscr{A}}, M) \simeq H^k(\hat{\Gamma}, M)$. Therefore, under Conjecture 4.1 we may as well compare the discrete and completed equivariant cohomologies.

The group Γ (resp. $\hat{\Gamma}$) acts on $\mathscr{A}$ (resp. $\hat{\mathscr{A}}$) virtually simplicially with finitely many orbits. We can pass to a cofinite subgroup which acts simplicially since this does not affect goodness. Somewhat daringly but for the sake of readability, we keep the notation Γ for such a subgroup. For $r \geq 0$ (and smaller than the dimension of $\mathscr{A}$), a finite set Σ_r of representatives for the action of Γ on the r-simplices of $\mathscr{A}$ plays the same role for the action of $\hat{\Gamma}$ on the r-simplices of $\hat{\mathscr{A}}$. Now we have two spectral sequences E and $\hat{E}$, converging to $H^k_\Gamma(\mathscr{A}, M)$ and $H^k_{\hat{\Gamma}}(\hat{\mathscr{A}}, M)$ respectively, and with E_1 terms

$$E_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, M)$$

and

$$\hat{E}_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} H^q(\hat{\Gamma}_\sigma, M).$$

Here Γ_σ (resp. $\hat{\Gamma}_\sigma$) denotes of course the stabilizer of the simplex σ in Γ (resp. $\hat{\Gamma}$). Having passed (if somewhat implicitly) to a small enough cofinite subgroup of Γ, these stabilizers are *pointwise* stabilizers, i.e. they are the intersections of the stabilizers of the vertices of σ.

Having assumed that the contractibility conjecture holds, we are operating under the congruence conjecture as well, so by Proposition 3.4 the notation above is unambiguous: The stabilizer $\hat{\Gamma}_\sigma$ of $\sigma \in \mathscr{A}$ in $\hat{\Gamma}$ coincides with the profinite completion of the discrete stabilizer Γ_σ.

We now proceed via the usual induction on the modular dimension. The starting point is not a problem since goodness holds in genus ≤ 2 with any number of marked points. Having assumed goodness until a certain dimension, the terms $E_1^{p,q}$ and $\hat{E}_1^{p,q}$ coincide because the stabilizers have strictly smaller modular dimension, so the abutments of the spectral sequences agree and we are done. Q.E.D.

Remark 7.5. The conjecture on the goodness of the Teichmüller groups, or equivalently the fact that the moduli stacks of curves are algebraic $K(\pi, 1)$, follows from the contractibility conjecture as explained above. However it may be that we have been a bit greedy with assumptions there because by assuming the validity of Conjecture 4.1 we also *ipso facto* assume the congruence property. It could be after all that one can prove that $\hat{\mathscr{A}}$ is contractible without showing that $\check{\mathscr{A}}$ is simply connected. What could we then conclude from the contractibility of the full completion $\hat{\mathscr{A}}$ alone?

Remark 7.6. As a side remark we add that it would be interesting to know whether the analog of Proposition 5.4 is true for the arc complex. Namely, is it true that for large enough $\lambda \in \Lambda$ and for $k > 0$, the rational cohomology $H^k(\mathscr{A}^\lambda(S), \mathbb{Q})$ of the corresponding quotient of the arc complex vanishes?

Here one encounters the fact that $\mathscr{A}(S)$ is more directly connected with the attending Teichmüller space $\mathfrak{T}(S)$, so that Hodge theory does not immediately apply.

§8. On the structure of the profinite Teichmüller groups

The last three sections of this text are based in part on the unpublished manuscripts [BL] and [L1]. From these there emerges a rather clear and tantalizing landscape with some but unfortunately preciously few unconditional statements to-date. Yet we feel that a description of this partly conjectural but quite detailed situation can be both enlightening and motivating for certain readers and this is why we include (part of) it here. As usual we have tried to keep track of the present status of the statements and to sort out their dependence on conjectures.

8.1. Dynamical rêveries: Where are the lions?

We start with a little bit of daydreaming about dynamics. Let S be a hyperbolic surface of finite type, $\Gamma = \Gamma(S)$, $\mathscr{C} = \mathscr{C}(S)$ etc. as above. What can one say about the natural action of $\hat{\Gamma}$ on—say—$\hat{\mathscr{C}}$? Unfortunately very little if anything at present, although this sounds like a very natural question indeed.

Let us review some basic ingredients of the discrete theory, especially as envisioned by Thurston; see [Iv2] for a very nice presentation. First one needs a *geometric* intersection number (α, β) assigning a positive natural number to two curves defined up to isotopy. This is not available in the profinite case i.e. for procurves (where would it take its values?). Note that there does exist an *algebraic* intersection number with values in $\hat{\mathbb{Z}}$, given by the cup product of elements in $H_1(S, \hat{\mathbb{Z}})$. Then given three simple curves α, β and γ 'in general position' one can study the sequence of integers $(t_\alpha^n(\beta), \gamma)$ $(n \geq 0)$ which gives information about the iteration of the twist t_α acting on $\mathscr{C}$ or for that matter on the Teichmüller space $\mathfrak{T}$. This sequence is exponentially increasing, measuring how the sequence $t_\alpha^n(\beta)$ escapes to infinity. That does not mean anything in the profinite case because there is no 'infinity' in the first place; $\hat{\mathscr{C}}$ is compact. As a baby example, one can think of the shift $S(z) = z + 1$, viewed on $\mathbb{Z}$, $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}$ respectively. This is indeed a 'baby' example, not least because the (full) profinite topology is highly non-abelian on top of being non archimedean or adelic. In fact one could first try to study the $pro-p$ dynamics, i.e. the action of Γ^{pro-p} on $\mathscr{C}^{pro-p}$.

So nothing works and we were not able to pull ourselves off the ground. Let us try a more direct path. Suppose you want to describe the elements of $\hat{\Gamma}(S)$. After all, any such $g \in \hat{\Gamma}(S)$ can be realized as the limit, for the profinite topology on $\hat{\Gamma}(S)$ of course, of a sequence $(g_n)_n$ $(n \geq 0)$ of elements of the discrete group $\Gamma(S)$. Write g as such a limit and use the Thurston–Bers classification. It tells us that, up to extracting a subsequence we can assume that either a) the g_n have finite orders or b) they are reducible, or c) they are pseudo-Anosov, the three possibilities being mutually exclusive.

Starting with case a), there are only finitely many conjugacy classes of torsion elements in $\Gamma(S)$. So we can assume that all the g_n's are conjugate and since the conjugacy classes in $\hat{\Gamma}(S)$ are closed, they converge to an element which is torsion and is again in the same $\hat{\Gamma}$-conjugacy classes. Conclusion: In case a) we simply recover $\hat{\Gamma}(S)$-conjugates of discrete finite order elements.

In case b), to each g_n we attach a multicurve, alias a simplex $\sigma_n \in \mathscr{C}(S)$ which is stabilized by g_n (setwise). By compactness the sequence of simplices $(\sigma_n)_n$ has a limit point $\sigma \in \hat{\mathscr{C}}(S)$ and the element $g \in \hat{\Gamma}(S)$ stabilizes σ. Now σ itself is in the $\hat{\Gamma}$-orbit of a *discrete* simplex $\tau \in \Gamma(S)$. So up to twisting by an element of $\hat{\Gamma}(S)$, g is reducible by a multicurve τ and in order to study g we can pass to $\hat{\Gamma}(S \setminus \tau)$, not forgetting about the multitwists supported on τ. Conclusion: In case b) we simply get $\hat{\Gamma}(S)$-conjugates of elements which are reducible in the usual sense.

There remains only case c), and this is of course where the unchartered territories lie wide open. But we can still eliminate a subcase.

Namely let $(g_n)_n$ be a converging sequence of pseudo-Anosov diffeomorphisms and let α_n (> 1) denote the stretching factor of g_n. Now if the sequence $(\alpha_n)_n$ is bounded, a result of N. Ivanov tells us that the g_n actually lie in finitely many conjugacy classes of $\Gamma(S)$, so g itself is again $\hat{\Gamma}(S)$-conjugate to a pseudo-Anosov of $\Gamma(S)$ and we find nothing new.

The only remaining case thus contains the core of the problem, which justifies asking the not so clearcut but important

Question 8.1. How can one 'describe' the limit in $\hat{\Gamma}(S)$ of a converging sequence of pseudo-Anosov diffeomorphisms whose stretching factors tend to infinity?

We note that $\Gamma(S)$ acts on measured foliations so it makes sense to say that two such foliations are close in the profinite topology. Does it help?

Although this may not be quite justified, the rest of this section is organized according to the classification in terms of hyperbolic, elliptic and parabolic elements.

8.2. Hyperbolic elements

In view of the question above, and to be completely honest, this subsection could very well consist of a big blank space. Yet.... Let us simply mention a problem which seems really hard although it obviously stands on the road to a deeper understanding of the situation:

Question 8.2. Let $\tau \in \Gamma$ be pseudo-Anosov and let $\langle\tau\rangle$ denote the procyclic group it generates inside $\hat{\Gamma}$. Prove or disprove that the centralizer $Z(\tau)$ of τ and the normalizer $N(\langle\tau\rangle)$ of $\langle\tau\rangle$ in $\hat{\Gamma}$ are virtually procyclic.

One reason this question looks important is that it is really step zero in trying to understand the group theoretic structure of the completion $\hat{\Gamma}$, in particular give group theoretic characterizations of certain classes of elements (see also §8.4 below). One reason it looks hard is that there does not seem to exist any 'elementary' proof in the discrete case (where on top of it $Z(\tau)$ has at most index 2 in $N(\langle\tau\rangle)$), that is not using at least a little bit of Thurston's theory, which in fact is used already when focusing on pseudo-Anosov diffeomorphisms.

8.3. Elliptic elements

Fortunately we are slightly better off for what concerns torsion elements in the profinite Teichmüller groups. Ideally we would like to vindicate the obvious guess, namely show that for any $\hat{\Gamma}(S) \simeq \hat{\Gamma}_{g,[n]}$ (with $g \geq 0$, $n \geq 0$, $2g-2+n > 0$ as usual...), every torsion element is conjugate to an element of the discrete group $\Gamma_{g,[n]}$. These elements 'come from geometry', reflecting automorphims of algebraic curves. We can remark that the analog of this expectation is *wrong* for the groups $Sp_{2g}(\mathbb{Z})$ and their completions ($g > 1$), and thus for p.p. abelian varieties (see e.g. [IN] for details). Note also that the groups $Sp_{2g}(\mathbb{Z})$ ($g > 1$) are *not* good.

Let us start with the following conditional statement:

Proposition 8.1. *If* $\hat{\Gamma}_{g,[n]}$ *is good (in particular if Conjecture 4.1 holds true) then any element of* $\hat{\Gamma}_{g,[n]}$ *of* prime power *order comes from geometry, i.e. is conjugate to an element of* $\Gamma_{g,[n]}$.

Thanks to Proposition 7.3 this statement is of course *unconditionally* true for $g = 0, 1, 2$. The proof of the proposition itself will appear as a rather special case of the observations below, which will also explain why we have to restrict to prime power order elements (meaning elements whose order is a power of a prime).

It seems reasonable to ask more generally: When is it that the conjugacy classes of finite order elements of a discrete group G and of its completion $\hat{G}$ coincide? Here we will definitely not be looking for maximal generality; in particular we assume that G is residually finite and sometimes view it as a subgroup of $\hat{G}$. Although this question does not seem have been raised as such in the literature, here are some clues as to the state of the art. We refer once and for all to the book of K. S. Brown ([Brown]) regarding the necessary inputs in group cohomology.

Let G be a group which is discrete and virtually of type FP; recall that the latter means that there is a cofinite (i.e. finite index) subgroup of G which is of type FP (see [Brown], §VIII.5). Such a group clearly has finite *v.c.d.* (virtual cohomological dimension) and is virtually torsionfree. Also the $\hat{\Gamma}_{g,[n]}$'s are easily seen to be of that type e.g. because using abelian level structures one finds a cofinite subgroup acting freely on an aspherical variety. The first question would be: Is the number of conjugacy classes of torsion elements in such a group finite? The answer is unknown in general (but there does not seem to be counterexamples around). It is positive if one restricts to prime power order elements, in which case one can say much more. A relatively elementary reference is again [Brown], Lemma IX.13.2, whose strategy remains the model for more sophisticated versions. The fact that only prime power order elements (more generally finite p-subgroups) can be dealt with is very deeply rooted in the currently available toolbox.

