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Arrangements, multiderivations, and adjoint quotient map of type ADE

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Abstract.

The first part of this paper is a survey on algebro-geometric aspects of sheaves of logarithmic vector fields of hyperplane arrangements. In the second part we prove that the relative de Rham cohomology (of degree two) of ADE-type adjoint quotient map is naturally isomorphic to the module of certain multiderivations. The isomorphism is obtained by the Gauss–Manin derivative of the Kostant–Kirillov form.

§1. Introduction

We begin with an example to illustrate how the structure of the module of logarithmic vector fields $D(\mathcal{A})$ is related to combinatorial problems of a hyperplane arrangement $\mathcal{A}$. Let $\mathcal{A}$ be a collection $\{H_{ij} \mid 1 \leq i < j \leq n\}$ of hyperplanes $H_{ij} = \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid x_i - x_j = 0\} \subset \mathbb{K}^n$, where $\mathbb{K}$ is a fixed field. According to the field $\mathbb{K}$, several enumerative problems appear for the complement $M(\mathcal{A}) = \mathbb{K}^n \setminus \bigcup H_{ij}$.

(i) If $\mathbb{K} = \mathbb{F}_q$ is a finite filed, then the complement $M(\mathcal{A})$ is a finite set, of cardinality $|M(\mathcal{A})| = q(q-1)(q-2)\dots(q-n+1)$.

(ii) If $\mathbb{K} = \mathbb{R}$ is the real numbers, then each connected component of $M(\mathcal{A})$ (the chamber) is expressed by the inequality $x_{i_1} < x_{i_2} < \dots < x_{i_n}$, where $(i_1, \dots, i_n)$ is a permutation of $(1, \dots, n)$. There are $n!$ chambers.

(iii) If $\mathbb{K} = \mathbb{C}$ is the complex numbers, then $M(\mathcal{A})$ is an affine complex variety of $\dim_{\mathbb{C}} = n$. Using the fibration $(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1})$ and the Leray–Hirsch theorem, the Poincaré

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polynomial is computed as $\sum_i b_i(M(\mathcal{A}))t^i = (1+t)(1+2t)\dots(1+(n-1)t)$.

The formulas in (i)–(iii) are similar in appearance. The general theory of arrangements [23] tells us that these invariants are combinatorial. Namely, they are determined from the poset $L(\mathcal{A})$ of subspaces obtained as intersections. Computations of the characteristic polynomial $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ unify these enumerative problems.

We also consider derivations

$$\delta_p = \sum_{i=1}^{n} x_i^p \partial_i,$$

($\partial_i = \frac{\partial}{\partial x_i}$) with $p = 0, 1, \dots, n-1$. These satisfy

$$\delta_p(x_i - x_j) = x_i^p - x_j^p \in (x_i - x_j), \tag{1}$$

for all i, j and the determinant of coefficients

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i) \tag{2}$$

is the product of defining equations. The properties (1) and (2) guarantee that the module

$$D(\mathcal{A}) = \{\delta \in \mathrm{Der}_S \mid \delta(x_i - x_j) \in (x_i - x_j), \forall i, j\}$$

is a free module over $S = \mathbb{K}[x_1, \dots, x_n]$ with basis $\delta_0, \dots, \delta_{n-1}$ (this is Saito's criterion [25]). Remarkably, the decomposition of $D(\mathcal{A})$ into a direct sum of rank one free modules implies the product formulas (i)–(iii) above (Terao's factorization theorem [38]). More generally, the algebraic structure of $D(\mathcal{A})$ determines the characteristic polynomial $\chi(\mathcal{A}, t)$ by Solomon–Terao's formula [35] (see also §2.2 below).

The graded S-module $D(\mathcal{A})$ can also be considered as a coherent sheaf $\widetilde{D}(\mathcal{A})$ on projective space $\mathbb{P}^{n-1}$. This fact enables us to employ algebro-geometric methods to study $\mathcal{A}$. The structures of these sheaves contain information on $\mathcal{A}$.

The purpose of this paper is to survey algebro-geometric aspects of $D(\mathcal{A})$ and give some related results. The paper is organized as follows. In §2, we start with recalling basic notions on logarithmic vector fields for a Cartier divisor. We also introduce the module $D(\mathcal{A}, \mathbf{m})$ of logarithmic vector fields for an arrangement with multiplicity (multiarrangements) in §2.2. In general, the logarithmic vector fields for multiarrangements are much more difficult to analyze than simple arrangements. However, freeness of rank ℓ simple arrangements is closely related to freeness of rank $\ell-1$ multiarrangements. We will describe freeness criteria for these objects in §2.3–§2.4. In §2.5, we give a new necessary condition for a 3-dimensional arrangement to be free, in terms of plane curves. In §3, we will review results on freeness of Coxeter multiarrangements. Coxeter arrangements are the best understood class of multiarrangements. In §4, we will give two applications of freeness of Coxeter multiarrangements. The first concerns the adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{g}//\operatorname{ad}(G)$ of a simple Lie algebra $\mathfrak{g}$ of ADE-type. To describe the relative de Rham cohomology of χ, the module $D(\mathcal{A}, \mathbf{m})$ naturally appears. In the second application, we will give another proof for the freeness of A_n-Catalan arrangements, which was first proved by Edelman and Reiner [15].

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§2. Algebraic geometry of logarithmic vector fields

2.1. Sheaf of logarithmic vector fields

Let X be a smooth complex variety and $D \subset X$ a Cartier divisor. Let $U \subset X$ be an open subset of X. Suppose that there exists $h \in \Gamma(U, \mathcal{O}_X)$ such that $U \cap D = \{h = 0\}$. Let $\delta \in \Gamma(U, \mathcal{T}_X)$ be a section of the tangent sheaf on an open subset $U \subset X$ (i.e., a holomorphic vector field on X). The section δ is said to be *logarithmic tangent to* D if $\delta h \in h \cdot \mathcal{O}_U$. The sheaf of vector fields logarithmic tangent to D is denoted by $\mathcal{T}_X(-\log D)$. The sheaves of logarithmic forms are also similarly defined as

$$\Omega^p_X(\log D) = \{\omega \in \frac{1}{h}\Omega^p_X \mid \omega \wedge dh \text{ is holomorphic}\}.$$

They were introduced by K. Saito in [25]. He proved that they are reflexive sheaves and if $\mathcal{T}_X(-\log D)$ (or $\Omega^1_X(\log D)$) is locally free then $\Omega^p_X(\log D) = \bigwedge^p \Omega^1_X(\log D)$. We also note that if $\dim X = 2$, $\mathcal{T}_X(-\log D)$ is a locally free sheaf.

Example 2.1. *Let* $X = \mathbb{P}^2_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[z_0, z_1, z_2]$. *Using the Euler sequence ([21])*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \longrightarrow \mathcal{T}_{\mathbb{P}^2} \longrightarrow 0,$$

we have the following.

(1) *If* $D_0 = \{z_0 = 0\} \subset \mathbb{P}^2$. *Then* $\mathcal{T}_{\mathbb{P}^2}(-\log D_0) \cong \mathcal{O}_{\mathbb{P}^2}(1)^2$.
(2) *If* $D_1 = \{z_0 z_1 = 0\} \subset \mathbb{P}^2$. *Then* $\mathcal{T}_{\mathbb{P}^2}(-\log D_1) \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$.
(3) *If* $D_2 = \{z_0 z_1 z_2 = 0\} \subset \mathbb{P}^2$. *Then* $\mathcal{T}_{\mathbb{P}^2}(-\log D_2) \cong \mathcal{O}_{\mathbb{P}^2}^2$.
(4) *If* $D_3 = \{z_0^2 + z_1^2 + z_2^2 = 0\} \subset \mathbb{P}^2$. *Then* $\mathcal{T}_{\mathbb{P}^2}(-\log D_3) \cong \mathcal{T}_{\mathbb{P}^2}(-1)$. *(Sketch:* $\bigoplus_d \Gamma(\mathcal{T}_{\mathbb{P}^2}(-\log D_3)(d))$ *is generated by* $\delta_0 = z_2\partial_1 - z_1\partial_2, \delta_1 = -z_2\partial_0 + z_0\partial_2$ *and* $\delta_2 = z_1\partial_0 - z_0\partial_1$ *with a relation* $z_0\delta_0 + z_1\delta_1 + z_2\delta_2 = 0$ *this induces a resolution which is isomorphic to shifted Euler sequence.)*

The examples above, $\mathcal{T}_X(-\log D)$ is always a uniform sheaf. However for "generic" divisors of higher degrees, we obtain "generic" sheaves. We can sometimes recover the original divisor D from the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D)$. (Dolgachev and Kapranov called this type of result a "Torelli-type" theorem in [14].) Let us recall two results in this direction. First one is due to Dolgachev and Kapranov, concerning the case of a union of generic hyperplanes.

Theorem 2.2. [14] *Let* $m \geq 2n+3$ *and* $\mathcal{A}_i = \{H_{i1}, H_{i2}, \dots, H_{im}\}$ *(*$i = 1, 2$*) be arrangements of generic* m *hyperplanes* $H_{ik} \subset \mathbb{P}^n_{\mathbb{C}}$ *in* n*-dimensional projective space. We denote the union by* $\cup\mathcal{A} = \bigcup_{H \in \mathcal{A}} H$. *If*

$$\mathcal{T}_{\mathbb{P}^n}(-\log(\cup\mathcal{A}_1)) \cong \mathcal{T}_{\mathbb{P}^n}(-\log(\cup\mathcal{A}_2)),$$

then $\cup\mathcal{A}_1 = \cup\mathcal{A}_2$.

For smooth divisors $D \subset \mathbb{P}^n$ defined by a homogeneous polynomial $\{f(z_0, \dots, z_n) = 0\}$, Torelli-type results are related to the following property.

Definition 2.3. *The homogeneous polynomial* $f(z_0, \dots, z_n)$ *is said to be* Thom–Sebastiani type *if there exists a linear transformation* $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *such that* $f(A(z_0, \dots, z_n)) = g(z_0, \dots, z_k) + h(z_{k+1}, \dots, z_n)$ *for some* $0 \leq k \leq n-1$.