Next if we want to compare G and $\hat{G}$ we clearly have to add some assumption; goodness at least enables us to compare cohomologies. It turns out that less will do. In fact one can show the following:

Theorem 8.2. (P. Symonds, [Sy]) *Let G be a discrete residually finite group which is virtually of type FP; assume that for q large enough the natural map $j : G \hookrightarrow \hat{G}$ induces an isomorphism $H^q(\hat{G}, \mathbb{Z}/p) \xrightarrow{\sim} H^q(G, \mathbb{Z}/p)$ between the cohomology groups with constant coefficients $\mathbb{Z}/p$. Then there are finitely many conjugacy classes of finite p-subgroups in G and $\hat{G}$ and they coincide.*

This result follows from a slight variation on the main result of [Sy]. We also refer to that paper for references on the subject, noting that one has to mobilize quite recent resources such as Lannes' T-functor in order to show this and related results. At any rate this in particular proves Proposition 8.1.

Let us try and go a little further, feeling the 'frontier' of the subject to-date, which will appear to stand perhaps surprisingly close. We retain the goodness assumption although it is conjectural for Teichmüller groups, simply because we do not know of other invariants in order to compare G and its completion. We are also willing to *assume* that the number of conjugacy classes of torsion elements in the *discrete* group G is finite, partly out of necessity, partly because this is plainly the case for Teichmüller groups. Thus start from a group G which is discrete, virtually of type FP, residually finite and good; variations are possible but relatively minor. *Assume* that the number of conjugacy classes of torsion elements is finite. Again all this holds true for Teichmüller groups, except of course for goodness which is conjectural if the genus is > 2.

Let $T(G) \subset G$ (resp. $T(\hat{G}) \subset \hat{G}$) denote the torsion of G (resp. $\hat{G}$). Consider again G as embedded in $\hat{G}$. For $g \in T(G)$, let $C_G(g)$ (resp. $C_{\hat{G}}(g)$) denote its conjugacy class in G (resp. $\hat{G}$). Then $C_G(g)$ is dense in $C_{\hat{G}}(g)$ which is a closed subset of $\hat{G}$. If $\overline{T(G)}$ denotes the closure of $T(G)$ in $\hat{G}$, our assumption on the finite number of conjugacy classes in $T(G)$ implies that in fact $\overline{T(G)} \subset T(\hat{G})$ and we have shown:

Lemma 8.3. *Under the above assumptions, every torsion element of $\hat{G}$ is conjugate to an element of G if and only if $T(G)$ is dense in $T(\hat{G})$.*

Let us then 'localize' the problem. Assume by contradiction that there is a $g \in T(\hat{G})$, a torsion element of order $n > 1$, which is *not* in the closure of $T(G)$. Then there exists $U \subset \hat{G}$ a normal open and torsionfree subgroup of $\hat{G}$ such that $gU \cap \overline{T(G)} = \emptyset$. Let $\mathcal{G} \subset \hat{G}$ be the subgroup generated by g and U: $\mathcal{G} \simeq U \rtimes \mathbb{Z}/n$ is a split extension of $\langle g \rangle \simeq \mathbb{Z}/n$ by U. Let $\mathcal{G}_0 = \mathcal{G} \cap G$ denote its discrete part.

By assumption $\mathcal{G}_0$ has torsion *strictly* dividing n i.e. the corresponding discrete extension is *not* split, whereas $\mathcal{G}$ contains an element of order n, namely g. Moreover $\mathcal{G}_0$ has finite index in G and it satisfies the same assumptions. Finally $\mathcal{G}$ is its profinite completion. If $n = p$ is a prime, $\mathcal{G}_0$ is torsionfree and good whereas its completion is not. This is not possible since then $\mathcal{G}_0$ has finite *c.d.* by a well-known result of Serre ([Brown], §VIII.3), whereas $c.d.(\mathcal{G}) = \infty$. In other words, *assuming* the finiteness of the number of conjugacy classes, we have reproved part of Theorem 8.2 in the *prime* order case. More modern technology enables one to deal with the *prime power* order case and prove the necessary finiteness result in that case.

For elements of general order n and assuming again the necessary finiteness result (which is unknown in general), the reasoning above reduces the problem, after renaming the ingredients, to the following question. Consider an extension:

$$1 \to H \to G \to \mathbb{Z}/n \to 1, \tag{7}$$

where H is discrete, of type FP (in particular torsionfree), residually finite and good. One can form the corresponding profinite extension by simply changing H and G into $\hat{H}$ and $\hat{G}$ respectively; call that extension *pro*$-(7)$. Note that torsion subgroups of G and $\hat{G}$ are cyclic of order dividing n. We arrive at the following

Question 8.3. Assume *pro*$-(7)$ splits; does (7) split as well?

This seems to be unknown, even for $n = pq$ the product of two primes p and q or if H is (finitely generated) free. It is apparently also unknown whether or not the number of conjugacy classes of elements of G of order n is necessarily finite.

8.4. Parabolic elements

We now come to the parabolic elements alias (multi)twists. They probably represent the most manageable type and can in principle serve to understand Teichmüller groups 'by induction', not unlike parabolic subgroups for linear algebraic groups. In the discrete setting, part of this program has been implemented and we will review some of the corresponding statements. As could be expected, the profinite case seems considerably harder. We will start with some statements which appear to-date to be out of reach. But again they look so natural and tantalizing that they deserve perhaps to be mentioned. One can of course skip them or consider that bit as sheer wishful thinking—however, see the epigraph. We keep the same notation as above. Occasionally the reader should assume that $d(S) > 1$ (i.e. not of type $(0,4)$ or $(1,1)$) in order to avoid trivial exceptions.

8.4.1. *A group theoretic characterization of twists?* Twists in the discrete case afford a purely group theoretic characterization. To wit:

Theorem 8.4. *Let $G \subset \Gamma = \Gamma(S)$ be a subgroup contained in some abelian level $\Gamma^{(m)}$, $m > 2$. Then $g \in G$ is a nontrivial power of a Dehn twist if and only if $Z(Z_G(g))$, i.e. the center of the centralizer of g in G, is isomorphic to $\mathbb{Z}$ and does not coincide with the whole centralizer $Z_G(g)$.*

This is Theorem 7.5.B of [Iv4] to which we refer. Considering a subgroup of Γ inside some $\Gamma^{(m)}$, rather than Γ itself, serves only to eliminate 'parasitic' finite groups and is not essential. Everything remains virtually true in Γ itself (see §8.4.3 below).

There are at least two points to be made here. First this characterization relies in an essential way on the (virtual) self-centralizing property of the pseudo-Anosov elements, coupled with the Thurston–Bers classification. Second this result looks strikingly similar to the basic tenet of the so-called 'local correspondence' of birational anabelian geometry, and for good reasons. We refer in particular to [Sz], Théorème 4.3. So for the record we raise the following

Question 8.4. Does anything like Theorem 8.4 hold in the profinite completion $\hat{\Gamma}$ or at least the geometric completion $\check{\Gamma}$ of Γ?

In some sense we already have at our disposal a topological and a birational statement. Can we prove a geometric version? Because of the first point above, this looks at first sight much harder than Question 8.2 above. But who knows? Of course a positive answer would show that *any* automorphism of $\hat{\Gamma}$ is 'inertia preserving', that is globally preserves conjugacy classes of twists (see also below, especially §10).

8.4.2. *A Galois theoretic characterization of twists?* We will be again very brief if not elliptic(!). Twists feature generators of inertia subgroups attached to the components of $\partial\mathfrak{M}$, the boundary (normal crossing) divisor of the stable completion of the moduli stack $\mathfrak{M}$. Now there is a natural outer action of the arithmetic Galois group: $G_{\mathbb{Q}} = Gal(\mathbb{Q}) \to Out(\hat{\Gamma})$. Twists being 'divisorial inertia' elements, they are acted on cyclotomically. Namely if $t \in \Gamma$ is a twist and $\sigma \in G_{\mathbb{Q}}$, one has $\sigma(t) \sim t^{\chi(\sigma)}$ where $\sim$ denotes conjugacy in $\hat{\Gamma}$ and χ is the cyclotomic character.

On the other hand torsion elements correspond to stack inertia in $\hat{\Gamma}$, viewed again as $\pi_1^{geom}(\mathfrak{M})$, the geometric fundamental group of $\mathfrak{M}$ (itself viewed over $\mathbb{Q}$). It may be interesting to study the Galois action on the torsion elements, or more generally the finite subgroups of $\hat{\Gamma}$ and this seems quite hard. One can however legitimately ask whether the cyclotomic action characterizes twists up to torsion (i.e. virtually). Recall that a multitwist is a product of commuting twists, that is a twist along a multicurve. We can thus ask the following

Question 8.5. Let $g \in \hat{\Gamma}$ be such that $\sigma(g) \sim g^{\chi(\sigma)}$ for all $\sigma \in G_Q$. Is it true that there is a power $N > 0$ such that g^N is conjugate to a multitwist?

Variant: Ask the same thing only virtually with respect to $G_{\mathbb{Q}}$ (arithmeticians rather say 'potentially'), that is require only that G_K acts cyclotomically on g, with K a finite extension of $\mathbb{Q}$ (i.e. a numberfield). Again this looks hard but tantalizing. Partial evidence can be produced, which however we skip here.

8.4.3. *Twists and curves (discrete)* Let us now compile a short list of facts in the discrete case, which will become a wishlist in the profinite case, to be discussed below. For the proofs we refer to [Iv2] (especially Chapters 4 and 7; see also [McC]). Note that these do use Thurston's theory in a crucial way. We will make use of a rather extended multiindex notation; in particular we write simply $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k)$ for a multicurve (or just a curve when $k = 0$). To α we associate $\sigma_\alpha \in \mathscr{C} = \mathscr{C}(S)$, a k-simplex of the curve complex. We also attach the multitwist $t_\alpha = \prod_i t_i$ where t_i is the twist along α_i, as well as $G_\alpha \simeq \mathbb{Z}^{k+1}$, the free abelian group spanned by the t_i. We will write $\sigma = \sigma_\alpha$ and G_σ instead of G_α when the meaning is clear; we set $G_\emptyset = (1)$. Note that $G_\alpha \subset \Gamma$ comes equipped with preferred generators, namely the t_i's, once S has been given an orientation. Our first item reads:

i) There is a one-to-one correspondence between a) multicurves, b) simplices of the curve complex and c) finitely generated free abelian subgroups of Γ generated by twists.

Note that the bijection between objects in b) and c) does require proof. All these objects are naturally equipped with an action of Γ and the correspondence is equivariant. Next if $\sigma \in \mathscr{C}$ is a simplex, $\Gamma_\sigma \subset \Gamma$ its stabilizer in Γ, from the equivalence between b) and c) we get that $\Gamma_\sigma = N_\Gamma(G_\sigma)$, where $N_\Gamma(\cdot)$ denotes of course the normalizer in Γ. This identification provides a basic equivalence between certain topological and group theoretic questions involving twists.

Here Γ_σ denotes the *setwise* stabilizer. Let us write $\Gamma_{\vec{\sigma}}$ for the subgroup stabilizing σ *pointwise* and preserving the orientation of the curves representing the vertices. Then $\Gamma_{\vec{\sigma}} \subset \Gamma_\sigma$ is a cofinite normal subgroup and the quotient is a subgroup of $\mathfrak{S}_\sigma(\pm)$, the group of signed permutations on the vertices of σ. Moreover $\Gamma_\sigma \cap \Gamma^{(m)} = \Gamma_{\vec{\sigma}} \cap \Gamma^{(m)}$ for $m > 2$ (cf. §2.3). Our second item is the following (easy) description of $\Gamma_{\vec{\sigma}}$:

ii) There is an exact sequence:

$$1 \to G_\sigma \to \Gamma_{\vec{\sigma}} \to \Gamma(S \setminus \sigma) \to 1,$$

in which $\Gamma(S \setminus \sigma)$ denotes as usual the Teichmüller group of S cut along σ; if that surface is disconnected permutation of the pieces is *not* allowed.

There is a great deal of flexibility in terms of centralizers and normalizers of (multi)twists and this is very nontrivial indeed. Everything can be so to speak understood virtually. More precisely, consider again $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k)$ a multicurve and $\sigma = \sigma_\alpha \in \mathscr{C}$ the associated simplex. Let $w = (w_0, \ldots, w_k)$ be a multiindex with $w_i \in \mathbb{Z} \setminus \{0\}$ and write $G_\sigma^w \subset G_\sigma$ for the cofinite subgroup generated by the $t_i^{w_i}$ $(i = 0, \ldots, k)$. Then for $m > 2$:

iii) $N_{\Gamma^{(m)}}(G_\sigma^w) = N_{\Gamma^{(m)}}(G_\sigma) = \bigcap_i Z_{\Gamma^{(m)}}(t_i)$, where $Z_{\Gamma^{(m)}}(\cdot)$ is the centralizer in $\Gamma^{(m)}$ and t_i the twist along α_i.

As usual, the fact that we have to introduce an abelian level should not obscure the situation. It serves only to avoid a possible finite automorphism group, this time a subgroup of the ordinary permutation group $\mathfrak{S}_\sigma$. In particular, if $k = 0$, i.e. for vertices, this says that if $t \in \Gamma$ is a twist and $w \in \mathbb{Z} \setminus \{0\}$, $N_\Gamma(\langle t^w \rangle) = N_\Gamma(\langle t \rangle) = Z_\Gamma(t)$. In the same vein let $\sigma, \sigma' \in \mathscr{C}$ be two *disjoint* simplices (no common vertex), not necessarily of the same dimension. Then:

iv) Under these assumptions the groups G_σ and $G_{\sigma'}$ have trivial intersection in Γ: $G_\sigma \cap G_{\sigma'} = (1)$.

This leads more generally to a lattice property for the groups G_σ, namely:

v) For all $\sigma, \sigma' \in \mathscr{C}$ we have: $G_\sigma \cap G_{\sigma'} = G_{\sigma \cap \sigma'}$, where $\sigma \cap \sigma' \in \mathscr{C}$ is the simplex spanned by the vertices common to σ and σ' (recall that $G_\emptyset = (1)$).

Concerning iv) and v), see Proposition 3.13 in [BL]; the proof uses Thurston's theory.

One can reasonably expect that all the above properties extend (*mutatis mutandis*) to the profinite case and they would have far reaching consequences. We now come to the—little—that is known to-date in this direction.

8.4.4. *Twists and curves (profinite)* The first and perhaps main stumbling block we come across, when passing into the profinite world, is the correspondence in i) of the last subsection, say between (multi)curves and (multi)twists (henceforth we will often omit the prefix). In other words, since the information contained in $\mathscr{C}(S)$ and $\mathscr{C}_P(S)$ can be retrieved from their 1-skeleton (cf. §9 for some more detail), what we need is to relate the *graph theoretic* and *group theoretic* types of information.

To this end, we first manufacture a group theoretic version $\hat{\mathscr{C}}_G(S)$ of the profinite completion $\mathscr{C}(S)$ which, as the name indicates, conveys the group theoretic information essentially by definition. Let $\mathcal{G} = \mathcal{G}(\hat{\Gamma}(S))$ denote the set of all *closed* subgroups of $\hat{\Gamma}$; then $\mathcal{G}$ has a natural structure of profinite set. Indeed $\mathcal{G} = \varprojlim_{\lambda} \mathcal{G}(\Gamma/\Gamma^\lambda)$, where Γ^λ runs along the cofinite normal subgroups of Γ and $\mathcal{G}(\Gamma/\Gamma^\lambda)$ denotes the finite set of the subgroups of Γ/Γ^λ. Moreover $\mathcal{G}$ comes equipped with a natural action of $\hat{\Gamma}$ by conjugation. Now define a map $r : \mathscr{C} \to \mathcal{G}$ by assigning to a simplex $\sigma \in \mathscr{C}$ the closed subgroup $\hat{G}_\sigma \simeq G_\sigma \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \simeq \hat{\mathbb{Z}}^{dim(\sigma)+1}$. So $r(\sigma) = \hat{G}_\sigma$— or more accurately the element of $\mathcal{G}$ representing that group. The map r is injective, say by v) in §8.4.3 above, and we often identify $\mathscr{C}$ with its image $r(\mathscr{C})$. Now we define $\hat{\mathscr{C}}_G(S) \subset \mathcal{G}$ as the closure of $\mathscr{C}$ in $\mathcal{G}$. Here we take the dimensionwise closure of $\mathscr{C}$ and obtain a simplicial profinite set, i.e. an element of $\hat{\mathcal{S}}$, with the face and degeneracy operators uniquely extending those of $\mathscr{C}$. The complex $\hat{\mathscr{C}}_G(S)$, which we call the *group theoretic completion* of the curve complex $\mathscr{C}(S)$, comes with a natural $\hat{\Gamma}$-action, namely the restriction of the conjugacy action on $\mathcal{G}$. In the discrete setting one can similarly consider the set $\mathcal{G}^{disc}$ of *all* subgroups of Γ and construct a complex $\mathscr{C}_G(S)$ in the same way. That the natural map $\mathscr{C}(S) \to \mathscr{C}_G(S)$ is an isomorphism comes down to a rephrasing of i) in §8.4.3.

Now recall the discussion before Proposition 3.3 and let us detail what is happening in the case at hand. Letting $\mathscr{C}_k$ denotes the k-dimensional simplices of $\mathscr{C}$, these fall into a finite number of Γ-orbits. Writing F_k for a finite set of representatives, we can break $\mathscr{C}_k$ as a disjoint union: $\mathscr{C}_k = \coprod_{\sigma \in F_k} \Gamma \cdot \sigma$. We will write $F = \coprod_{k \geq -1} F_k$ for the finite union of the F_k's; including $k = -1$ corresponding to the empty simplex comes handy. In topological terms F enumerates the *topological types* of multicurves (including the empty one), whereas in modular terms it enumerates the strata of the stable stratification of $\overline{\mathfrak{M}}$, with F_{-1} corresponding to the generic stratum $\mathfrak{M}$ of smooth curves. We can still decompose $\hat{\mathscr{C}}_k$ $(k \geq 0)$ as $\hat{\mathscr{C}}_k = \bigcup_{\sigma \in F_k} \hat{\Gamma} \cdot \sigma$ but this time we do not know *a priori* whether or not this is a *disjoint* sum. In other words we do not know whether or not it can happen that two (discrete) multicurves of distinct topological types actually lie in the same $\hat{\Gamma}$-orbit. The same holds true for the group theoretic completion $\hat{\mathscr{C}}_G$ with σ replaced by $\hat{G}_\sigma$. In particular the group representing a simplex of $\hat{\mathscr{C}}_G$ is a conjugate in $\hat{\Gamma}$ of $\hat{G}_\sigma$ for some $\sigma \in F$ ($=$ (1) if $k=-1$, $\sigma=\emptyset$), but again it could *a priori* happen that G_σ and $G_{\sigma'}$ which are *not* conjugate in Γ (i.e. σ and σ' in $\mathscr{C}$ are of distinct topological types) give rise to $\hat{G}_\sigma$ and $\hat{G}_{\sigma'}$ which *are* conjugate in $\hat{\Gamma}$. In other words, in order to extend the notion of topological type to the profinite case we should answer at least part of the following

Question 8.6. Are the profinite Teichmüller groups conjugacy separable?

Is it true that two non conjugate elements in $\Gamma(S)$ remain so in $\hat{\Gamma}(S)$? Note that this is indeed the case in (modular) dimension 1 because then $\Gamma(S)$ is virtually free. Here we need only separate conjugacy classes of twists and moreover we could perhaps do it already in the geometric completion $\check{\Gamma}(S)$, which is a *stronger* statement. Finally, one way to do it is to determine the centralizer of a twist in the completion and show that if two twists are not conjugate, their centralizers are not isomorphic. All this very much ties up with our current theme, namely again the connection between graph and group theoretic information.

Returning to the group theoretic complex $\hat{\mathscr{C}}_G$, one of its main virtues consists in the fact that the stabilizer of any simplex σ for the $\hat{\Gamma}$-action coincides with $N_{\hat{\Gamma}}(\hat{G}_\sigma)$, the normalizer of the attached group in $\hat{\Gamma}$. Here we run into a slight notational problem, which does reflect the situation. The group attached to a discrete simplex $\sigma \in \mathscr{C}$ is indeed the profinite completion $\hat{G}_\sigma$ of the group G_σ attached to that simplex in $\mathscr{C}_G \simeq \mathscr{C}$, the discrete version of $\hat{\mathscr{C}}_G$. For a general $\sigma \in \hat{\mathscr{C}}_G$, this is of course meaningless. So we rather boldly (from a notational viewpoint) but hopefully unambiguously switch to the profinite viewpoint. From now on we write simply $G_\sigma \in \mathcal{G}$ for the group attached to *any* $\sigma \in \hat{\mathscr{C}}_G$ and confuse groups and elements of $\mathcal{G}$ in the notation. For $\sigma \in \mathscr{C}_G \simeq \mathscr{C}$ we write $G_\sigma^{disc} \simeq \mathbb{Z}^{dim(\sigma)+1}$ for the discrete group (previously G_σ), and G_σ for its completion (previously $\hat{G}_\sigma$).

On the graph theoretic side now, Proposition 3.4 enables us to describe the stabilizer $\check{\Gamma}_\sigma$ of a discrete simplex σ of the *geometric* completion $\check{\Gamma}(S)$. Indeed one defines the open subgroup $\check{\Gamma}_{\vec{\sigma}}$ of $\check{\Gamma}_\sigma$ and one gets, in complete analogy with the discrete case (see §8.4.3, ii)) a short exact sequence:

$$1 \to G_\sigma \to \check{\Gamma}_{\vec{\sigma}} \to \check{\Gamma}(S \setminus \sigma) \to 1. \tag{8}$$

Recall that in our present notation G_σ is the group attached to σ and for a discrete simplex, this is the full profinite completion of the group attached in the discrete case. The fact that this group *injects* into the geometric stabilizer results from monodromy computations for the congruence levels (see e.g. [B1], Theorem 2.1). However we cannot *a priori* identify the stabilizer for the completed complex with a group theoretic ingredient, whereas this is immediately the case if we use a group theoretic version of the complexes. Indeed, let for simplicity $\sigma = v \in \mathscr{C}$ be a vertex of the *discrete* complex and write $\hat{\Gamma}_v$ for the stabilizer. Then, by Proposition 3.3, $\hat{\Gamma}_v = \overline{Z_\Gamma(t_v)} \subset Z_{\hat{\Gamma}}(t_v)$, the closure of the centralizer of the attached twist. We do get an inclusion, but not an identification. By contrast, the stabilizer of the vertex $v \in \mathscr{C}$ considered in $\hat{\mathscr{C}}_G$ is indeed the normalizer $N_{\hat{\Gamma}}(\hat{G}_v)$, where G_v is the procyclic group generated by the twist t_v.

The above makes it clear that comparing $\hat{\mathscr{C}}$ and $\hat{\mathscr{C}}_G$ is a natural task. Let us first make use of the universal property of the $\hat{\Gamma}$-completion $\hat{\mathscr{C}}$. It tells us that since $\hat{\mathscr{C}}_G$ is a profinite complex with $\hat{\Gamma}$-action, the map r factors through $\hat{\mathscr{C}}$. In other words there exists a natural map, still denoted r for simplicity, $r : \hat{\mathscr{C}}(S) \twoheadrightarrow \hat{\mathscr{C}}_G(S)$, which is surjective for the usual reason that its image is both dense and closed. Our problem about connecting graph theoretic and group theoretic pieces of information can now be formulated precisely as the following

Conjecture 8.5. The natural comparison map $r : \hat{\mathscr{C}}(S) \twoheadrightarrow \hat{\mathscr{C}}_G(S)$ provides an isomorphism between the profinite curve complex and its group theoretic version.

This statement can be regarded as our second serious conjecture after Conjecture 4.1. It looks difficult as it stands but perhaps not so much *modulo* the congruence conjecture (or *a fortiori* Conjecture 4.1). Before listing some of its consequences, let us see where we stand with respect to a proof.

A first and useful observation is that the group theoretic version $\hat{\mathscr{C}}_G(S)$ is a *flag* complex almost by definition, that is a simplex is determined by its faces, indeed (by induction) by its vertices. If $\sigma \in \hat{\mathscr{C}}_G$, $G_\sigma \subset \hat{\Gamma}$ is nothing but the group generated by the procyclic subgroups attached to the vertices of σ. By contrast, it is by no means *a priori* clear that there cannot be—say—two edges joining the same vertices in $\hat{\mathscr{C}}$. Next we record the following easy properties of the map r:

Proposition 8.6. *The natural epimorphism* $r : \hat{\mathscr{C}}(S) \twoheadrightarrow \hat{\mathscr{C}}_G(S)$ *satisfies:*
i) It is an isomorphism if $d(S) = 1$*;*

ii) For any S with $d(S) > 1$, r is an isomorphism if (and only if) it induces an isomorphism between the 1-skeleta.