Theorem 2.4. [42, 43]

(i) *Let* $D_1, D_2 \subset \mathbb{P}^n$ *be degree* d *smooth divisors which are not Thom–Sebastiani type. Then* $\mathcal{T}_{\mathbb{P}^n}(-\log D_1) \cong \mathcal{T}_{\mathbb{P}^n}(-\log D_2)$ *if and only if* $D_1 = D_2$.
(ii) *Let* $D_1, D_2 \subset \mathbb{P}^2$ *be smooth cubic curves with non-zero* j*-invariant* $j(D_i) \neq 0$. *Then* $\mathcal{T}_{\mathbb{P}^n}(-\log D_1) \cong \mathcal{T}_{\mathbb{P}^n}(-\log D_2)$ *if and only if* $D_1 = D_2$.

2.2. Log vector fields for multiarrangements

The main theme of this paper is logarithmic vector fields for arrangements of hyperplanes. Freeness is one of the important properties for arrangements. Ziegler [55] showed that a free arrangement of rank ℓ induces several free multiarrangements of rank $\ell - 1$ (see §2.4). This means that freeness of multiarrangements will be necessary for that of simple arrangements. Recently, several results on free simple arrangements have been generalized to free multiarrangements.

Let $V = \mathbb{C}^\ell$ be a complex vector space with coordinate $(x_1, \cdots, x_\ell)$, $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes. Let us denote by $S = \mathbb{C}[x_1, \dots, x_\ell]$ the polynomial ring and fix $\alpha_i \in V^*$ a defining equation of H_i, i.e., $H_i = \alpha_i^{-1}(0)$. We also put $Q(\mathcal{A}, \mathbf{m}) = \prod_{i=1}^n \alpha_i^{\mathbf{m}(H_i)}$ and $|\mathbf{m}| = \sum_i \mathbf{m}(H_i)$.

Definition 2.5. *A multiarrangement is a pair* $(\mathcal{A}, \mathbf{m})$ *of an arrangement* $\mathcal{A}$ *with a map* $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$*, called the multiplicity.*

An arrangement $\mathcal{A}$ can be identified with a multiarrangement with constant multiplicity $m \equiv 1$, which is sometimes called a *simple arrangement.* With this notation, the main object is the following module of S-derivations which has contact to each hyperplane of order m.

Definition 2.6. *Let* $(\mathcal{A}, \mathbf{m})$ *be a multiarrangement, and define*

$$D(\mathcal{A}, \mathbf{m}) = \{\delta \in \mathrm{Der}_S \,|\delta\alpha_i \in (\alpha_i)^{\mathbf{m}(H_i)}, \forall i\},$$

and

$$\Omega^p(\mathcal{A}, \mathbf{m}) = \left\{ \omega \in \frac{1}{Q}\Omega_V^p \,\middle|\, d\alpha_i \wedge \omega \text{ does not have pole along } H_i, \forall i \right\},$$

The module $D(\mathcal{A}, \mathbf{m})$ is obviously a graded S-module. A multiarrangement $(\mathcal{A}, \mathbf{m})$ is said to be *free* with exponents $(e_1, \dots, e_\ell)$ if and only if $D(\mathcal{A}, \mathbf{m})$ is an S-free module and there exists a basis $\delta_1, \dots, \delta_\ell \in D(\mathcal{A}, \mathbf{m})$ such that $\det \delta_i = e_i$. Here note that the degree $\deg \delta$ of a derivation δ is the polynomial degree, in other words, $\deg(\delta f) = \deg \delta + \deg f - 1$ for a homogeneous polynomial f. An arrangement $\mathcal{A}$ is said to be free if $(\mathcal{A}, 1)$ is free.

Let $\delta_1, \dots, \delta_\ell \in D(\mathcal{A}, \mathbf{m})$. Then $\delta_1, \cdots, \delta_\ell$ form a S-basis of $D(\mathcal{A}, \mathbf{m})$ if and only if

$$\delta_1 \wedge \cdots \wedge \delta_\ell = c \cdot Q(\mathcal{A}, \mathbf{m}) \cdot \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_\ell},$$

where $c \in \mathbb{C}^*$ is a non-zero constant (Saito's criterion [25]). From Saito's criterion, we also obtain that if a multiarrangement $(\mathcal{A}, \mathbf{m})$ is free with exponents $(e_1, \dots, e_\ell)$, then $|\mathbf{m}| = \sum_{i=1}^{\ell} e_i$.

The Euler vector field $\theta_E = \sum_{i=1}^{\ell} x_i\partial_i$ is always contained in $D(\mathcal{A})$. Thus it is natural to define $D_0(\mathcal{A}) := D(\mathcal{A})/S\cdot\theta_E$. Since $D_0(\mathcal{A})$ is a graded S-module, it determines a coherent sheaf $\widetilde{D_0}(\mathcal{A})$ on $\mathbb{P}^{\ell-1}$. On the other hand, an arrangement $\mathcal{A}$ defines a Cartier divisor $\cup\overline{\mathcal{A}} = \bigcup \overline{H} \subset \mathbb{P}^{\ell-1}$. The logarithmic sheaf $\mathcal{T}_{\mathbb{P}^{\ell-1}}(-\log(\cup\overline{\mathcal{A}}))$ determined by the divisor $(\cup\overline{\mathcal{A}})$ is related to $\widetilde{D_0}(\mathcal{A})$ by the following formula.

$$\mathcal{T}_{\mathbb{P}^{\ell-1}}(-\log(\cup\overline{\mathcal{A}})) \cong \widetilde{D_0}(\mathcal{A})[+1].$$

From the sheaf $\widetilde{D}(\mathcal{A})$, we can reconstruct the graded module $D(\mathcal{A})$ as global sections $\Gamma_*(\mathbb{P}^{\ell-1}, \widetilde{D}(\mathcal{A})) := \bigoplus_{d\in\mathbb{Z}}\Gamma(\mathbb{P}^{\ell-1}, \widetilde{D}(\mathcal{A}))$. More generally, we have the following.

Proposition 2.7. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement. Then the natural map*

$$\Omega^p(\mathcal{A}, \mathbf{m}) \longrightarrow \Gamma_*(\mathbb{P}^{\ell-1}, \widetilde{\Omega}^p(\mathcal{A}, \mathbf{m}))$$

is an isomorphism.

Proof. We prove the surjectivity. Since $\bigcup_{i=1}^{\ell} U_i = \mathbb{P}^{\ell-1}$, where $U_i = \{z_i \neq 0\} \subset \mathbb{P}^{\ell-1}$, is an affine open covering, any element of the right hand side $\frac{\omega}{Q} \in \Gamma_*(\mathbb{P}^{\ell-1}, \widetilde{\Omega}^p(\mathcal{A}, \mathbf{m}))$ can be expressed as

$$\frac{\omega}{Q} = \frac{\omega_1}{z_1^{d_1}Q} = \frac{\omega_2}{z_2^{d_2}Q} = \cdots = \frac{\omega_\ell}{z_\ell^{d_\ell}Q},$$

with $\frac{\omega_i}{Q} \in \Omega^p(\mathcal{A}, \mathbf{m})$. Using the fact that S is a UFD, it is easily seen that ω is a regular differential form. Assume that z_i and α_H are linearly independent. Taking the wedge with $d\alpha_H$, $d\alpha_H \wedge \frac{\omega_i}{Q}$ does not have pole along H. So $d\alpha_H \wedge \frac{\omega}{Q} = d\alpha_H \wedge \frac{\omega_i}{z_i^{d_i}Q}$. Hence $d\alpha_H \wedge \frac{\omega}{Q} \in \Omega^p(\mathcal{A}, \mathbf{m})$.

Q.E.D.

Combining the above proposition with a sheaf theoretic property of reflexive sheaves, we can prove that $\Omega^p(\mathcal{A}, \mathbf{m})$ is determined by $\Omega^1(\mathcal{A}, \mathbf{m})$ in general.

Proposition 2.8. *Assume that $\Omega^1(\mathcal{A}_1, m_1) \cong \Omega^1(\mathcal{A}_2, m_2)$. Then $\Omega^p(\mathcal{A}_1, m_1) \cong \Omega^p(\mathcal{A}_2, m_2)$.*

Proof. Let us denote $\mathcal{E}^p = \widetilde{\Omega}^p(\mathcal{A}, \mathbf{m})$. Let $U := \mathbb{P}^{\ell-1} \setminus \bigcup_{H,H'\in\mathcal{A}, H\neq H'}(\overline{H}\cap\overline{H'})$ be the complement to the union of codimension ≥ 2 strata. We denote the inclusion map $i : U \hookrightarrow \mathbb{P}^{\ell-1}$. The restriction $i^*\mathcal{E}^p$ is locally free. Hence we have $i^*\mathcal{E}^p = \bigwedge^p i^*\mathcal{E}^1$. Since $\mathcal{E}^p$ is reflexive, hence normal, we have $\mathcal{E}^p = i_*(i^*(\mathcal{E}^p))$. Thus $\mathcal{E}^p = i_*(\wedge^p i^*(\mathcal{E}^1))$. (See [17, §1] for basic properties of reflexive sheaves.)

Thus $\mathcal{E}^p$ is determined by $\mathcal{E}^1$. Then by Proposition 2.7, we obtain the graded module $\Omega^p(\mathcal{A}, \mathbf{m})$ from its sheafification. Q.E.D.

The following result shows that $D(\mathcal{A})$ determines the characteristic polynomial $\chi(\mathcal{A}, t)$. Corollary 2.10 is known as Terao's factorization theorem.

Theorem 2.9. [35] *Denote by $H(\Omega^p(\mathcal{A}), x) \in \mathbb{Z}[[x]][x^{-1}]$ the Hilbert series of the graded module $\Omega^p(\mathcal{A})$. Define*

$$\Phi(\mathcal{A}; x, y) = \sum_{p=0}^{\ell} H(\Omega^p(\mathcal{A}), x) y^p. \tag{3}$$

Then

$$\chi(\mathcal{A}, t) = \lim_{x \to 1} \Phi(\mathcal{A}; x, t(1-x) - 1). \tag{4}$$

Corollary 2.10. [38] *Suppose that $\mathcal{A}$ is a free arrangement with exponents $(e_1, \ldots, e_\ell)$. Then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - e_i). \tag{5}$$

The notion of freeness has a geometric interpretation. It is equivalent to a splitting $\widetilde{D}(\mathcal{A}) = \bigoplus_{i=1}^{\ell} \mathcal{O}(-e_i)$ of the sheaf $\widetilde{D}(\mathcal{A})$. Then the formula (5) indicates that the characteristic polynomial is related to the Chern polynomial $c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + \ldots$ (recall that $c_t(\mathcal{O}(-e)) = 1 - et$) [32]. Indeed, for locally free arrangements, Mustaţă and Schenck gave a beautiful formula connecting $\chi(\mathcal{A}, t)$ and the Chern polynomial.

Theorem 2.11. [20] *If $\widetilde{D}(\mathcal{A})$ is a locally free sheaf on $\mathbb{P}^{\ell-1}$, then*

$$c_t(\widetilde{D_0}(\mathcal{A})) = t^{\ell-1} \chi_0(\mathcal{A}, 1/t), \tag{6}$$

where $\chi_0(\mathcal{A}, t) = \chi(\mathcal{A}, t)/(t-1)$.

Note that in the case $\ell \leq 3$, the local freeness is always satisfied. Thus the Chern polynomial is essentially equivalent to $\chi(\mathcal{A}, t)$ and combinatorially computable [28].

2.3. Characterizing freeness

A vector bundle on $\mathbb{P}^1$ is always a direct sum of line bundles (Grothendieck). The splitting of vector bundles on $\mathbb{P}^n (n \geq 2)$ is also a well studied subject, e.g., see [21]. There are several criterion to be split. The next result is known as Horrocks' criterion.

Theorem 2.12. *Let E be a rank r holomorphic vector bundle on $\mathbb{P}^n (n \geq 2)$. The following conditions are equivalent.*

(i) $E = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(d_i)$ *for some* $d_1, \dots, d_r \in \mathbb{Z}$.

(ii) $H^i(\mathbb{P}^n, E(d)) = 0$ *for* $\forall 1 \leq i \leq n-1$ *and* $\forall d \in \mathbb{Z}$.

(iii) *(If $n \geq 3$) $\exists H \subset \mathbb{P}^n$ a hyperplane such that the restriction splits as* $E|_H = \bigoplus_{i=1}^{r} \mathcal{O}_H(d_i)$.

Yuzvinsky [52, 53, 54] developed sheaf theory on the intersection lattice $L(\mathcal{A})$ and gave a cohomological criterion for an arrangement $\mathcal{A}$ to be free which is similar to Theorem 2.12 (ii). As an application he proved that the set of free arrangements form a Zariski open subset in the moduli space of all arrangements having the fixed combinatorial type.

Here we describe a criterion similar to Theorem 2.12 (iii). We begin with recalling Ziegler's restriction [55].

Choose a hyperplane $H \in \mathcal{A}$ and coordinate $(z_1, \dots, z_\ell)$ such that $H = \{z_\ell = 0\}$. Define a submodule $D_0^H(\mathcal{A})$ of $D(\mathcal{A})$ as follows:

$$D_0^H(\mathcal{A}) := \{\delta \in D(\mathcal{A}) \mid \delta z_\ell = 0\}.$$

Lemma 2.13. $D(\mathcal{A}) = S \cdot \theta_E \oplus D_0^H(\mathcal{A})$.

Proof. Let $\delta \in D(\mathcal{A})$. The assertion is obvious from $\delta = \left(\frac{\delta z_\ell}{z_\ell}\right) \theta_E + \left(\delta - \frac{\delta z_\ell}{z_\ell}\theta_E\right)$. Q.E.D.

The arrangement $\mathcal{A}$ determines the restricted arrangement $\mathcal{A}^H = \{H \cap H' \mid H' \in \mathcal{A}, H' \neq H\}$ on H. The restricted arrangement $\mathcal{A}^H$ possesses a natural multiplicity

$$\begin{array}{rccl} m^H: & \mathcal{A}^H & \longrightarrow & \mathbb{Z} \\ & X & \longmapsto & \sharp\{H' \in \mathcal{A} \mid X = H \cap H'\}. \end{array}$$

Ziegler [55] proved that the freeness of $\mathcal{A}$ implies that of $(\mathcal{A}^H, m^H)$.

Theorem 2.14. [55]

(1) *If $\delta \in D_0^H(\mathcal{A})$, then $\delta|_{z_\ell=0} \in D(\mathcal{A}^H, m^H)$.*

(2) *If $\mathcal{A}$ is free with exponents $(1, e_2, \dots, e_\ell)$, then $(\mathcal{A}^H, m^H)$ is free with exponents $(e_2, \dots, e_\ell)$.*

Corollary 2.15. *$\mathcal{A}$ is free with exponents $(1, e_2, \dots, e_\ell)$ if and only if the following are satisfied.*

- *$(\mathcal{A}^H, m^H)$ is free with exponents $(e_2, \dots, e_\ell)$.*
- *The restriction induces the surjection $D_0^H(\mathcal{A}) \longrightarrow D(\mathcal{A}^H, m^H)$.*

Using Corollary 2.15, we can establish a Horrocks' type criterion for freeness. Namely, we will characterize freeness by using the freeness of the restriction $D(\mathcal{A}^H, m^H)$. We first consider the case $\ell = 3$. By analyzing the Hilbert series of these graded modules using the restriction map and Solomon–Terao's formula (Theorem 2.9), we have the following.

Theorem 2.16. [51] *If $\ell = 3$, then the cokernel of the restriction map is finite dimensional. Furthermore, suppose that $\exp(\mathcal{A}^H, m^H) = (e_1, e_2)$, then*

$$\dim_{\mathbb{C}} \mathrm{Coker} = b_3(\mathbb{C}^3 \backslash \bigcup_i H_i) - e_1 e_2.$$

Corollary 2.17. [51] *Suppose $\ell = 3$. Then the following conditions are equivalent.*

- *$\mathcal{A}$ is free with exponents $(1, e_2, e_3)$.*
- *$\chi(\mathcal{A}, t) = (t-1)(t-e_2)(t-e_3)$ and there exists $H \in \mathcal{A}$ such that $\exp(\mathcal{A}^H, m^H) = (e_2, e_3)$.*

Remark 2.18. *Recently a higher dimensional version of Corollary 2.17 has been obtained by Schulze* [31].

The characterization in the case $\ell \geq 4$ is the following.

Theorem 2.19. [50] *Suppose $\ell \geq 4$. Then an arrangement $\mathcal{A}$ is free with exponents $(1, e_2, \ldots, e_\ell)$ if and only if there exists $H \in \mathcal{A}$ such that*

(a) *$(\mathcal{A}^H, m^H)$ is free with exponents $(e_2, \ldots, e_\ell)$, and*
(b) *the localization $\mathcal{A}_x = \{H \in \mathcal{A} \mid x \in H\}$ is free for any $x \in H \setminus \{0\}$.*

2.4. Freeness for multiarrangements

The notion of multiarrangement is a natural generalization of simple arrangement. For a 2-dimensional simple arrangement $\mathcal{A}$, it is easy to construct explicit basis of $D(\mathcal{A})$. However, for the case of multiarrangements, describing an explicit basis for $D(\mathcal{A}, \mathbf{m})$ is difficult even for $\ell = 2$. Wakamiko [44] gave an explicit basis for $D(\mathcal{A}, \mathbf{m})$ with $\ell = 2, |\mathcal{A}| = 3$. Wakefield and Yuzvinsky [46] computed the exponents for $\ell = 2$ and generic $\mathcal{A}$. Both results show that the exponents tend to $(\lfloor \frac{|\mathbf{m}|}{2} \rfloor, \lceil \frac{|\mathbf{m}|}{2} \rceil)$, where $|\mathbf{m}| = \sum_{H \in \mathcal{A}} \mathbf{m}(H)$.

Remark 2.20. *The above mentioned results remind the author results of Dolgachev and Kapranov (cf. §2.1) and Schenck* [29] *on stability of $\mathcal{T}_{\mathbb{P}^n}(-\log(\cup \mathcal{A}))$. It seems natural to ask whether for generic $\mathcal{A}$ with $\ell = 3$, $\widetilde{D}(\mathcal{A}, \mathbf{m})$ is a stable rank 3 vector bundle on $\mathbb{P}^2$.*

Recently several results on $D(\mathcal{A})$ has been generalized to multiarrangements. Abe, Terao and Wakefield [3] proved that Solomon–Terao's formula (4) (and (3)) gives a polynomial $\chi((\mathcal{A}, \mathbf{m}), t)$ for any multiarrangement $(\mathcal{A}, \mathbf{m})$. The polynomial $\chi((\mathcal{A}, \mathbf{m}), t)$ is called the characteristic polynomial of a multiarrangement $(\mathcal{A}, \mathbf{m})$, which is a basic tool for proving non-freeness for multiarrangements.

Another important result on free multiarrangements is the Addition-Deletion Theorem [4]. Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement. Choose a hyperplane $H_0 \in \mathcal{A}$ with $\mathbf{m}(H_0) > 0$. One can associate two multiarrangements to $(\mathcal{A}, \mathbf{m}, H_0)$ as follows.

- The deletion $(\mathcal{A}', m')$: $\mathcal{A}' = \mathcal{A}$ and the multiplicity $m' : \mathcal{A}' \to \mathbb{Z}_{\geq 0}$ is defined by
$$m'(H) = \begin{cases} \mathbf{m}(H) & \text{if } H \neq H_0, \\ \mathbf{m}(H) - 1 & \text{if } H = H_0. \end{cases}$$
- The restriction $(\mathcal{A}'', \mathbf{m}^*)$: $\mathcal{A}'' = \{H \cap H_0 \mid H \in \mathcal{A}, H \neq H_0\}$. Let $X \in \mathcal{A}''$. Then X has codimension two. Thus the multiarrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\}$ with the multiplicity $\mathbf{m}|_{\mathcal{A}_X}$ is free. We can choose the basis $\theta_X, \psi_X, \partial_3, \dots, \partial_\ell$ with $\theta_X \notin \alpha_{H_0} \cdot \mathrm{Der}_S$ and $\psi_X \in \alpha_{H_0} \cdot \mathrm{Der}_S$. Define the multiplicity $\mathbf{m}^* : \mathcal{A}'' \to \mathbb{Z}_{\geq 0}$ by $\mathbf{m}^*(X) = \deg \theta_X$.