Sketch of proof. To i): If $d(S) = 1$, $\mathscr{C}(S) \simeq \mathscr{C}(S_{1,1}) \simeq \mathscr{C}(S_{0,4})$ and the assertion reduces to a known fact about modular curves (see [B1], below Theorem 7.8). To ii): $\mathscr{C}(S)$ can be explicitly recovered from its 1-skeleton (see below, beginning of §9.1) and the recipe extends verbatim to the profinite versions. Q.E.D.

So we would like to perform an induction on the modular dimension $d = d(S)$ and we can restrict to (pro)graphs, although this reduction is not so useful. Let us first deal with the question *locally* in the sense that we will compare the structures of $\hat{\mathscr{C}}$ and $\hat{\mathscr{C}}_G$ near a given vertex, which as usual we may assume to be in the discrete complex. In other words, let $v \in \mathscr{C}(S)$ be a vertex and let $S(v)$ denote the simplicial closure of $St(v)$, the star of v in $\mathscr{C}(S)$. So $S(v)$ is the union of the closed top dimensional (non degenerate) simplices of $\mathscr{C}(S)$ containing v among their vertices; it is also the union of the star and the link of v in $\mathscr{C}(S)$. We write $\hat{S}(v)$ (resp. $\hat{S}_g(v)$) for the closure of $S(v)$ in $\hat{\mathscr{C}}(S)$ (resp. $\hat{\mathscr{C}}_G(S)$); one can also regard $\hat{S}(v)$ (resp. $\hat{S}_g(v)$) as the simplicial closure of the star of v in $\hat{\mathscr{C}}$ (resp. $\hat{\mathscr{C}}_G$). We will say that r is a *local isomorphism* (or is 'étale') if the induced epimorphism $\hat{S}(v) \twoheadrightarrow \hat{S}_g(v)$ is an isomorphism for every vertex v of $\mathscr{C}(S)$, hence also for every vertex of $\hat{\mathscr{C}}(S)$ by $\hat{\Gamma}$-equivariance. At this point, let us show:

Proposition 8.7. *Assume the validity of the congruence conjecture and that $r : \hat{\mathscr{C}}(S) \twoheadrightarrow \hat{\mathscr{C}}_G(S)$ is an isomorphism for $d(S) < d$ ($d > 1$ an integer). Then it is a* local *isomorphism for $d(S) = d$.*

Proof. Fix v in $\mathscr{C}(S)$ ($d(S) = d$) corresponding to a curve γ on S. For definiteness we may assume that γ is nonseparating. The separating case is completely analogous, or even identical if one coherently extends the notions pertaining to complexes to disconnected surfaces (as in [BL], §2). Of course if $g(S) \leq 2$, we *know* that the congruence property holds true but in general we simply assume its validity.

In the discrete setting it is easy to describe $S(v)$; it is naturally isomorphic to $K(\mathscr{C}(S \setminus \gamma))$, the cone (with vertex γ) over the curve complex of $S \setminus \gamma$, the surface S slit along γ, which has dimension $d(S) - 1$. More precisely it is of type $(g-1, n+2)$ if S is of type (g, n).

The exact sequence (8) above tells us, assuming the congruence subgroup property holds true, that $\hat{\Gamma}_v / \langle t_\gamma \rangle$ is isomorphic to $\hat{\Gamma}(S \setminus \gamma)$ up to a possible $\mathbb{Z}/2$ symmetry group which plays no role here. This implies that $\hat{S}(v) \simeq K(\hat{\mathscr{C}}(S \setminus \gamma))$, $\hat{S}_g(v) \simeq K(\hat{\mathscr{C}}_G(S \setminus \gamma))$ and thus the desired isomorphism $\hat{S}(v) \simeq \hat{S}_g(v)$ since $\hat{\mathscr{C}}(S \setminus \gamma) \simeq \hat{\mathscr{C}}_G(S \setminus \gamma)$ by the inductive assumption. Q.E.D.

Before moving on we remark that it is not obvious to work with the geometric completion $\check{\mathscr{C}}(S)$ all along, without assuming the congruence conjecture, because there is no guarantee that the map r factors through that completion.

So there would remain to show that r is actually of degree 1, i.e. it is one-to-one on vertices, a property which can of course be detected on the 1-skeleta. We unfortunately do not know how to prove it at the time of this writing. From the arguments developed in [B1], §7 one should be able to derive other properties of r, still modulo the congruence conjecture (or unconditionally for $g(S) \leq 2$), namely that the stabilizers of $\sigma \in \hat{\mathscr{C}}(S)$ and $r(\sigma)$ for the $\hat{\Gamma}$-action coincide, that if σ and σ' are distinct in the same $\hat{\Gamma}$-orbit their images $r(\sigma)$ and $r(\sigma')$ are distinct and, as a consequence, that r has finite degree. In other words, modulo the congruence conjecture, one should be able to establish that r is finite étale (i.e. here simply unramified).

As a sample of the consequences of the validity of Conjecture 8.5 we mention:

Proposition 8.8. *Assume the validity of Conjecture 8.5 and let $v \in \mathscr{C}(S)$ be a vertex with associated twist $t_v \in \Gamma$. Let $\hat{\Gamma}_v$ denote the stabilizer of v for the action of $\hat{\Gamma}$ on $\hat{\mathscr{C}}(S)$. Then:*

$$\hat{\Gamma}_v \simeq \overline{Z_\Gamma(t_v)} \simeq Z_{\hat{\Gamma}}(t_v) \simeq N_{\hat{\Gamma}}(\langle t_v \rangle).$$

Proof. We have already mentioned the first equality, stemming from Proposition 3.3. The notation $\hat{\Gamma}_v$ could appear ambiguous as it stands. Disambiguation would result from answering Question 3.1 positively, since it says that $\hat{\Gamma}_v$ should indeed also be the full completion of Γ_v, the stabilizer of v under the Γ-action on $\mathscr{C}(S)$. Under the assumed isomorphism, $\hat{\Gamma}_v$ is also the stabilizer of v in $\hat{\mathscr{C}}_G(S)$, namely $N_{\hat{\Gamma}}(\langle t_v \rangle)$, and $Z_{\hat{\Gamma}}(t_v)$ is squeezed between that group and the closure of the discrete stabilizer. Q.E.D.

One can draw many more consequences from Conjecture 8.5. In [BL], §3 the reader will find a—as yet partly conjectural—weighted version of $\hat{\mathscr{C}}_G$ which should enable one to adapt to the profinite case 'virtual' statements much as the *first* equality of that item iii) in §8.4.3 above. More precisely, let G_σ be as above and let $U \subset G_\sigma$ be an open subgroup. The coincidence of the normalizers ($N_{\hat{\Gamma}}(U) \simeq N_{\hat{\Gamma}}(G_\sigma)$) is expected and would represent an important and useful piece of information. Because they use (bits of) Thurston's theory, the second equality in that same item iii), as well as items iv) and v) may prove even more difficult to transfer to the profinite setting. At any rate we see that combining the (as yet hypothetical) isomorphism of the graph and group theoretic profinite curve complexes with some form of the congruence subgroup property would yield a fairly complete description of the centralizers and normalizers of (multi)twists or the free abelian groups they generate as well as of their open subgroups.

In closing we mention that it would be very desirable to make the detailed connection between the relatively classical language used here and the theory developed in [HM2], especially its §5; see in particular Theorem 14 there.

§9. Automorphisms of profinite complexes

In these two final sections we make the junction with Grothendieck–Teichmüller theory, which was one of the initial motivations of the project. Again we find the emerging landscape rather compelling but we will have to underline remaining gaps which essentially stem from the ones we have already encountered. This and the next section have been divided into rather short subsections in order to emphasize the main ideas. They are detailed in [BL] and [L1]; caution should be exerted however when perusing these texts which contain 'optimistic' (although probably true and often enlightening) statements.

There are two main clusters of facts to be mentioned from the start. First and pursuing the analogy with arithmetic groups, there exists a close connection between, on the one hand the automorphisms of curve complexes which, as mentioned above, feature the analogs of buildings in the linear situation, and on the other hand the automorphisms of the attending groups. This point will be shortly reviewed for the discrete case at the beginning of the next section, and will then be partly extended to the profinite case. The second main fact is the *contrast* between the discrete and the profinite case. To put it very simply, the discrete complexes and groups are essentially rigid, whereas a huge deformation group, namely the Grothendieck–Teichmüller group (arguably in its 'richest' incarnation) appears in the profinite case. It features the main theme of these sections. Before plunging into particulars, let us add a remark which simply reinforces the first point above. Although we have kept a full section about *group* automorphisms, it will be plain that most of the work goes into understanding the geometry of the complexes, and indeed retrieve the information contained in their 1-skeleta.

So in the sequel, profinite *graph* theoretic considerations are in some sense overshadowing *group* theoretic ones.

9.1. Automorphisms of discrete complexes

We will review the discrete situation in a way which is not only taylored to our needs but actually adds to the existing presentation. We refer essentially to [Iv3], [McC] and [M] for the latter; §2 of [BL] contains a more detailed version of the incomplete summary below.

In these two sections §9 and §10 we will be interested in the discrete and profinite curve and pants complexes. For the time being we consider again $\mathscr{C}_P(S)$ as a *two* dimensional complex, but will eventually return to the pants *graph* (Cf. §2.1). Arc complexes will hardly appear, and for good reasons. Finally recall that for $k \geq 0$, $\mathscr{C}(S)^{(k)}$ denotes the k-skeleton of $\mathscr{C}(S)$ and idem for $\mathscr{C}_P(S)$. The automorphisms of the curve complex are described as follows:

Theorem 9.1. *Let S be hyperbolic of type (g,n) with $d(S) > 1$ and $(g,n) \neq (1,2)$. Then there is an exact sequence:*

$$1 \to Inn(\Gamma(S)) \to Aut(\mathscr{C}(S)) \to \mathbb{Z}/2 \to 1.$$

Moreover $Aut(\mathscr{C}(S)^{(1)}) = Aut(\mathscr{C}(S))$.

For proofs, comments and attributions, see the references quoted above. The exception for type $(1,2)$ is not serious and is well-understood (see e.g. [BL] §2). In particular $\mathscr{C}(S_{1,2}) \simeq \mathscr{C}(S_{0,5})$ so it is not even a new object. The fact that the curve *graph* determines the automorphisms is important and easy. More generally, as mentioned already, the whole of $\mathscr{C}(S)$ can be reconstructed from its 1-skeleton. A moment contemplation will convince the reader of the validity of the following concrete recipe: The k-dimensional (nondegenerate) simplices of $\mathscr{C}(S)$ are in one-to-one correspondence with the *complete* subgraphs of $\mathscr{C}(S)^{(1)}$ with $k+1$ vertices.

The exact sequence in the statement embodies the *rigidity* of the curve complex. As orientation preserving automorphisms it admits only those coming from the natural action of $\Gamma(S)$. The center acts trivially but under the assumptions of the theorem it is itself trivial (i.e. $Inn(\Gamma(S)) \simeq \Gamma(S)$) except for type $(2,0)$ where it has order 2. The sequence is split by the existence of a reflection, that is an orientation reversing involution of S. Although at this point it looks far fetched, one can think of $\mathbb{Z}/2$ as $Gal(\mathbb{C}/\mathbb{R})$; in the profinite case, this factor will be replaced by the huge Grothendieck–Teichmüller group, containing a copy of $Gal(\mathbb{Q})$, the absolute Galois group of $\mathbb{Q}$, whose only nontrivial 'geometric' element is precisely complex conjugacy.

Coming to the pants complex the situation looks similar at first sight. Indeed we have the exact analog of the above result, namely:

Theorem 9.2. *Let S be hyperbolic of type (g,n) with $d(S) > 1$. Then there is an exact sequence:*

$$1 \to Inn(\Gamma(S)) \to Aut(\mathscr{C}_P(S)) \to \mathbb{Z}/2 \to 1.$$

Moreover $Aut(\mathscr{C}_P(S)^{(1)}) = Aut(\mathscr{C}_P(S))$.