The following theorem generalizes the classical Addition-Deletion Theorem [37] to multiarrangements.

Theorem 2.21. [4] *With the notations above, any two of the following statements imply the third:*

(i) $(\mathcal{A}, \mathbf{m})$ *is free with exponents* $(d_1, \dots, d_\ell)$.
(ii) $(\mathcal{A}', \mathbf{m}')$ *is free with exponents* $(d_1, \dots, d_\ell - 1)$.
(iii) $(\mathcal{A}'', \mathbf{m}^*)$ *is free with exponents* $(d_1, \dots, d_{\ell-1})$.

Using Theorem 2.21, one can construct a lot of free multiarrangements inductively.

2.5. Free arrangements and intersection of plane curves

In this section we consider a 3-dimensional free arrangement $\mathcal{A}$ with exponents $(1, e_1, e_2)$. Choose $H_0 \in \mathcal{A}$. Then the deconing $\mathbf{d}_{H_0}\mathcal{A}$ is an affine line arrangement in $\mathbb{C}^2$. Freeness of $\mathcal{A}$ imposes strong conditions on the positions of intersections $L_2(\mathcal{A}) := \{L \cap L' \in \mathbb{C}^2 \mid L, L' \in \mathbf{d}_{H_0}\mathcal{A}, L \neq L'\}$ and their multiplicities. Let $\mu(p) := \sharp\{L \in \mathbf{d}_{H_0}\mathcal{A} \mid L \ni p\} - 1$.

Theorem 2.22. *Assume that $\mathcal{A}$ is free. With notation as above, there exist plane curves $C_1, C_2 \subset \mathbb{C}^2$ with degrees e_1 and e_2 respectively* *such that* $C_1 \cap C_2 = L_2(\mathcal{A})$ *and the intersection multiplicity is*

$$\mathrm{mult}_p(C_1, C_2) = \mu(p).$$

Remark 2.23. *If* $\mathcal{A}$ *is a fiber-type arrangement, we can find easily such* C_1 *and* C_2 *as union of lines.*

Proof. Choose coordinates (z_0, z_1, z_2) so that $H_0 = \{z_0 = 0\}$. We can choose a basis $\theta_E, \delta_1, \delta_2 \in D(\mathcal{A})$ such that $\delta_1 z_0 = \delta_2 z_0 = 0$ (see Lemma 2.13). Let $\alpha = a_1 z_1 + a_2 z_2$ be a linear form such that the line $\{\alpha = 0\} \subset \mathbb{C}^2$ is not parallel to any line $L \in \mathbf{d}_{H_0}\mathcal{A}$. By definition $\delta_i \alpha \in \mathbb{C}[z_1, z_2]$ is a polynomial of degree e_i. Note that

$$\delta_i \alpha = 0 \text{ at } p \in \mathbb{C}^2 \Longleftrightarrow \begin{cases} \delta_i(p) = 0 \text{ or} \\ \delta_i(p) \text{ is parallel to } \{\alpha = 0\}. \end{cases}$$

If $p \notin \bigcup_{L \in \mathbf{d}_{H_0}\mathcal{A}} L$, $\delta_1(p)$ and $\delta_2(p)$ are linearly independent. If $p \in L$ and $p \notin L_2(\mathcal{A})$, $\delta_1(p)$ and $\delta_2(p)$ span the tangent space T_pL. In any case, either $\delta_1(p)\alpha \neq 0$ or $\delta_2(p)\alpha \neq 0$. Hence $\delta_1(p)\alpha = \delta_2(p)\alpha = 0$ precisely when $p \in L_2(\mathbf{d}_{H_0}\mathcal{A})$. Fix $p \in L_2(\mathcal{A})$ and choose the coordinate (z_1, z_2) such that $p = (0, 0)$ and $\{z_1 = 0\} \in \mathbf{d}_{H_0}\mathcal{A}$. Let Q be the product of defining equations which contain p. Then

$$\eta_1 = z_1 \partial_{z_1} + z_2 \partial_{z_2} \text{ and } \eta_2 = \frac{Q}{z_1} \partial_{z_2}$$

form a basis of $D(\mathbf{d}_{H_0}\mathcal{A}_p)$. It is easily seen that $\mathrm{mult}_p(\eta_1\alpha, \eta_2\alpha) = \dim \mathbb{C}[[z_1, z_2]]/(\eta_1\alpha, \eta_2\alpha) = \mu(p)$. Germs of δ_i at p is expressed as $\delta_i = f_{i1}\eta_1 + f_{i2}\eta_2$ with det $= f_{11}f_{22} - f_{12}f_{21}$ is contained in the unit $\mathbb{C}[[z_1, z_2]]^\times$. Thus intersection multiplicity is

$$\mathrm{mult}_p(C_1, C_2) = \dim \mathbb{C}[[z_1, z_2]]/(\delta_1\alpha, \delta_2\alpha) = \mu(p).$$

Q.E.D.

Remark 2.24. *Although there exists a free arrangement which has non-vanishing homotopy group* $\pi_2(M(\mathcal{A}))$, [16], *it is challenging to see the homotopy types of free arrangements.*

2.6. An example of a non-free arrangement

Factorization of the characteristic polynomial (Corollary 2.10) is a necessary combinatorial condition for an arrangement to be free. However the converse is not true. Indeed, there are non-free arrangements which have factored characteristic polynomials.

Example 2.25. *(Stanley's example [23]) Let $\mathcal{A} = \{H_0, H_1, \ldots, H_6\}$ be an arrangement of 7 planes in $\mathbb{R}^3$ defined as Figure 1 (real lines). The characteristic polynomial is $\chi(\mathcal{A}, t) = (t-1)(t-3)^2$. However $\mathcal{A}$ is not free. We shall give three proofs.*

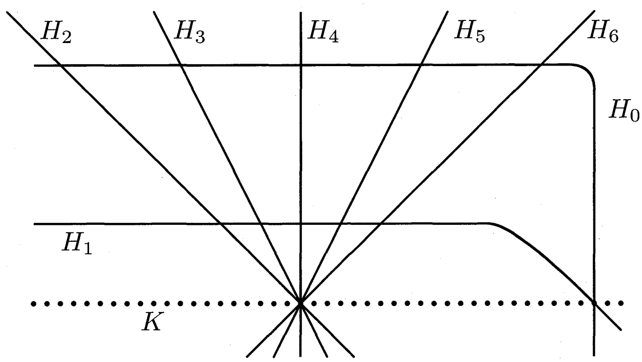

Fig. 1. $\mathcal{A} = \{H_0, \ldots, H_6\}$

First note that by 2.10, if $\mathcal{A}$ is free, then the exponents should be $(1, 3, 3)$.

(1) Consider another hyperplane K (dotted line). The extended arrangement $\mathcal{A} \cup \{K\}$ is of fiber-type and hence free with exponents $(1, 2, 5)$ (also easily proved by using Addition-Deletion Theorem 2.21). Hence $D(\mathcal{A} \cup \{K\})$ has degree 2 element δ which is linearly independent from the Euler vector field θ_E. By definition, $\delta \in D(\mathcal{A})$. However this contradicts the fact that $D(\mathcal{A})$ does not have a basis element of degree ≤ 2 other than θ_E.

(2) Consider the restriction to H_0. Then $(\mathcal{A}^{H_0}, m^{H_0})$ is free with exponents $(1, 5)$. From Corollary 2.17, $\mathcal{A}$ is not free.

(3) Consider the deconing $\mathbf{d}_{H_0}\mathcal{A}$ with respect to H_0. If $\mathcal{A}$ is free, then by Theorem 2.22, the intersections satisfy $L(\mathbf{d}_{H_0}\mathcal{A}) = C_1 \cap C_2$, where C_i is a cubic curve. We may assume that C_1 does not have H_1 as a component. Then $H_1 \cap C_1$ consists of five points. This contradicts Bezout theorem.

§3. Coxeter multiarrangements

Coxeter multiarrangements are a well-studied class of multiarrangements. Using the notion of primitive derivation, we can construct a basis for several Coxeter multiarrangements. Here we give a brief review.

The importance of the primitive derivation was first realized by K. Saito [27] in the context of singularity theory. K. Saito's theory of primitive forms reveals that the parameter space B of semi-universal deformation $X \to B$ of an isolated singularity $0 \in X_0$ possesses rich geometric structures [26, 19]. On the other hand, Grothendieck–Brieskorn–Slodowy's theory [11] shows that for simple singularities, the semi-universal family can be described in terms of Lie theory. In particular, the parameter space B can be canonically identified with the Weyl group quotient $\mathfrak{h}/W$ of an ADE-type Cartan subalgebra $\mathfrak{h}$ (see also §4.1). In [27], Saito describes the flat structure for any finite reflection group $W \curvearrowright V$ in purely invariant theoretic way by using the primitive derivation. Later Terao [40, 41] pulled back the theory to V via the natural projection $\pi : V \to V/W$ and proved freeness of Coxeter multiarrangements with constant multiplicity.

In this section, we will describe the structure of $D(\mathcal{A}, \mathbf{m})$ for a Coxeter arrangement $\mathcal{A}$ based on [40, 41, 49, 6].

Let V be an ℓ-dimensional Euclidean space over $\mathbb{R}$ with inner product $I : V \times V \to \mathbb{R}$. Fix a coordinate $(x_1, \cdots, x_\ell)$ and put $S = S(V^*) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x_1, \ldots, x_\ell]$. Let $W \subset O(V, I)$ be a finite irreducible reflection group with the Coxeter number h. Let $\mathcal{A}$ be the corresponding Coxeter arrangement, i.e., the collection of all reflecting hyperplanes of W. Fix a defining linear form $\alpha_H \in V^*$ for each hyperplane $H \in \mathcal{A}$.

It is proved by Chevalley [13] that the invariant ring S^W is a polynomial ring $S^W = \mathbb{C}[P_1, \ldots, P_\ell]$ with $P_1, \ldots, P_\ell$ are homogeneous generators. Suppose that $\deg P_1 \leq \cdots \leq \deg P_\ell$. Then it is known that $\deg P_1 = 2 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell = h$. Note that we may choose $P_1(x) = I(x, x)$. Then $\frac{\partial}{\partial P_i}$ $(i = 1, \ldots, \ell)$ can be considered as a rational vector field on V with order one poles along $H \in \mathcal{A}$. Indeed by using the fact

$$\Delta := \det \left(\frac{\partial P_i}{\partial x_j} \right)_{i,j=1,\ldots,\ell} \doteq \prod_{H \in \mathcal{A}} \alpha_H,$$

we may define the action of the differential operator $\frac{\partial}{\partial P_i}$ to $f \in S$ by

$$\frac{\partial f}{\partial P_i} = \frac{1}{\Delta} \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial P_\ell}{\partial x_1} \\ \frac{\partial P_1}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial P_\ell}{\partial x_2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \cdots & \frac{\partial f}{\partial x_\ell} & \cdots & \frac{\partial P_\ell}{\partial x_\ell} \end{pmatrix}.$$

Obviously, we have $\frac{\partial P_i}{\partial P_i} = 1$ and $\frac{\partial P_j}{\partial P_i} = 0$ for $i \neq j$.

Definition 3.1. *We denote* $D = \frac{\partial}{\partial P_\ell}$ *and call it the* primitive derivation.

Since $\deg P_i < \deg P_\ell$ for $i \le \ell - 1$, the primitive derivation D is uniquely determined up to nonzero constant multiple independent of the choice of the generators $P_1, \ldots, P_\ell$.

Next we define the affine connection ∇.

Definition 3.2. *For a given rational vector field* $\delta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$ *and a rational differential k-form* $\omega = \sum_{i_1,\ldots,i_k} g_{i_1,\ldots,i_k} dx_{i_1,\ldots,i_k}$ *(where* $dx_{i_1,\ldots,i_k} = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$*), define* $\nabla_\delta \omega$ *by*

$$\nabla_\delta \omega = \sum_{i_1,\ldots,i_k} \delta(g_{i_1,\ldots,i_k}) dx_{i_1,\ldots,i_k}.$$

Let $\mathbf{m} : \mathcal{A} \longrightarrow \{0,1\}$ be a map. The differentiation ∇_D by the primitive derivation changes the degree by h. This action connects $D(\mathcal{A}, \mathbf{m})$ with $D(\mathcal{A}, 2k + \mathbf{m})$ and $\Omega^1(\mathcal{A}, 2k - \mathbf{m})$.

Theorem 3.3. *Fix notation as above, and let k be a positive integer.*

(1) *The map*

$$\begin{array}{rccc} \Phi_k : & D(\mathcal{A}, \mathbf{m})(kh) & \longrightarrow & \Omega^1(\mathcal{A}, 2k - \mathbf{m}) \\ & \delta & \longmapsto & \nabla_\delta \nabla_D^k dP_1 \end{array}$$

gives an S-isomorphism of graded modules.

(2) *The map*

$$\begin{array}{rccc} \Psi_k : & D(\mathcal{A}, \mathbf{m})(-kh) & \longrightarrow & D(\mathcal{A}, 2k + \mathbf{m}) \\ & \delta & \longmapsto & \nabla_\delta \nabla_D^{-k} E \end{array}$$

gives an S-isomorphism of graded modules.

Corollary 3.4. *For a* $\{0,1\}$*-valued multiplicity* $\mathbf{m} : \mathcal{A} \to \{0,1\}$ *and an integer* $k > 0$*, the following conditions are equivalent.*

- $(\mathcal{A}, \mathbf{m})$ *is free with exponents* $(e_1, \ldots, e_\ell)$.
- $(\mathcal{A}, 2k + \mathbf{m})$ *is free with exponents* $(kh + e_1, \ldots, kh + e_\ell)$.
- $(\mathcal{A}, 2k - \mathbf{m})$ *is free with exponents* $(kh - e_1, \ldots, kh - e_\ell)$.

If $\mathbf{m} \equiv 0$, then $(\mathcal{A}, \mathbf{m})$ is free with exponents $(0, \ldots, 0)$. Hence $(\mathcal{A}, 2k)$ is free with exponents $(kh, kh, \ldots, kh)$. If $\mathbf{m} \equiv 1$, then $(\mathcal{A}, \mathbf{m})$ is free with exponents $(e_1, \ldots, e_\ell)$, where $e_i = \deg P_i - 1$ (by [25, 27]). Hence $(\mathcal{A}, 2k + 1)$ is free with exponents $(e_1 + kh, \ldots, e_\ell + kh)$. In particular, Coxeter multiarrangements with constant multiplicities are free [40].

The primitive derivation acts on W-invariant forms. The following will be used in the next section.

Theorem 3.5. [41] *With notation as above, the set of W-invariant derivations $D(\mathcal{A}, 2k+1)^W$ is a free S^W-module. Furthermore, if $k > 0$,*

$$\nabla_{\frac{\partial}{\partial P_i}} D(\mathcal{A}, 2k+1)^W \subset D(\mathcal{A}, 2k-1)^W,$$

$$\nabla_{\frac{\partial}{\partial P_\ell}} D(\mathcal{A}, 2k+1)^W = D(\mathcal{A}, 2k-1)^W.$$

Remark 3.6. *Recently Theorem 3.3 and Corollary 3.4 are generalized for $\mathbf{m} : \mathcal{A} \longrightarrow \{-1, 0, +1\}$ by Abe* [1].

§4. Applications of freeness of $D(\mathcal{A}, \mathbf{m})$

In this section we will describe two applications of freeness of $D(\mathcal{A}, \mathbf{m})$.

4.1. Relative de Rham cohomology of adjoint quotient maps

Let $\mathfrak{g}$ be a simple Lie algebra of type ADE over $\mathbb{C}$. The categorical quotient map $\chi : \mathfrak{g} \to B := \mathfrak{g}//G$ of the adjoint group action on $\mathfrak{g}$ is called the adjoint quotient map. The purpose of this section is to investigate the $\mathcal{O}_B$-module structure of the relative de Rham cohomology $H^2(\Omega^\bullet_\chi)$ (see below the definition of $\Omega^\bullet_\chi$) of $\chi : \mathfrak{g} \to B$ through an action of vector fields Der_B on B (the Gauß–Manin connection).

The study of the relative de Rham cohomology for an affine morphism goes back to E. Brieskorn [10] who proved the coherence of relative de Rham cohomology for any polynomial map $f : \mathbb{C}^n \to \mathbb{C}$ with isolated critical point $0 \in \mathbb{C}^n$ (and M. Sebastiani proved $\mathcal{O}_\mathbb{C}$-freeness of rank μ, where μ is the Milnor number of f). Further, K. Saito proved the freeness for the semi-universal deformation $F : X \to B := \mathbb{C}^\mu$ of an isolated hypersurface singularity defined by f. More precisely, he gave an isomorphism between a certain submodule of vector fields Der_B on B and $H^n(\Omega^\bullet_{X/B})$. The isomorphism is given by the following correspondence, we first fix a special cohomology class ζ called a primitive form, then for given vector field $\delta \in \mathrm{Der}_B$ take a lift up $\tilde{\delta} \in \mathrm{Der}_X$ of the vector field on the total space X, and differentiate ζ by $\tilde{\delta}$, we have a new cohomology class $\mathcal{L}_{\tilde{\delta}}\zeta$, where $\mathcal{L}$ is the Lie derivative. On the other hand, the semi-universal deformation of a simple singularity is constructed by using the adjoint quotient map χ of type ADE [11, 33].

Indeed, if we restrict the map χ to a certain affine subspace $X \subset \mathfrak{g}$, we have the semi-universal deformation of a simple singularity. In this case H. Yamada [48] showed that the restriction of the Kostant–Kirillov form ζ to X becomes the primitive form which generates the relative de Rham cohomology $H^2(\Omega^\bullet_{X/B})$ by differentiation by means of vector fields $\delta \in \mathrm{Der}_B$.

Let us introduce some notation. Let $\mathfrak{g}$ a simple Lie algebra over $\mathbb{C}$ (later we will restrict $\mathfrak{g}$ to ADE-type). Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha, (\Phi \subset \mathfrak{h}^*)$ a Cartan decomposition with respect to a Cartan algebra $\mathfrak{h}$ with $\ell = \dim \mathfrak{h}$, G the adjoint group of $\mathfrak{g}$ and T the maximal torus of G with Lie algebra $\mathfrak{h}$. We denote by W the Weyl group $N_G(T)/T$. The classical Chevalley's restriction theorem states that the restriction $\rho : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ of polynomial functions induces an isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W \tag{7}$$

of algebras of invariants. We also denote $\mathbb{C}[\mathfrak{h}] = S$ and $S^W = \mathbb{C}[P_1, \dots, P_\ell]$ as in §3. The categorical quotient of the adjoint action is $B = \mathfrak{g}//G \cong \mathfrak{h}//W \cong \mathrm{Spec}\, S^W$. We call the quotient map $\chi : \mathfrak{g} \to B$ the adjoint quotient map as mentioned above. The construction is summarized in the following diagram.

(8)

$$\begin{array}{ccccccc} & & \mathfrak{g} & & & & \mathbb{C}[\mathfrak{g}] \\ & & \downarrow & & & & \uparrow \\ V = \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W = B = \mathfrak{g}//G & , & S & \hookleftarrow & S^W = \mathbb{C}[B] = \mathbb{C}[\mathfrak{g}]^G. \end{array}$$

Definition 4.1. *Define the relative de Rham complex $\Omega^\bullet_\chi$ for the adjoint quotient map $\chi : \mathfrak{g} \to B$ by*

$$\Omega^\bullet_\chi = \frac{\Omega^\bullet_\mathfrak{g}}{\chi^*\Omega^1_B \wedge \Omega^{\bullet-1}_\mathfrak{g}} = \frac{\Omega^\bullet_\mathfrak{g}}{\sum_{i=1}^\ell dP_i \wedge \Omega^{\bullet-1}_\mathfrak{g}}.$$

By the formula $d(P \cdot \omega) = dP \wedge \omega + P \cdot d\omega$, the differential $d_\chi : \Omega^\bullet_\chi \to \Omega^{\bullet+1}_\chi$ is a S^W-module homomorphism. Hence the cohomology group $H^k(\Omega^\bullet_\chi)$ possesses S^W-module structure.