It turns out that type $(1,2)$ is not exceptional here but this is a detail. Much more important is the fact that the similarity between Theorems 9.1 and 9.2 is largely *deceptive.* For one it will break in the profinite case and this is one of the seeds of the very existence of the Grothendieck–Teichmüller group, viewed as a *deformation* group in the full *profinite* setting (whereas deformation theory is usually developed in a *prounipotent* context). In the present discrete setting, retrieving $\mathscr{C}_P(S)$ from its 1-skeleton is much harder than for $\mathscr{C}(S)$ and it is the main result of [M]. It may also be interesting to remark that one can read both theorems above so to speak backward, and conclude that Teichmüller groups are nothing but the orientation preserving automorphism groups of the pants or curve graphs.

In [BL] (§2), we developed an alternative approach (in the discrete case) for comparing the pants graph $\mathscr{C}_P(S)^{(1)}$, the full pants complex $\mathscr{C}_P(S)$, the curve graph $\mathscr{C}(S)^{(1)}$ and thereby also $\mathscr{C}(S)$. Apart from reproving the results of [M], one gets a better grasp on the situation, and one which should largely extend to the profinite setting. Here we give a very brief and terse summary for further use and reference.

We first introduce a graph $\mathscr{C}_*$ which does not seem to have attracted much attention but is well adapted to our needs, including in the profinite case. Given S as usual, the vertices of $\mathscr{C}_*(S)$ are the pants decomposition of S. So $\mathscr{C}_*$ and $\mathscr{C}_P$ share the same set of vertices. Now two vertices of $\mathscr{C}_*$ are joined by an edge if (and only if) the corresponding decompositions differ by just one curve, i.e. they have $d(S) - 1$ curves in common. So with respect to $\mathscr{C}_P$ we simply drop the condition of minimal intersection. In particular the pants *graph* $\mathscr{C}_P$ is a subgraph of $\mathscr{C}_*$. It is interesting to spell out the conventions in the low dimensional cases. We set $\mathscr{C}_*(S_{0,3}) = \mathscr{C}_P(S_{0,3}) = \{*\}$, the one-point set, not the empty set. If $d(S) = 1$, $\mathscr{C}_P(S) = F$, the familiar Farey graph, whereas $\mathscr{C}_*(S) = G$ is the complete graph on the vertices of F. If $d(S) > 1$, $\mathscr{C}_*(S)$ is nothing but the 1-skeleton of the dual of $\mathscr{C}(S)$. Another easy point is that one should extend all these notions to *non connected* surfaces, where however each connected component is hyperbolic. This is essentially routine but quite useful in practice (see [BL]).

Let us now describe the basic results; proofs are essentially combinatorial in this discrete setting but they seem particularly apt to be transfered to the profinite setting. First $\mathscr{C}_*$ is obtained from $\mathscr{C}_P$ by replacing every *maximal* copy of F by a copy of G. Mark that this operation is irreversible; $\mathscr{C}_P$ carries strictly more information than $\mathscr{C}_*$, namely it specifies the *minimal intersection rule*, something which becomes crucial in the profinite setting. This description also ensures that an automorphism of $\mathscr{C}_P$ determines a unique automorphism of $\mathscr{C}_*$: $Aut(\mathscr{C}_P(S)) \subset Aut(\mathscr{C}_*(S))$. The next result reads:

Theorem 9.3. *([BL], §2) The full curve complex $\mathscr{C}(S)$ can be graph theoretically reconstructed from the graph $\mathscr{C}_*(S)$. In particular $Aut(\mathscr{C}_*(S)) = Aut(\mathscr{C}(S))$.*

Before making a remark on the proof, let us draw an easy consequence. Since $Aut(\mathscr{C}_P(S)) \subset Aut(\mathscr{C}_*(S))$, the result implies that in fact $Aut(\mathscr{C}_P(S)) \subset Aut(\mathscr{C}(S))$. So Theorem 9.2 becomes an immediate corollary of Theorem 9.1 and the pants graph and complex do indeed carry the same information. As for the proof, Theorem 9.3 follows from an explicit reconstruction statement which is significant by itself. Namely we have:

Theorem 9.4. *([BL], Theorem 2.10) Let $C \subset \mathscr{C}_*(S)$ be a subgraph which is (abstractly) isomorphic to $\mathscr{C}_*(\Sigma)$ for a certain surface Σ and which is maximal with this property. Then there exists a unique $\sigma \in \mathscr{C}(S)$ such that $C = \mathscr{C}_*(S \setminus \sigma)$.*

We naturally refer to [BL] for the proof but also for some remarks about the statement. In order to reconstruct $\mathscr{C}(S)$, one then builds a complex from $\mathscr{C}_*$ by considering subgraphs as in the statement of the theorem and with inclusion maps as face operators. The result ensures that this complex is isomorphic to $\mathscr{C}(S)$, which is thus encoded in the graph $\mathscr{C}_*(S)$.

9.2. Rigidity of the profinite pants complex

Coming to the problem of determining the automorphism groups of $\hat{\mathscr{C}}(S)$ and $\hat{\mathscr{C}}_P(S)$, we first have to topologize them. Working with $\hat{\mathscr{C}}(S)$ for definiteness, it is defined as an inverse limit of finite quotient complexes $\mathscr{C}^\lambda(S)$ $(\lambda \in \Lambda)$. A continuous automorphism, which is also open, coherently defines for every $\lambda \in \Lambda$ a finite simplicial map $\mathscr{C}^\lambda \to \mathscr{C}^\mu$ for some $\mu \in \Lambda$ $(\lambda \geq \mu)$. When varying $\lambda \in \Lambda$, a basis of neighborhoods of the identity in $Aut(\hat{\mathscr{C}}(S))$ is given by those automorphisms which induce the natural projection; note that these neighborhoods are not subgroups.

In the sequel, one can use the full complexes or their 1-skeleta (graphs) indifferently and we do not record the choice in the notation. That the profinite pants complex (or graph) is rigid means that the literal analog of Theorem 9.2 holds true:

Theorem 9.5. *Let S be hyperbolic of type (g,n) with $d(S)>1$ and $(g,n)\neq(1,2)$. There is an exact sequence:*

$$1 \to Inn(\hat{\Gamma}(S)) \to Aut(\hat{\mathscr{C}}_P(S)) \to \mathbb{Z}/2 \to 1.$$

This is Theorem 4.15 in [BL], whose proof there is essentially self-contained. We have excluded type $(1,2)$ here because the notion of topological type in the profinite case is still not well-established (see Question 8.6) even in the presence of the congruence property which does hold true in genus 1. Apart from that, the result says that any orientation preserving automorphism of $\hat{\mathscr{C}}_P(S)$ comes from the $\hat{\Gamma}$-action, where one first has to show that the notion of orientation is meaningful for $\hat{\mathscr{C}}_P(S)$ (see below); this is not the case of $\hat{\mathscr{C}}(S)$.

In this short subsection we will content ourselves to present the seed of the rigidity of the profinite pants graph by describing its geometric (modular) interpretation which is important in itself. So let $\overline{\mathfrak{M}}^\lambda$ denote again the Deligne–Mumford completion of the level structure of level $\lambda\in\Lambda$ and let $\partial\mathfrak{M}^\lambda=\overline{\mathfrak{M}}^\lambda\setminus\mathfrak{M}^\lambda$ be the divisor at infinity. We now define a curve, or rather a one dimensional D-M stack (resp. complex orbicurve) $\mathcal{F}^\lambda$ sitting inside the boundary $\partial\mathfrak{M}^\lambda$ as in:

Definition 9.6. Let S be hyperbolic with $d(S)>1$; $\mathcal{F}(S)\subset\overline{\mathfrak{M}}(S)$ is the one dimensional D-M stack whose (closed complex) points represent curves (Riemann surfaces) with at least $d(S)-1$ singularities (nodes). For an arbitrary level $\lambda\in\Lambda$, we let $\mathcal{F}^\lambda(S)$ denote the preimage of $\mathcal{F}(S)$ via the canonical projection $\overline{\mathfrak{M}}^\lambda(S)\to\overline{\mathfrak{M}}(S)$.

In other words $\mathcal{F}$ is the closure of the one dimensional stratum in the stable stratification of $\overline{\mathfrak{M}}$. A complex point of $\mathcal{F}$ parametrizes an algebraic curve which is a stable graph of copies of $\mathbb{P}^1\setminus\{0,1,\infty\}$, save for an irreducible component of type $(0,4)$ or $(1,1)$. Algebraically, the $\mathcal{F}^\lambda$'s are stable stack curves, i.e. they have nodal singularities and finite automorphism groups. The letter $\mathcal{F}$ stands for Farey or Fulton, as this curve is connected to a conjecture of W. Fulton (see [G] and its references).

Each component of $\mathcal{F}$ is a moduli space of dimension 1 and can be naturally triangulated into two triangles, so that by lifting that triangulation to the corresponding Teichmüller space one gets the Farey tessellation F. That triangulation of $\mathcal{F}$ lifts uniquely to $\mathcal{F}^\lambda$ for any level $\lambda \in \Lambda$. Moreover, and this is where the connection between $\mathcal{F}(S)$ and $\mathscr{C}_P(S)$ comes in, the dual of that triangulation is naturally isomorphic to the graph $\mathscr{C}_P^\lambda(S)$, where as usual we identify the Farey graph and the corresponding tessellation. This first makes it possible to define an orientation on $\hat{\mathscr{C}}_P(S)$. Indeed the complex curves $\mathcal{F}^\lambda$ are oriented, and this defines an orientation on $\mathscr{C}_P^\lambda(S)$; the natural projections $\mathcal{F}^\lambda \to \mathcal{F}^\mu$ for $\lambda \geq \mu$ are complex maps and thus preserve the orientation. So in turn $\hat{\mathscr{C}}_P(S)$ inherits a natural orientation. Finally the rigidity of the profinite pants graph comes from the fact that the above shows that a morphism $\mathscr{C}_P^\lambda(S) \to \mathscr{C}_P^\mu(S)$ uniquely determines an analytic, and in fact algebraic morphism: $\mathcal{F}^\lambda \to \mathcal{F}^\mu$ between the corresponding curves (for more, see [BL]).

9.3. Galois action and weak anabelianity

Given S, $\Gamma = \Gamma(S)$ and $\mathfrak{M} = \mathfrak{M}(S)$ as usual, there is a canonical outer action $G_{\mathbb{Q}} \to Out(\hat{\Gamma})$, stemming from the fact that $\hat{\Gamma} = \pi_1^{geom}(\mathfrak{M}) = \pi_1(\mathfrak{M} \otimes \bar{\mathbb{Q}})$. Here we write $G_{\mathbb{Q}}$ (or $Gal(\mathbb{Q})$) for the Galois group of $\mathbb{Q}$ and we regard $\mathfrak{M}$ as a $\mathbb{Q}$-stack. One can also get a *bona fide* action by picking a rational basepoint, possibly tangential at infinity, on $\mathfrak{M}$. The action is faithful but in fact the *outer* action is faithful already (the case of S compact was recently settled in [HM1]).

One of the only known general facts about this arithmetic action (for general schemes or even stacks) is that it preserves divisorial inertia (Grothendieck–Murre). Here this translates into the fact that it permutes the conjugacy classes of the procyclic groups associated to twists. Assume Conjecture 8.5; by the above we get an (outer) action of $G_{\mathbb{Q}}$ on $\hat{\mathscr{C}}_G(S)$, hence on $\hat{\mathscr{C}}(S)$. Here we have really cut short and the reader may want to go to [BL] §4 and especially [L1] in order to get a more detailed picture. Much more can be said, including on the faithfulness of this action, but we do not want to unravel the rather complex net of—most likely spurious—assumptions and implications which precise statements would require to-date. For instance, assuming Conjecture 8.5 *and* that $\hat{\Gamma}$ is centerfree, one gets, after picking a rational basepoint, a faithful action:

$$G_{\mathbb{Q}} \hookrightarrow Aut(\hat{\mathscr{C}}(S)).$$

This natural arithmetic galois action on the profinite curve complexes represents a vast generalization of the action on 'dessins d'enfants'; indeed these correspond to the case of $\mathfrak{M}_{0,4}$. Informally speaking, $G_{\mathbb{Q}}$ acts on the tower of all the finite étale covers of $\mathfrak{M} \otimes \bar{\mathbb{Q}}$ and $\hat{\mathscr{C}}$ retains some kind of homotopical information at infinity from this tower of geometric covers, on which the Galois action can then be recorded.

We will not repeat here the little bit of anabelian philosophy which is necessary to put the statement below in its proper context. That input can be found in [BL] (see the Introduction there and §5) along with some of the classical references. We refer particularly to [IN] for a relevant discussion. Stripped to a bare minimum, here is however a (residue of a) main idea. If X/k is a scheme (geometrically connected of finite type) over a field k, let again $\pi_1^{geom}(X) = \pi_1(X \otimes \bar{k})$ ($\bar{k}$ a separable closure of k) denote its geometric fundamental group. It is acted on by $G_k = Gal(k)$ and one can define $Out_{Gal(k)}(\pi_1^{geom}(X))$, the group of the Galois invariant outer automorphisms of $\pi_1^{geom}(X)$. On the other hand, one has the group $Aut_k(X)$ of the k-automorphisms of X, giving rise to a map:

$$Aut_k(X) \to Out_{Gal(k)}(\pi_1^{geom}(X)),$$

which is 'often' injective e.g. for reasons of hyperbolicity and *is* indeed injective in the cases we are interested in. If X is to be deemed 'anabelian', it should be somehow determined by the profinite group $\pi_1^{geom}(X)$ equipped with its G_k-outer action. In particular this would mean that the map above should be an isomorphism. So a *necessary* condition for anabelianity reads (see [IN] for more):

$$Aut_k(X) \stackrel{?}{=} Out_{Gal(k)}(\pi_1^{geom}(X)).$$

Returning to the moduli spaces $\mathfrak{M}_{g,n}$, these are $K(\pi,1)$'s viewed as complex orbifolds and perhaps also algebraically $K(\pi,1)$'s (see §7 above). They are indeed good candidates for being 'anabelian'; in fact the $\mathfrak{M}_{0,n}$ surely deserve to be called anabelian, being globally iterated fibrations ('Artin good neighborhoods'). Royden's theorem tells us that they have no nontrivial automorphisms: $Aut_{\mathbb{C}}(\mathfrak{M}) = \{1\}$. As a result, and since $\pi_1^{geom}(\mathfrak{M}) = \hat{\Gamma}$, the criterion above for being anabelian becomes:

$$Out_{Gal(\mathbb{Q})}(\hat{\Gamma}) \stackrel{?}{=} (1).$$

This we can prove modulo strong extra assumptions. We will state the result for the sake of clarity and because it exemplifies how the Galois action already fits into the picture. Let us not try to economize on assumptions because they are probably all spurious, although seriously new ideas are called for in order to remove them. So let us assume Conjecture 8.5 and the congruence subgroup conjecture (or Conjecture 4.1). Moreover, we have to restrict from the start to *inertia preserving* (outer) automorphisms, i.e. those which preserve (globally) the conjugacy classes of procyclic groups associated to twists. As usual we use a star to denote this inertia preserving condition. We can state:

Proposition 9.7. *Let S be hyperbolic, assume Conjecture 8.5 and the congruence conjecture; then* $Out^*_{Gal(\mathbb{Q})}(\hat{\Gamma}(S)) = (1)$.

For the proof, details and more elaborate statements we refer to [BL] §5. Again, at this stage, several strong assumptions are usually needed to make things work but the main line of reasoning still deserves to be mentioned. First the result is true in dimension 1 by 'standard' anabelian results (H. Nakamura, A. Tamagawa, S. Mochizuki) for curves. Next and using the assumed validity of Conjecture 8.5, an element $f \in Out^*(\hat{\Gamma}(S))$ induces an element of $Aut(\hat{\mathscr{C}}(S))$ which we call by the same name. By the rigidity of the pants graph (Theorem 9.5), it is enough to show that if f is Galois invariant, it induces an element of $Aut(\hat{\mathscr{C}}_P(S))$. This can be done by induction on $d(S)$ as usual, modulo the congruence conjecture. Here the real point lies in the connection between Galois invariance on the one hand and preservation of the pants graph, i.e. of the 'intersection rule' (see §9.4 below), on the other. Clearly it would be highly desirable to elaborate a much better understanding of this phenomenon.

9.4. Intersection rules and topological associators

We now would like to investigate the deformations or the automorphisms of the profinite *curve* complex $\hat{\mathscr{C}}$. Perhaps it is best to first state rather bluntly the conjecture which will occupy us for the rest of the present §9. It predicts the structure of the groups $Aut(\hat{\mathscr{C}}(S))$ and thus potentially determines the automorphism groups of the Teichmüller groups, and more; see §10.1 below for the network of established connections in the discrete setting. Before stating it, we should also make it clear again that although we do have to label certain statements as 'conjectures' for lack of a complete proof, they do not all share the same status. The overarching contractibility conjecture is wide open, but most of the rest actually should follow from it. In particular the reader will find in [L1] ample material strongly backing most statements which will appear from here to the end of the text. We could in fact pile up assumptions as was done above several times and produce a 'conditional statement' rather than a 'conjecture' but, rightly or wrongly, that seemed to us rather awkward at this stage.

Return to $S \simeq S_{g,n}$ hyperbolic of type (g, n). We say that S is *topologically generic* if it contains a piece of type $(1, 3)$, i.e. if one can cut S along a multicurve so that one of the resulting pieces is a copy of

$S_{1,3}$. Clearly this is equivalent to $g = g(S) > 0$ and $n > 2$ if $g = 1$, $n > 0$ if $g = 2$, so that the exceptional types $(1,1)$, $(1,2)$ and $(2,0)$ are ruled out (these are those where for instance the center of $\Gamma(S)$ is nontrivial). One can also say that S is (topologically) generic if $g(S) > 0$, $d(S) > 2$ and S is not of type $(2,0)$. With this terminology we state:

Conjecture 9.8. Let S be a topologically generic (hence hyperbolic) surface of type (g, n). Then there is an exact sequence:

$$1 \to Inn(\hat{\Gamma}(S)) \to Aut(\hat{\mathscr{C}}(S)) \to \mathbb{I}\Gamma \to 1$$

where $\mathbb{I}\Gamma$ denotes the (profinite) higher genus Grothendieck–Teichmüller group (cf. [L1]).

If $g = 0$, $n > 4$, in which case $d(S) > 1$ but S is *not* generic, the same exact sequence takes place, but with $\mathbb{I}\Gamma$ replaced by $\widehat{GT}$, the profinite genus 0 Grothendieck–Teichmüller group (cf. [D]).

The cases with $d(S) > 1$ which *a priori* are *not* covered by the statement are types $(1,2)$ and $(2,0)$. But since $\hat{\mathscr{C}}(S_{1,2}) \simeq \hat{\mathscr{C}}(S_{0,5})$ and $\hat{\mathscr{C}}(S_{2,0}) \simeq \hat{\mathscr{C}}(S_{0,6})$ this is not serious and is detailed in [L1]. The group $\mathbb{I}\Gamma$ was developed in [HLS] (with a different notation), with refinements in [NS] and further improvements in [L1] §5. At this point we need only know that it is a well-defined and 'big' profinite group; e.g. it contains $Gal(\mathbb{Q})$. But is is contained in $\widehat{GT}$, defined in the landmark paper by Drinfel'd ([D]) and which, in the present terms, covers the genus 0 situation. Some (rather tiny) inputs from Grothendieck–Teichmüller theory can be found below in §10.2; the reading of this and the next subsections definitely benefits from an acquaintance with the theory, but we will mostly try to pinpoint two fundamental and defining phenomena. In this subsection we define the analog of associators, recalling that 'deforming the associativity constraint in a universal way' can be seen as *the* defining motto of the whole theory in [D]. In the next subsection we go to the locality or *two-level principle* ('principe des deux premiers étages' in Grothendieck's *Esquisse d'un programme*) in our modular profinite setting, this being *the* reason why the Grothendieck–Teichmüller group, which one can define completely abstractly, turns out to be at all manageable (e.g. defined by finitely many 'equations').

Let S be (connected) hyperbolic with $d(S) > 0$. Recall from the end of §9.1 the graph $\mathscr{C}_*(S)$ and the natural embedding $j_0 : \mathscr{C}_P(S) \to \mathscr{C}_*(S)$ of the pants graph into it. We also have an isomorphism $Aut(\hat{\mathscr{C}}(S)) \simeq Aut(\hat{\mathscr{C}}_*(S))$. We can then define:

Definition 9.9. A *topological associator* is an oriented embedding of $\hat{\mathscr{C}}_P(S)$ into $\hat{\mathscr{C}}(S)$, that is a $\hat{\Gamma}(S)$-orbit for the natural right action of

$\hat{\Gamma}(S)$ on the set of injective maps $j : \hat{\mathscr{C}}_P(S) \hookrightarrow \hat{\mathscr{C}}_*(S)$. The set of these associators is denoted $J(S)$.

More explicitly we identify two embeddings j and $j' = j \circ g$, $g \in \hat{\Gamma}$. Recall that $\hat{\mathscr{C}}_P(S)$ is equipped (contrary to $\hat{\mathscr{C}}_*(S)$ or $\hat{\mathscr{C}}(S)$) with a natural orientation, once the surface S has itself been given an orientation, which we assume once and for all. It is natural to define the set of oriented embeddings by identifying two injective maps j and j' as above if the composite map $j^{-1} \circ j$ exists and lies in $Aut^+(\hat{\mathscr{C}}_P(S))$, the group of oriented automorphisms of $\hat{\mathscr{C}}_P(S)$. The rigidity of the profinite pants graph identifies this group with $Inn(\hat{\Gamma}(S))$.

Next, there is a natural action of $Aut(\hat{\mathscr{C}}(S))$ on $J(S)$ obtained by first identifying $Aut(\hat{\mathscr{C}}(S))$ with $Aut(\hat{\mathscr{C}}_*(S))$ and then postcomposing: $\phi{\cdot}j = \phi{\circ}j$ for $j \in J(S)$, $\phi \in Aut(\hat{\mathscr{C}}_*(S))$. This action factors through the quotient $Aut(\hat{\mathscr{C}}(S))/\hat{\Gamma}(S) = Out(\hat{\mathscr{C}}(S))$ of $Aut(\hat{\mathscr{C}}(S))$ by $\hat{\Gamma}(S)$, where $\hat{\Gamma}(S)$ acts effectively via $Inn(\hat{\Gamma}(S))$. That this is a *normal* subgroup of $Aut(\hat{\mathscr{C}}(S))$ is part of Conjecture 9.8. Why this should be true (and more) is detailed in [L1], §3.

Finally, this action of $Out(\hat{\mathscr{C}}(S))$ on $J(S)$ should be free and transitive. This amounts to showing that an embedding of $\hat{\mathscr{C}}_P$ into $\hat{\mathscr{C}}_*$ uniquely extends to an automorphism of $\hat{\mathscr{C}}_*$. Since $\hat{\mathscr{C}}_*$ and $\hat{\mathscr{C}}_P$ share the same vertices, the question is only about edges. Moreover, assuming Conjecture 8.5, we are dealing with flag complexes, so *if* the extension exists, it is unique since edges are determined by their boundary vertices. Finally, existence can be reduced to a local problem, that is to modular dimension 1 and the embedding of the profinite Farey graph into the profinite complete graph on the same vertices, in which case it is obvious. For details we refer to [L1] (Proposition 0.3) and especially [BL], §4. Again we refrain from a formal statement as more foundational work is required to make the above arguments watertight without assuming too much.

The upshot is that the set $J(S)$ of topological associators should be a *torsor* under the natural action of $Out(\hat{\mathscr{C}}(S))$ and Conjecture 9.8 predicts that for $d(S) > 1$, this last group is nothing but a version of the Grothendieck–Teichmüller group, in particular is essentially independent of the type (g, n). Here we should add that $J(S)$ has a privileged basepoint, namely the completion of j_0 defined by the topological embedding $\mathscr{C}_P(S) \hookrightarrow \mathscr{C}_*(S)$. So the torsor $J(S)$ has a natural trivialization and for most purposes can be identified with the attending group. Still the above serves to underline the close parallel (but with significant differences as well) with the situation in [D], which gave rise to the original definition of the Grothendieck–Teichmüller group. In [D], §4, the prounipotent genus 0 version of the Grothendieck–Teichmüller group $GT(k)$ (k a field of characteristic 0) appears via universal deformations of quasi-Hopf quasitriangular (i.e. braided) universal enveloping algebras. The profinite version $\widehat{GT}$ is then introduced by analogy (top of p.846). Note that the prounipotent (or pronilpotent) version is not deduced from the profinite version via the natural (functorial) procedure. The point we would like to make here is that we are in fact exploring a deformation theory in the full *profinite* modular setting. How can one interpret J as a set of deformations? Classically, if α and β are simple (isotopy classes of) curves on a surface they are said to have *minimal intersection* if either they are supported on a subsurface of type $(1,1)$ and intersect at 1 point only, or they are supported on a subsurface of type $(0,4)$ and intersect at 2 points This elementary topological notion is the essential ingredient in the definition of the graph $\mathscr{C}_P(S)$. That graph *and* its profinite completion are essentially rigid. The curve complex (or graph) $\hat{\mathscr{C}}(S)$, however turns out to have a lot of (non inner) automorphisms and these are parametrized by the *profinite deformations of the minimal intersection rule.* Moreover the graphs $\hat{\mathscr{C}}_P(S)$ and $\hat{\mathscr{C}}_*(S)$ share the same set of vertices and in [L1], §3, it is explained how a kind of *transversality* property should be valid. Namely given two embeddings $j, j' \in J$, either their images coincide or they have *no* edge in common. An embedding is thus entirely specified by giving one of its edges, and such an edge in turn deserves to be called a rule for minimal intersection. The topological rule recalled above corresponds to the topological embedding j_0 and one could conclude that in some sense much of the mystery is (still) hidden in the profinite completion of the Farey graph.