Let $D \subset B$ be the set of critical points of the quotient map $\pi : \mathfrak{h} \to B$. It is proved in [25] that π induces an isomorphism

$$D(\mathcal{A})^W \xrightarrow{\cong} \mathrm{Der}_B(-\log D).$$

Thus for $\delta \in D(\mathcal{A})^W$, we may differentiate the Kostant-Kirillov form ζ by δ and obtain a relative 2-form $\nabla_\delta\zeta$ (which has poles along D in general).

After Yamada's result, it was naturally conjectured that $H^2(\Omega^\bullet_\chi)$ is a free S^W-module of rank ℓ.

Theorem 4.2. *Let $\mathfrak{g}$ a simple Lie algebra of type ADE, with a Cartan subalgebra $\mathfrak{h}$ and the Weyl group W. Let $\mathcal{A}$ be the corresponding Weyl arrangement on $\mathfrak{h}$. The map $D(\mathcal{A})^W \ni \delta \longmapsto \nabla_\delta \zeta$ induces a natural isomorphism*

$$H^2(\Omega^\bullet_\chi) \cong D(\mathcal{A}, 5)^W,$$

of S^W-modules.

The rest of this section is devoted to a proof of this theorem.

We first recall a result due to J. Vey [47], which is an analogue of Weyl's unitary trick.

Theorem 4.3. *Let G be a connected reductive algebraic group over $\mathbb{C}$ with a linear action on a finite dimensional $\mathbb{C}$-vector space E. Let $\Omega^\bullet_E$ be the de Rham complex of holomorphic differential forms on E and $\mathcal{I}^\bullet$ the ideal of $\Omega^\bullet$ generated by differentials $df_1, df_2, \cdots, df_r$, where $f_1, f_2, \cdots, f_r$ are G-invariant homogeneous polynomials on E. Then the morphism*

$$(\Omega^\bullet)^G/(\mathcal{I}^\bullet)^G \to \Omega^\bullet/\mathcal{I}^\bullet$$

is a quasi-isomorphism.

By this, we can compute cohomology of $\Omega^\bullet_\chi$ by using the complex $\Omega^{G,\bullet}_{\mathfrak{g}}/(\sum_i dP_i \wedge \Omega^{\bullet-1}_{\mathfrak{g}})^G$ of G-invariant relative forms. Next we shall describe $\Omega^{G,\bullet}_{\mathfrak{g}}$.

Broer [12] considered a generalization of Chevalley's restriction theorem (7) in the following setting. Let M be a finite dimensional G-module and $\mathrm{Mor}(\mathfrak{g}, M)$ (resp. $\mathrm{Mor}_G(\mathfrak{g}, M)$) the space of polynomial (resp. G-equivariant polynomial) morphisms of $\mathfrak{g}$ into M. It is isomorphic to $\mathbb{C}[\mathfrak{g}] \otimes M$ (resp. $(\mathbb{C}[\mathfrak{g}] \otimes M)^G$). For any G-module M the restriction map ρ induces a homomorphism

$$\rho_M : \mathrm{Mor}_G(\mathfrak{g}, M) \longrightarrow \mathrm{Mor}_W(\mathfrak{h}, M^T).$$

Since the union of all Cartan subalgebras is Zariski dense in $\mathfrak{g}$, ρ_M is injective for all M. If $M = \mathbb{C}$ is a trivial G-module, ρ_M is bijective because of Chevalley's theorem. However it is not necessarily bijective in general. Broer [12] proved that

Theorem 4.4. *Let M be a G-module. Restriction induces an isomorphism*

$$\rho_M : \mathrm{Mor}_G\left(\mathfrak{g}, M\right) \xrightarrow{\cong} \mathrm{Mor}_W\left(\mathfrak{h}, M^T\right)$$

if and only if the weights 2α ($\alpha \in \Phi$ is a root of $\mathfrak{g}$) do not occur as T-weights in M. (We shall call M small *if it satisfies this assumption.)*

We need this theorem to describe the set of G-invariant differential forms $\Omega_{\mathfrak{g}}^{\bullet,G}$ on $\mathfrak{g}$ below. By definition the set of all differential p-forms on $\mathfrak{g}$ is $\Omega_{\mathfrak{g}}^p = \mathbb{C}[\mathfrak{g}] \otimes \overset{p}{\wedge} \mathfrak{g}^*$. Thus we apply Theorem 4.4 for $M = \overset{p}{\wedge} \mathfrak{g}^*$. If $p = 1$, since $\mathfrak{g} \cong \mathfrak{g}^*$ by Killing form, the T-weights of $\mathfrak{g}^*$ are nothing but the roots of $\mathfrak{g}$, so $\mathfrak{g}^*$ is small. Thus we have an isomorphism

$$\rho_1 : \Omega_{\mathfrak{g}}^{1,G} \cong (\mathbb{C}[\mathfrak{h}] \otimes \mathfrak{h}^*)^W \cong \Omega_{\mathfrak{h}}^{1,W}.$$

It follows from a result of Solomon [34] that

$$\pi^* : \Omega_B^p \xrightarrow{\cong} \Omega_{\mathfrak{h}}^{p,W}. \tag{9}$$

Thus we conclude that G-invariant 1-forms $\Omega_{\mathfrak{g}}^{1,G}$ on $\mathfrak{g}$ are nothing but the pull back $\chi^*\Omega_B^1$ of 1-forms on B. In particular, $\Omega_{\chi}^{1,G} = 0$, we have

$$H^2(\Omega_{\chi}^{\bullet,G}) \cong \mathrm{Ker}(d_{\chi} : \Omega_{\chi}^{2,G} \to \Omega_{\chi}^{3,G}). \tag{10}$$

From the classification of simple root systems, it is easily seen that $M = \overset{2}{\wedge} \mathfrak{g}^*$ is small if and only if $\mathfrak{g}$ is of type ADE, since the set of weights of $\overset{2}{\wedge} \mathfrak{g}^*$ is

$$\{0\} \cup \Phi \cup \{\alpha + \beta | \alpha, \beta \in \Phi, \alpha \neq \beta\}.$$

Furthermore, $(\overset{2}{\wedge} \mathfrak{g}^*)^T \cong \overset{2}{\wedge} \mathfrak{h}^* \otimes (\overset{2}{\wedge} \mathfrak{h}^{\perp})^T$ is a direct sum decomposition of W-submodules. From (9), we obtain,

Proposition 4.5. *Let $\mathfrak{g}$ be a simple Lie algebra of type ADE. Then*

$$\Omega_{\mathfrak{g}}^{2,G} \cong \chi^*\Omega_B^2 \oplus \Omega_{\chi}^{2,G}. \tag{11}$$

$$\rho_2 : \Omega_{\chi}^{2,G} \xrightarrow{\cong} \left(\mathbb{C}[\mathfrak{h}] \otimes (\overset{2}{\wedge} \mathfrak{h}^{\perp})^T\right)^W. \tag{12}$$

By Proposition 4.5, we can identify G-invariant relative 2-forms $\Omega_{\chi}^{2,G}$ with the submodule $(\mathbb{C}[\mathfrak{h}] \otimes (\overset{2}{\wedge} \mathfrak{h}^{\perp})^T)^W$ of $\Omega_{\mathfrak{g}}^{2,G}$. Now we define two submodules $\mathcal{H}_{\chi} \subset \mathcal{H}'_{\chi} \subset \Omega_{\mathfrak{g}}^{2,G}$ which are related to $H^2(\Omega_{\chi}^{\bullet})$.

Definition 4.6.

$$\mathcal{H}_{\chi} := \{\omega \in \Omega_{\chi}^{2,G} \mid d_{\mathfrak{g}}\omega \in \sum_i dP_i \wedge \Omega_{\mathfrak{g}}^{2,G}\}, \tag{13}$$

(14) $$\mathcal{H}'_\chi := \{\omega \in \Omega_\chi^{2,G} \mid d_{\mathfrak{g}}\omega \wedge dP_1 \wedge \cdots \wedge dP_\ell = 0 \ in \ \Omega_{\mathfrak{g}}^{3+\ell}\},$$

where $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \cdots, P_\ell]$.

By (10) and definition above,

(15) $$\mathcal{H}_\chi \cong H^2(\Omega_\chi^{\bullet,G})$$

and obviously $\mathcal{H}_\chi \subset \mathcal{H}'_\chi$. Later it will be proved that $\mathcal{H}_\chi \subsetneqq \mathcal{H}'_\chi$.

Let $e_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Phi$) be non-zero root vectors such that $I([e_\alpha, e_{-\alpha}], h) = \alpha(h)$ (for all $\alpha \in \Phi,\ h \in \mathfrak{h}$), where $I(\bullet, \bullet)$ is the Killing form, and $e_\alpha^* \in \mathfrak{g}_\alpha^*$ be the dual basis. Then each element of $(\mathbb{C}[\mathfrak{h}] \otimes (\overset{2}{\wedge}\, \mathfrak{h}^\perp)^T)^W$ can be expressed in the form

$$\omega = \sum_{\alpha \in \Phi^+} f_\alpha \cdot e_\alpha^* \wedge e_{-\alpha}^* \in \left(\mathbb{C}[\mathfrak{h}^*] \otimes (\overset{2}{\wedge}\, \mathfrak{h}^\perp)^T\right)^W.$$

Since ω is W-invariant, if we apply the simple reflection $s_\alpha \in W$ with respect to a root $\alpha \in \Phi^+$ to ω we have $s_\alpha f_\alpha = -f_\alpha$. Hence f_α is divisible by α.

Next let us recall the definition of the Kostant–Kirillov form. The Kostant–Kirillov form ζ is a symplectic form on the (co)adjoint orbit $G \cdot x \subset \mathfrak{g}$ of $x \in \mathfrak{g}$. Let $Y, Z \in \mathfrak{g}$. Then $[Y, x] = \frac{d}{dt}\big|_{t=0} \mathrm{ad}(e^{tX})x \in T_x(G \cdot x)$. For two tangent vectors $[Y, x], [Z, x] \in T_x(G \cdot x)$, the 2-form ζ is given by the formula

$$\zeta([Y, x], [Z, x]) = I(x, [Y, Z]),$$

where $[Y, Z]$ is the bracket in $\mathfrak{g}$.