9.5. A graph theoretic view of the two-level principle

As mentioned already above, the 'two-level principle' is one of the, if not *the* founding principle of Grothendieck–Teichmüller theory. In a nutshell and shunning serious motivations, one is interested in the automorphism group of the so-called Teichmüller tower, meaning the collection of all the $\hat{\Gamma}_{g,[n]}$'s with varying (g,n) equipped with morphisms 'coming from geometry'. An interesting subtower is obtained by restricting to the case of genus 0, i.e. fixing $g=0$ and letting n vary. According to the context this already encompasses e.g. braided categories or mixed Tate motives over $\mathbb{Z}$. Very roughly speaking, the two-level principle says that the group of automorphisms of the whole structure, i.e. the automorphism group of the 'tower' (or category) is determined by the first two levels, that is the four cases with modular dimension $3g-3+n \leq 2$ connected by a few geometric maps. It will however emerge from this and the next section that even the notion of 'tower' is actually superfluous and that the objects can be considered one by one.

In genus 0, as can be gathered from [D], this principle is derived, in a nontrivial way, from McLane's coherence relations for what is nowadays called braided categories. It was implemented by Y. Ihara (starting with his paper in the *Grothendieck Festschrift*) in the same context, that is genus 0, prounipotent (or pronilpotent) and this paved the way to many important papers by the Japanese school. That same principle in genus 0 is reflected in the geometry at infinity of the moduli schemes $\mathfrak{M}_{0,n}$. In higher genus it was stated without proof in Grothendieck's *Esquisse* and vindicated in [L2] (see also [B1], Theorem 3.2) precisely by using (discrete) curve complexes and their homotopy types (cf. Theorem 5.1 above). Again it reflects the geometry at infinity of the moduli stacks. Although it is quite a bit more subtle than that, one can still profitably recall at this point the elementary fact that the fundamental group of a simplicial set or CW-complex depends only on its 2-skeleton.

The two-level principle can be decomposed into two statements expressing injectivity and surjectivity respectively. The first says that two automorphisms of the tower which coincide at the first two levels coincide. The second says that any automorphism of the two-level truncated tower can be extended to an automorphism of the full tower. Here we are interested only in the first statement (injectivity) because by now, owing to work of Y. Ihara (see especially his article in *Israel J. of Math.*, 1992) and to [HLS] and [NS] in the profinite case for all genus, the surjectivity part comes for free. Actually the *Grothendieck–Teichmüller lego*, recalled and somewhat improved in [L1] §5, enables one to *explicitly* describe the extension from the *first* level upward, provided one knows there is no obstruction at the second level. For much more material we refer the reader to [L1], [LS] and e.g. the homepage of P.L., which contain a hoard of references.

Here we will state precisely the graph theoretic version of the two-level principle; [L1] contains a proof which is valid for the *geometric completion.* Let us start with an elementary and useful definition. Given a connected surface surface S and a curve γ on it (by which we mean as usual a simple loop considered up to isotopy), we say that γ is *complex theoretically non separating* if it is either nonseparating in the usual sense or if $S \setminus \{\gamma\}$ decomposes into two components, one of which is a trinion i.e. has type $(0,3)$. Because trinions (or 'tripods' in the terminology of [HM1,2]) are rigid, this seems to be the right notion of separability and it enjoys curious elementary topological properties (see [L1], §1).

The *graph theoretic version of the (injectivity part) of the two-level principle* reads as follows:

Let S be hyperbolic with $d(S) > 1$ and let γ be a complex theoretically non separating (discrete) curve of S. Let then $F \in Aut(\hat{\mathscr{C}}(S))$ be an automorphism of the profinite curve complex fixing γ, so that F restricts to an automorphism of $\hat{\mathscr{C}}(S \setminus \{\gamma\})$. Then, if that restriction reduces to the identity, $F \in \langle t_\gamma \rangle$, the procyclic group generated by the twist along γ.

At present we are not able to garantee that inside $\hat{\mathscr{C}}(S)$, the star of γ is indeed isomorphic to (the cone over) $\hat{\mathscr{C}}(S \setminus \{\gamma\})$. The analog is true for the geometric completion and so if $g(S) \leq 2$ or if we assume the congruence conjecture. We refer again to [L1], Remarks 2.10, for comments on this graph theoretic avatar of the principle and its relation to previous incarnations. An attempted proof naturally proceeds by ascending induction on the modular dimension $d = d(S)$. In fact let us denote by (P_d) (principle in dimension $\leq d$) the statement above for $1 < d(S) \leq d$, and by $(\check{P}_d)$ the statement obtained by replacing the full profinite by the geometric completion everywhere. Then we have:

Proposition 9.10. *(see [L1], Proposition 1.3) Assertion $(\check{P}_d)$ implies $(\check{P}_{d+1})$ for all $d > 1$.*

If the congruence property holds this induction step is also true for the profinite completion. There remains to establish the base case $d = 2$, which contrary to what often happens in inductive proofs, is very far from being 'trivial'. First types $(0,5)$ and $(1,2)$ give rise to isomorphic curve complexes (and their completions) so that we need only study the case of $\hat{\mathscr{C}}(S_{0,5})$. Second, the congruence property does hold in genus 0 (or indeed for $d(S) \leq 2$) so that (P_2) and $(\check{P}_2)$ coincide. Finally, §2 of [L1] is devoted to this graph theoretic version of the pentagonal story and shows that (P_2) holds true, which completes the proof of:

Theorem 9.11. *The (injectivity part of the) graph theoretic version of the two-level principle holds true for geometric completions.*

In particular the principle holds in general modulo the congruence conjecture. As mentioned above the surjectivity part of that principle is not really a problem any longer so that this represents in some sense the major step in unraveling the structure of $Aut(\hat{\mathscr{C}}(S))$ (Conjecture 9.8) modulo the congruence conjecture.

We close this paragraph by stating a consequence of the two-level principle in genus 0 coupled with the computation of the automorphism group of the profinite graph $\hat{\mathscr{C}}(S_{0,5})$ (cf. [L1], §§3, 4):

Theorem 9.12. *Conjecture 9.8 holds true in genus 0.*

§10. Automorphisms of the profinite Teichmüller groups

10.1. Automorphisms of discrete Teichmüller groups

Let us very briefly review the situation in the discrete case, highlighting important and (for us) relevant statements. In a few sentences, and without paying due attention to a few low dimensional exceptions it looks as follows:
i) Teichmüller groups are rigid; for $d(S) > 1$:

$$Out(\Gamma(S)) = \mathbb{Z}/2,$$

where the nontrivial element corresponds of course to taking a mirror image, alias orientation reversing involution or complex conjugacy (see [Iv3]);

ii) This is true universally; if $\Gamma^\lambda \subset \Gamma$ is normal and cofinite, $Out(\Gamma^\lambda)$ can be described as an extension of $\mathbb{Z}/2$ by the geometric group Γ/Γ^λ acting by conjugation (see [Iv3]);

iii) Every automorphism is inertia preserving, that is permutes the cyclic subgroups generated by the twists:

$$Aut(\Gamma(S)) = Aut^*(\Gamma(S));$$

iv) Every automorphism of the complex $\mathscr{C}(S)$ is induced by an automorphism of the group $\Gamma(S)$.

Several remarks are in order. The most obvious one is that the above tersely and incompletely encapsulates the results obtained by a number of people over many years and does not really do justice to the situation. More to the point, iii) is rarely stated explicitly because there was no particular motivation to emphasize it but it is proved on the way to proving i) and ii). See [Iv5] and [McC] which also summarizes and develops a nice theory of the *abelian* subgroups of Γ. As usual, a profinite (or even $pro-\ell$) version would be welcome. One can also regard iii) as a consequence of Theorem 8.4; see §8.4.1 for an all too brief discussion. Next there is a natural map $Aut^*(\Gamma(S)) \to Aut(\mathscr{C}(S))$, recalling from §8.4.4 that, essentially trivially, $\mathscr{C}(S) \simeq \mathscr{C}_G(S)$. By iii) we get the whole automorphism group and iv) says this map is an isomorphism. Of course, given Theorem 9.1 and i) above, iv) immediately follows, but it is still significant, e.g. as an analog of a result of Tits which states that (under certain conditions) every automorphism of a building comes from an automorphism of the group.

Finally ii) is both deep and relatively easy. It stems from the fact that if $f \in Aut^*(\Gamma^\lambda)$ with Γ^λ cofinite (not necessarily normal), f also induces an automorphism of $\mathscr{C}(S)$. The point is that if two curves α and β are distinct, the associated cyclic groups $\langle t_\alpha \rangle$ and $\langle t_\beta \rangle$ are *not* commensurable; they do not intersect along cofinite subgroups. As a result any cofinite subgroup of such a cyclic group determines the associated curve uniquely. The reader probably already noticed the connection with some of the questions discussed in §8.4.4.

The morale of the discrete tale thus sounds as follows. All group automorphisms are inertia preserving and any such, indeed any automorphism of a cofinite subgroup induces an automorphism of the curve complex. As a result, the automorphisms of the curve complex $\mathscr{C}(S)$ control the automorphisms of the group $\Gamma(S)$, indeed of all its cofinite subgroups. Finally the curve complex itself is rigid (Theorem 9.1) and this essentially completes the part of the discrete story which is relevant here.

10.2. A few inputs in Grothendieck–Teichmüller theory

We have managed hitherto to avoid giving precise definitions of the Grothendieck–Teichmüller group(s), or say of its profinite versions $\widehat{GT}$ and $\mathbb{I}\Gamma$ partly because they are cumbersome, partly because we wanted to insist on the phenomena which motivate these definitions and make them viable. In this subsection we recall less than a minimum; there now exist numerous references as far as $\widehat{GT}$ is concerned and we will partly rely on the interest of the reader in terms of digging them out, which is very easy. Again the homepage of P.L. and e.g. [LS] or [L1] will provide her/him with bibliographical entries of all kinds depending on her/his taste and background.

As is well-known, $\widehat{GT}$ was properly introduced in [D], first in its prounipotent version, as a universal deformation group for braided quasi-Hopf algebras (these objects were of course also introduced by V. Drinfel'd). Then §4 of [D] switches—somewhat abruptly!—to the profinite case to which we confine ourselves here. There is a nested sequence of inclusions:

$$Gal(\mathbb{Q}) \subset \widehat{GT} \subset \mathrm{Aut}^*(\hat{F}_2),$$

where $Gal(\mathbb{Q})$ is the Galois group of $\mathbb{Q}$, $\widehat{GT}$ will be defined presently and $\mathrm{Aut}^*(\hat{F}_2)$ is the group of continuous inertia preserving automorphism of $\widehat{F}_2$, the profinite completion of the discrete free group $F_2 = \langle x, y \rangle$ on the generators x and y. In this context 'inertia preserving' means that the procyclic groups $\langle x \rangle$ and $\langle y \rangle$ are respectively mapped to conjugate groups by an element $F \in \mathrm{Aut}^*(\hat{F}_2)$ and so is $\langle z \rangle$, with $xyz = 1$. We add that the above inclusions are 'almost natural', depending only on the choice of a rational (tangential) basepoint.