Proposition 4.7. *By restricting the Kostant–Kirillov form ζ to $\mathfrak{h}$, we have the following expression*

(16) $$\rho_2(\zeta) = -\sum_{\alpha \in \Phi^+} \frac{e_\alpha^* \wedge e_{-\alpha}^*}{\alpha}.$$

Proof. Let $h \in \mathfrak{h} \backslash \bigcup H_\alpha$. We compute $\zeta([e_\alpha, h], [e_\beta, h])$ in two ways. First, using the property $[h, e_\alpha] = \alpha(h)e_\alpha$, we have $\zeta([e_\alpha, h], [e_\beta, h]) = \alpha(h)\beta(h)\zeta(e_\alpha, e_\beta)$. On the other hand, using the definition of ζ, we have $\zeta([e_\alpha, h], [e_\beta, h]) = I(h, [e_\alpha, e_\beta])$. Note that it is non-zero only if $\beta = -\alpha$, and in this case, we have $I(h, [e_\alpha, e_{-\alpha}]) = \alpha(h)$. Hence we have $\zeta(e_\alpha, e_{-\alpha}) = \frac{-1}{\alpha(h)}$, which implies (16). Q.E.D.

The generic fiber of $\chi : \mathfrak{g} \to B$ is isomorphic to G/T, which is homotopy equivalent to the flag manifold of G. We recall the Borel–Hirzebruch description of the de Rham cohomology of G/T in degree 2 [9]. A G-invariant differential form on G/T can be seen as a G-invariant section of the vector bundle $\overset{\bullet}{\wedge} T^*(G/T)$. Hence the evaluation at the base point $[T] \in G/T$ induces an isomorphism

$$(17) \qquad \lambda : \Omega^{\bullet,G}_{G/T} \xrightarrow{\cong} \left(\overset{\bullet}{\wedge} T^*_{[T]}(G/T)\right)^T \cong \left(\overset{\bullet}{\wedge} \mathfrak{h}^{\perp}\right)^T.$$

For degree 2, the above map induces the isomorphism $\Omega^{\bullet,G}_{G/T} \cong (\wedge^2 \mathfrak{h}^{\perp})^T = \bigoplus_{\alpha\in\Phi^+}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Using the map λ, we can show that $H^2(G/T, \mathbb{C}) \cong \mathfrak{h}$.

Theorem 4.8. [9]

$$\begin{array}{rccc} \omega : & \mathfrak{h} & \longrightarrow & \Omega^{2,G}_{(G/T)} \\ & h & \longmapsto & \sum_{\alpha\in\Phi} \alpha(h)\cdot\lambda^{-1}(e^*_\alpha \wedge e^*_{-\alpha}) \end{array}$$

induces an isomorphism of $\mathbb{C}$-vector spaces $\mathfrak{h} \xrightarrow{\cong} H^2(G/T, \mathbb{C})$. In particular, $\omega(h)$ is closed,

$$(18) \qquad \sum_{\alpha\in\Phi} \alpha(h)\cdot d\lambda^{-1}(e^*_\alpha \wedge e^*_{-\alpha}) = 0.$$

Next we consider the relative de Rham cohomology for the projection

$$\mathrm{pr} : \mathfrak{h} \times G/T \to \mathfrak{h}.$$

We may consider G acts on each fiber of pr from the left. Since $\Omega^2_{\mathrm{pr}} \cong \mathbb{C}[\mathfrak{h}] \otimes \Omega^2_{G/T}$, the set of G-invariant relative 2-forms Ω^2_{pr} is described by the following isomorphism

$$(19) \qquad 1 \otimes \lambda : \Omega^2_{\mathrm{pr}} \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}] \otimes (\overset{2}{\wedge} \mathfrak{h}^{\perp})^T.$$

By definition and Theorem 4.8,

$$H^2(\Omega^\bullet_{\mathrm{pr}}) \cong \mathbb{C}[\mathfrak{h}] \otimes H^2(\Omega^\bullet_{(G/T)}) \cong \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{h} \cong \mathrm{Der}_{\mathfrak{h}}\,.$$

The isomorphism is given by

Theorem 4.9.

$$(20) \qquad \begin{array}{rccc} 1\otimes\omega : & \mathrm{Der}_{\mathfrak{h}} & \longrightarrow & \mathbb{C}[\mathfrak{h}] \otimes \Omega^{2,G}_{(G/T)} \cong \Omega^{2,G}_{\mathrm{pr}} \\ & \delta & \longmapsto & (1\otimes\omega)\delta = \sum_{\alpha\in\Phi} (\delta\alpha)\cdot\lambda^{-1}(e^*_\alpha \wedge e^*_{-\alpha}) \end{array}$$

induces a $\mathbb{C}[\mathfrak{h}]$*-module isomorphism* $\mathrm{Der}_{\mathfrak{h}} \cong H^2(\Omega^\bullet_{\mathrm{pr}})$*, where* $\delta\alpha$ *is the differentiation of a function* α *by a vector field* δ*.*

We define a submodule $\mathcal{H}_{\mathrm{pr}} \subset \Omega^{2,G}_{(G/T)\times\mathfrak{h}}$ as

$$\mathcal{H}_{\mathrm{pr}} := \left\{ \sum_{\alpha\in\Phi} (\delta\alpha)\cdot\lambda^{-1}(e^*_\alpha \wedge e^*_{-\alpha}) \in \Omega^{2,G}_{(G/T)\times\mathfrak{h}} \,\middle|\, \delta \in \mathrm{Der}_{\mathfrak{h}} \right\}.$$

From the decomposition $\Omega^{2,G}_{(G/T)\times\mathfrak{h}} = \bigoplus_{i+j=2} \Omega^{i,G}_{G/T} \wedge \Omega^j_{\mathfrak{h}}$ and $\Omega^{2,G}_{\mathrm{pr}} \cong \mathbb{C}[\mathfrak{h}] \otimes \Omega^{2,G}_{G/T}$, we may consider

$$\Omega^{2,G}_{(G/T)\times\mathfrak{h}} \supset \Omega^{2,G}_{\mathrm{pr}} \supset \mathcal{H}_{\mathrm{pr}} \ \left(\cong H^2(\Omega^\bullet_{\mathrm{pr}})\right). \tag{21}$$

Let $x_1, \cdots, x_\ell$ be a coordinate system of $\mathfrak{h}$, then $\mathcal{H}_{\mathrm{pr}}$ has another expression as in (14):

$$\mathcal{H}_{\mathrm{pr}} = \left\{ \omega \in \Omega^{2,G}_{\mathrm{pr}} \,\middle|\, d\omega \wedge dx_1 \wedge \cdots \wedge dx_\ell = 0 \text{ in } \Omega^{3+\ell,G}_{(G/T\times\mathfrak{h})} \right\}. \tag{22}$$

To study the relative de Rham cohomology of $\chi : \mathfrak{g} \to B$, we consider the following diagram

$$\begin{array}{ccc} (G/T)\times\mathfrak{h} & \xrightarrow{\widetilde{\pi}} & \mathfrak{g} \\ \mathrm{pr}\downarrow & & \downarrow\chi \\ \mathfrak{h} & \xrightarrow{\pi} & B \end{array} \quad \left(\begin{array}{ccc} (g[T],h) & \longmapsto & \mathrm{ad}(g)h \\ \downarrow & & \downarrow \\ h & \longmapsto & \overline{h} \end{array} \right). \tag{23}$$

More precisely, from diagram (23), there is a natural homomorphism

$$\widetilde{\pi}^* : H^2\left(\Omega^\bullet_\chi\right) \hookrightarrow H^2\left(\Omega^\bullet_{\mathrm{pr}}\right),$$

which is injective because we have realized these cohomology groups as subspaces of absolute differential forms (see (15) and (21)). We consider the image of $H^2(\Omega^\bullet_\chi)$ in $H^2(\Omega^\bullet_{\mathrm{pr}}) \cong \mathrm{Der}_{\mathfrak{h}}$. Note that if we define a W action on $G/T \times \mathfrak{h}$ by

$$w \cdot (g[T], h) = (gn_w^{-1}, \mathrm{ad}(n_w)h),$$

where $n_w \in N_G(T)$ is a representative of $w \in W = N_G(T)/T$, then obviously $\widetilde{\pi}$ is a W-invariant map and the pull back of differential form (resp. relative cohomology class) on $\mathfrak{g}$ by $\widetilde{\pi}$ becomes a W-invariant differential form (resp. W-invariant cohomology class).

$$\widetilde{\pi}^*\mathcal{H}_\chi \subset \mathcal{H}^W_{\mathrm{pr}}. \tag{24}$$

Now recall two expressions of relative 2-forms (12) in Proposition 4.5 and (19), we have a diagram:

(25)
$$\begin{array}{ccccc} \Omega^{2,G}_{\mathcal{X}} & \overset{\widetilde{\pi}^*}{\hookrightarrow} & \Omega^{2,G\times W}_{\mathrm{pr}} & \subset & \Omega^{2,G}_{\mathrm{pr}} \\ \rho_2 \downarrow \wr & & 1\otimes\lambda \downarrow \wr & & \downarrow \wr \\ \left(\mathbb{C}[\mathfrak{h}^*]\otimes(\overset{2}{\wedge}\,\mathfrak{h}^{\perp})^T\right)^W & \overset{(1\otimes\lambda)\circ\widetilde{\pi}^*\circ\rho_2^{-1}}{\longrightarrow} & \left(\mathbb{C}[\mathfrak{h}^*]\otimes(\overset{2}{\wedge}\,\mathfrak{h}^{\perp})^T\right)^W & \subset & \mathbb{C}[\mathfrak{h}^*]\otimes(\overset{2}{\wedge}\,\mathfrak{h}^{\perp})^T. \end{array}$$

We compute the map $(1\otimes\lambda)\circ\widetilde{\pi}^*\circ\rho_2^{-1}$.