Twisting by inner automorphisms of $\hat{F}_2$ one can normalize the elements of $\mathrm{Aut}^*(\hat{F}_2)$ by requiring that the group $\langle x \rangle$ be globally fixed. Behind this normalization and in greater generality are again such notions as tangential basepoints, splitting of certain sequences etc. The long and the short is that, concretely speaking, the elements of $\mathrm{Aut}^*(\hat{F}_2)$ we are interested in are given as pairs $F = (\lambda, f)$ with $\lambda \in \hat{\mathbb{Z}}^*$ (the invertible elements of $\hat{\mathbb{Z}}$) and $f \in \hat{F}_2'$ (the topological derived subgroup of $\hat{F}_2$). The action on $\hat{F}_2$ is defined by:

$$F(x) = x^\lambda, \qquad F(y) = f^{-1} y^\lambda f.$$

One requires that these formulas define an automorphism, that is an *invertible* morphism, but in contrast with the pronilpotent case there is no effective way to test invertibility here, which simply has to be assumed (or 'imposed' which is the same).

Multiplication is given by composition in the automorphism group $\mathrm{Aut}(\widehat{F}_2)$. This leads to the following formula for the product of $F = (\lambda, f)$ and $F' = (\lambda', f')$: $F' \circ F = (\lambda\lambda', f'F'(f))$.

Then for an automorphism F as above to define an element of $\widehat{GT}$, it has to extend to an automorphism of the first two levels i.e. to the group $\hat{\Gamma}_{0,[4]}$ (recall that $\hat{\Gamma}_{0,4} \simeq \hat{F}_2$) and especially to $\hat{\Gamma}_{0,[5]}$. This entails that the associated pair (λ, f) has to satisfy the following 3 relations

(I) (2-cycle) $f(x, y)f(y, x) = 1$;

(II) (3-cycle) $f(x, y)x^\mu f(z, x)z^\mu f(y, z)y^\mu = 1$ where $xyz = 1$ and $\mu = (\lambda - 1)/2$;

(III) (5-cycle) $f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}{,}_{51}) = 1$;

In these formulas one uses the fact that one can 'change variables' in the proword f and in the last of these, $x_{ij} \in \Gamma_{0,5}$ represents a pure braid twisting strands i and j once ($i, j \in \mathbb{Z}/5$, $i \neq j$). A more omplete description is available in virutally any reference on the subject.

Thus $\widehat{GT} \subset \mathrm{Aut}^*(\hat{F}_2)$ is the subgroup whose elements are defined by pairs $F = (\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2'$, acting on $\hat{F}_2$ as above and satisfying (I), (II) and (III). These are often refered to as 'relations' but 'equations' would be more correct: $\widehat{GT}$ is a subgroup, not a quotient of $\mathrm{Aut}^*(\hat{F}_2)$.

Remark 10.1. It was noted by H. Furusho that (I) is in fact an easy consequence of (III). We nevertheless retain (I) in the definition because of its geometric meaning. Recently, the same author proved the surprising and beautiful result that (II) is a consequence of (III) *in the* *prounipotent setting*, that is for $GT(k)$, where k is a field of characteristic zero (you may need a quadratic extension in order to define μ). Furusho's result immediately implies that this is also the case in the pro-ℓ setting (ℓ a prime) and it raises the question as to whether this implication holds true in the present full profinite case. The group $\widehat{GT}$ would then be characterized by the 5-cycle or pentagon relation only.

If $F = \sigma \in Gal(\mathbb{Q})$, we denote the parameters by $(\lambda_\sigma, f_\sigma)$ and in fact $\lambda_\sigma = \chi(\sigma)$ coincides with the value of the cyclotomic character $\chi : Gal(\mathbb{Q}) \to \hat{\mathbb{Z}}^*$. In particular the first projection map $\widehat{GT} \to \hat{\mathbb{Z}}^*$, defined by $F = (\lambda, f) \to \lambda$, is surjective since it is already surjective when restricted to $Gal(\mathbb{Q})$. Its kernel is an important subgroup of $\widehat{GT}$, containing the Galois group of $\mathbb{Q}^{ab}$, the maximal abelian extension of $\mathbb{Q}$.

The only "discrete" elements of $\widehat{GT}$, that is those given by pairs $(\lambda, f) \in \mathbb{Z}^* \times F_2'$ are $(\pm 1, 1)$ ([D], Proposition 4.1) and the only nontrivial element among these, given by the pair $c = (-1, 1)$, corresponds to complex conjugacy. About the second projection $F = (\lambda, f) \to f$, it is interesting to note here that it is exactly two-to-one. Namely if $F = (\lambda, f)$, then $F' = F \circ c = (-\lambda, f)$ is the only other element with the same f, as can be readily infered from the formula for the multiplication law of $\widehat{GT}$ mentioned above. So f determines λ up to a sign, or to put it more geometrically, up to reflection; see [L1] for an elaboration on this theme.

We will be even more sketchy about the group $\mathbb{\Gamma}$ although this should be the 'overarching' avatar of the Grothendieck–Teichmüller group, since it corresponds to considering the full Teichmüller tower (all hyperbolic types (g, n)) and the full profinite completion, which carries the most information. We refer to [L1], §5 for a detailed description and comments, beyond the terse indications below; see also [HLS] and [NS] for a slightly different viewpoint. By definition an element of $\widehat{GT}$ acts on $\hat{\Gamma}_{0,[5]}$ and also on the almost isomorphic group $\hat{\Gamma}_{1,[2]}$; $\mathfrak{M}_{1,2}$ is the second piece at the second level i.e. it has dimension 2. Now consider the topological surface $S = S_{1,3}$. Drawing a picture, it is plain that one can find two curves α_0 and α_1 such that $S_0 = S \setminus \alpha_0$ is of type $(0, 5)$ and $S_1 = S \setminus \alpha_1$ is of type $(1, 2)$. Consider a pair (F_0, F_1) of elements of $\widehat{GT}$ and let F_0 (resp. F_1) act on $\hat{\Gamma}(S_0)$ (resp. $\hat{\Gamma}(S_1)$). Using the locality of the action and the Teichmüller lego (mark that these are highly nontrivial ingredients), one can test whether these actions paste into an action on the full group $\hat{\Gamma}(S)$. By the way, we are in the range where the congruence subgroup property is known to be valid, which is relevant here. Whether or not this compatibility of the local actions takes place can be expressed as an explicit relation between the parameters of F_0 and F_1. If it does, we say that (F_0, F_1) constitutes a compatible pair and $\mathbb{I}\Gamma$ is nothing but the set of such pairs. One then checks that it is a group, which is almost by definition and that F_0 and F_1 actually determine each other, which shows that $\mathbb{I}\Gamma$ can be realized (in at least two ways) as a subgroup of $\widehat{GT}$.

We can now add an item in our nested sequence, to get:

$$Gal(\mathbb{Q}) \subset \mathbb{I}\Gamma \subset \widehat{GT} \subset \mathrm{Out}^*(\hat{F}_2).$$

This does prompt remarks, some of which the reader will find at the end of [L1]. As to the strictness of the inclusions, whether $Gal(\mathbb{Q})$ and $\mathbb{I}\Gamma$ coincide or not is clearly an important question, with loads of potential consequences. For the reader who is used to the prounipotent setting, this is of course the (full profinite, any genus) counterpart of the possible isomorphism between the Deligne–Ihara algebra (over $\mathbb{Q}_\ell$) and the graded Lie algebra $\mathfrak{grt} \otimes \mathbb{Q}_\ell$.

At the other end we used $\mathrm{Out}^*(\hat{F}_2)$ rather than $\mathrm{Aut}^*(\hat{F}_2)$ because the former is somewhat 'smaller' and does not require using a splitting (sometimes called a Belyi lifting) via a tangential base point. But that is a detail and the last containment is quite ample—$\mathrm{Out}^*(\hat{F}_2)$ is a huge and somewhat amorphous group. The middle inclusion however is quite interesting. *A priori* $\widehat{GT}$ and $\mathbb{I}\Gamma$ differ only by one relation taking place on a surface of type $(1, 2)$ and they may look very similar if not equal. But from a motivic viewpoint, the prounipotent group $\widehat{GT}(\mathbb{Q})$ should control (i.e. determine the Galois group of the Tannakian category of) the mixed Tate motives over $\mathbb{Z}$, whereas the prounipotent avatar of $\mathbb{I}\Gamma$, still to be investigated, should have to do with 'modular motives', i.e. mixed motives attached to the $\mathfrak{M}_{g,n}$'s, which are not only not Tate, but are not even defined to-date (at least for $g > 2$).

10.3. Automorphisms of profinite Teichmüller groups

We have seen in §10.1 that the determination of the automorphism group of the curve *complex* (Theorem 9.1), coupled with the fact that all the group automorphisms are inertia preserving (assertion iii) in §10.1) is the key to unlocking the structure, not only of the automorphisms of the Teichmüller groups, but also of all their cofinite subgroups.

In the profinite situation, Conjecture 9.8, which by now the reader may perhaps find more 'natural' and which is true in genus 0, should play the same role. But first we do not know how to prove that all automorphisms are inertia preserving and this looks really hard, even in genus 0 (see however [HM2], especially Introduction, Theorem A). Of course it would be a consequence of a positive answer to Question 8.4 but that also seems quite hard (yet, see the epigraph as usual...). Let us just list the question for the sake of the record:

Question 10.1. Let S be hyperbolic with $d(S) > 1$; is it true that all the automorphisms of the attached profinite Teichmüller group $\hat{\Gamma}(S)$ are inertia preserving: $Aut^*(\hat{\Gamma}(S)) \stackrel{?}{=} Aut(\hat{\Gamma}(S))$.
Does this hold more generally for the open subgroups of $\hat{\Gamma}(S)$?

So per force we limit ourselves to exploring the subgroup $Aut^*(\hat{\Gamma}(S))$. Let us now mention a result whose proof is completely independent of the above. Indeed by building on the work of a number of people (especially H. Nakamura, P.L.–L. Schneps, D. Harbater–L. Schneps; see [BL] for references), one can show:

Theorem 10.2. *([BL], Proposition 4.14) For every $n \geq 5$:*

$$Out^*(\hat{\Gamma}_{0,[n]}) = \widehat{GT}.$$

This is certainly a nice piece of information, featuring the profinite counterpart of what Y. Ihara elaborated around 1990 in the pronilpotent framework. However it seems clear that even in this restricted context, the proof is not really satisfactory and for instance cannot deal with the automorphisms of the open subgroups.

In fact we have already encountered in §8.4 the obstacles which stand in the way of applying Conjecture 9.8 (granted its validity) to the group automorphisms, i.e. of comparing graph theoretic and group theoretic automorphism groups. Without going into detail and in group theoretic parlance, a basic point is again that one would need to recognize a simplex $\sigma \in \hat{\mathscr{C}}(S)$ from the associated group, and indeed 'virtually' so, e.g. from the normalizer of any open subgroup of the associated group (see §8.4.4).

The reader will find in [L1], §5 more material on the subject, including the by now rather natural if conjectural description of the automorphism groups of all the open subgroups of the profinite Teichmüller groups. Let us content ourselves here with stating the obvious, recalling that the notion of topological genericity has been defined above Conjecture 9.8:

Conjecture 10.3. Let S be topologically generic (e.g. $S \simeq S_{1,3}$), then: $Out^*(\hat{\Gamma}(S)) \simeq \mathbb{I}\Gamma$.

Of course one expects that $Out^*(\hat{\Gamma}(S))$ coincides with $Out(\hat{\Gamma}(S))$ (see Question 10.1 above). One may note that the notion of 'tower' ('tour de Teichmüller' in Grothendieck's *Esquisse*) has disappeared completely, whether in the graph or the group theoretic setting. In particular, in Theorem 10.2 and Conjecture 10.3, the Grothendieck–Teichmüller group appears as the (outer) automorphism group of a single group, and ditto for the complexes in Conjecture 9.8 or Theorem 9.12. The point is that the structure of the tower, or at least its first few levels, which suffices by the two-level principle, can be recovered *from within* a given group or especially graph (see [BL], beginning of §4, for the graph theoretic viewpoint).

In closing I formulate the hope that this guided tour may inspire some readers to undertake a deeper foray into a nacsent field where there obviously remains a lot to be done, including from a fundational viewpoint, and where new and specific techniques are sorely needed.

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Here is an approximate translation of the epigraph: "I have noted on other occasions that when oracles (here myself!) declare with an air of deep understanding (or doubt) that such or such problem is 'out of reach', it is actually an entirely subjective assertion. It simply means, apart from the fact that the problem is supposed not to have yet been solved, that the person speaking is short of ideas on the question [...]."

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