Lemma 4.10. *The map* $(1\otimes\lambda)\circ\widetilde{\pi}^*\circ\rho_2^{-1} : \left(\mathbb{C}[\mathfrak{h}^*]\otimes(\overset{2}{\wedge}\,\mathfrak{h}^{\perp})^T\right)^W \to \left(\mathbb{C}[\mathfrak{h}^*]\otimes(\overset{2}{\wedge}\,\mathfrak{h}^{\perp})^T\right)^W$ *can be expressed as*

$$\sum_{\alpha\in\Phi^+}\alpha f_\alpha\cdot e_\alpha^*\wedge e_{-\alpha}^* \longmapsto -\sum_{\alpha\in\Phi^+}\alpha^3 f_\alpha\cdot e_\alpha^*\wedge e_{-\alpha}^*.$$

Proof. The derivation of $\widetilde{\pi}$ is given by

$$\begin{array}{cccc} (d\widetilde{\pi})_{([T],h)}: & T_{([T],h)}(G/T\times\mathfrak{h}) & \longrightarrow & T_h\mathfrak{g} \\ & \|\wr & & \|\wr \\ & \mathfrak{g}/\mathfrak{h}\oplus\mathfrak{h} & & \mathfrak{g} \\ & & & \\ & (\bar{X}_1, X_2) & \longmapsto & [X_1,h]+X_2. \end{array}$$

Indeed

$$(d\widetilde{\pi})_{([T],h)}(\bar{X}_1,X_2) = \left.\frac{d}{dt}\right|_{t=0}\mathrm{ad}(\exp(tX_1))(h+tX_2) = [X_1,h]+X_2.$$

In particular $\mathfrak{g}/\mathfrak{h}\oplus\mathfrak{h}\ni(\bar{e}_\alpha,0)\longmapsto -\alpha(h)e_\alpha\in\mathfrak{g}_\alpha$. Hence we have

$$\widetilde{\pi}^*(e_\alpha^*) = -\alpha(h)e_\alpha^*,\ \widetilde{\pi}^*(e_{-\alpha}^*) = \alpha(h)e_{-\alpha}^*.$$

Q.E.D.

Example 4.11. *From Proposition 4.7 and the preceding lemma, the pull back of the Kostant–Kirillov form* ζ *is*

$$\widetilde{\pi}^*(\zeta) = \sum_{\alpha\in\Phi^+}\alpha\cdot\lambda^{-1}(e_\alpha^*\wedge e_{-\alpha}^*).$$

Using the Euler vector field $\theta_E := \sum_{i=1}^{\ell} x_i\frac{\partial}{\partial x_i}$, *it is expressed as* $\widetilde{\pi}^*(\zeta) = (1\otimes\omega)(\theta_E)$.

As a corollary of Lemma 4.10, we can characterize the image of the map $\widetilde{\pi}^* : \Omega_{\mathcal{X}}^{2,G} \to \Omega_{\mathrm{pr}}^{2,G\times W}$.

Corollary 4.12.

$$(1\otimes\lambda)\circ\widetilde{\pi}^*(\Omega_{\mathcal{X}}^{2,G}) = \left\{ \sum_{\alpha\in\Phi^+} F_\alpha \cdot e_\alpha^* \wedge e_{-\alpha}^* \;\middle|\; F_\alpha \textit{ can be divisible by } \alpha^3 \right\}^W .$$

Proof. If $\sum_{\alpha\in\Phi^+} F_\alpha \cdot e_\alpha^* \wedge e_{-\alpha}^*$ is contained in the left hand side above, F_α have to be divisible by α^3 from the preceding lemma. Conversely, if F_α is divisible by α^3 for all $\alpha \in \Phi^+$, it is the image of

$$\rho_2^{-1}\left(\sum_{\alpha\in\Phi^+} \frac{F_\alpha}{\alpha^2} \cdot e_\alpha^* \wedge e_{-\alpha}^*\right) \in \Omega_{\mathcal{X}}^{2,G}.$$

Q.E.D.

Here it is possible to characterize the image of $\widetilde{\pi}^*$. We have a diagram deduced from (25),

$$\begin{array}{lccc} \mathcal{H}_{\mathcal{X}} \subset \mathcal{H}'_{\mathcal{X}} \overset{\widetilde{\pi}^*}{\hookrightarrow} & \mathcal{H}_{\mathrm{pr}}^W & \subset & \mathcal{H}_{\mathrm{pr}} \\ & (1\otimes\omega)^{-1}\downarrow\wr & & \downarrow\wr \\ & \mathrm{Der}_{\mathfrak{h}}^W & \subset & \mathrm{Der}_{\mathfrak{h}} . \end{array} \tag{26}$$

Combining (20) and Corollary 4.12, we have

Theorem 4.13. $(1\otimes\omega)^{-1}\circ\widetilde{\pi}^*$ *induces an isomorphism*

$$\mathcal{H}'_{\mathcal{X}} \cong D(\mathcal{A},3)^W.$$

Proof. For any $\delta \in \mathrm{Der}_{\mathfrak{h}}^W$, since

$$\begin{aligned} d\left((1\otimes\omega)\delta\right)\wedge dP_1\wedge\cdots\wedge dP_\ell &= Q\cdot d\left((1\otimes\omega)\delta\right)\wedge dx_1\wedge\cdots\wedge dx_\ell \\ &= 0, \end{aligned}$$

$(1\otimes\omega)\delta \in \widetilde{\pi}^*\mathcal{H}'_{\mathcal{X}}$ if and only if $\delta\alpha$ is divisible by α^3 for all $\alpha \in \Phi^+$.

Q.E.D.

Now we are in a position to prove our main result.

Theorem 4.14. $(1\otimes\omega)^{-1}\circ\widetilde{\pi}^*$ *induces an isomorphism*

$$\mathcal{H}_{\mathcal{X}} \cong D(\mathcal{A},5)^W.$$

Hence $H^2(\Omega_{\mathcal{X}}^{\bullet})$ is a free $\mathbb{C}[B]$-module of rank ℓ.

Proof. Suppose $\eta \in \mathcal{H}_\chi$ and put $\widetilde{\pi}^*\eta = \sum_{\alpha\in\Phi}(\delta\alpha)\cdot\lambda^{-1}(e_\alpha^*\wedge e_{-\alpha}^*)$ for $\delta \in D(\mathcal{A},3)^W$, then by definition there exist $\eta_1,\cdots,\eta_\ell \in \Omega_{\mathfrak{g}}^{2,G}$ such that

$$d\eta = \sum_{i=1}^{\ell} dP_i \wedge \eta_i.$$

Applying the operator d and multiplying by $dP_1\wedge\cdots\wedge\widehat{dP_i}\wedge\cdots\wedge dP_\ell$,

$$dP_1\wedge\cdots\wedge dP_i\wedge\cdots\wedge dP_\ell\wedge d\eta_i = 0.$$

Hence $\eta_i \in \mathcal{H}'_\chi$ for all $i=1,\cdots,\ell$ and $\widetilde{\pi}^*\eta_i = \sum_{\alpha\in\Phi}(\delta_i\alpha)\cdot\lambda^{-1}(e_\alpha^*\wedge e_{-\alpha}^*)$ for some $\delta_i \in D(\mathcal{A},3)^W$. Using (18), we have

$$\begin{aligned} d\widetilde{\pi}^*\eta &= \sum_{\alpha\in\Phi} d(\delta\alpha)\wedge\lambda^{-1}(e_\alpha^*\wedge e_{-\alpha}^*) \\ &= \sum_{\alpha\in\Phi}\sum_i \frac{\partial}{\partial P_i}(\delta\alpha)dP_i\wedge\lambda^{-1}(e_\alpha^*\wedge e_{-\alpha}^*) \\ &= \sum_{\alpha\in\Phi}\sum_i \left((\nabla_{\frac{\partial}{\partial P_i}}\delta)\alpha\right)dP_i\wedge\lambda^{-1}(e_\alpha^*\wedge e_{-\alpha}^*), \end{aligned}$$

and $\delta_i = \nabla_{\frac{\partial}{\partial P_i}}\delta$. By Theorem 3.5, $\eta \in \mathcal{H}_\chi$ if and only if $\delta \in D(\mathcal{A},5)^W$.

Q.E.D.

4.2. Freeness of A_n-Catalan arrangements

As another application, we prove that Catalan arrangements of type A are free.

Let $(x_1,\ldots,x_n,z)$ be a coordinate of $\mathbb{C}^{n+1}$. (The cone of) Catalan arrangement Cat_n is defined by

$$z\times\prod_{1\le i<j\le n}((x_i-x_j)(x_i-x_j-z)(x_i-x_j+z)) = 0.$$

The terminology "Catalan arrangement" comes from the fact that the number of chambers divided by $2n!$ is equal to n-th Catalan number. See [7] for more combinatorial aspects of Cat_n. (Note that the definition of Cat_n in this article is the coning of that of [7].)

Theorem 4.15. *The Catalan arrangement* Cat_n *is free with exponents* $(0,1,n+1,n+2,\ldots,2n-1)$.

Remark 4.16. *This result was first proved by Edelman and Reiner* [15] *using Addition-Deletion Theorem 2.21. It can be also proved by* *using the freeness of $D(\mathcal{A},3)$ and Theorem 2.19. We give another proof which is also based on the freeness of $D(\mathcal{A},3)$. However, instead of using Theorem 2.19, we will directly show the existence of basis of $D(\mathrm{Cat}_n)$ by invariant theoretic arguments.*

Lemma 4.17. *For non-negative integers $p, i \geq 0$, define a symmetric polynomial $F_{p,i}(x_1, x_2)$ of two variables by $F_{p,i} := \frac{x_1^{p+1} - x_2^{p+1}}{x_1 - x_2} \cdot (x_1 - x_2)^{2i}$. Then*

$$\mathbb{C}[x_1, x_2]^{\mathfrak{S}_2} = \bigoplus_{p,i \geq 0} \mathbb{C} \cdot F_{p,i}.$$

Proof. Let $F(x_1, x_2)$ be a homogeneous symmetric polynomial. We will prove F is expressed a linear combination $\{F_{p,i}\}$ by induction on $\deg F$. If $\deg F = 1$, then $F = x_1 + x_2 = F_{1,0}$. Consider the case $\deg F \geq 2$. If $F(x, x) = 0$, then we have $F = (x_1 - x_2)^2 \cdot G(x_1, x_2)$ with G symmetric. Thus by inductive hypothesis, G is a linear combination of $\{F_{p,i}\}$, and so is F. Suppose $F(x, x) = ax^n \neq 0$. Consider $G(x_1, x_2) = F - \frac{a}{n+1} F_{n,0}$. Then $G(x, x) = 0$, and is reduced to the previous case. Thus $\mathbb{C}[x_1, x_2]^{\mathfrak{S}_2}$ is spanned by $\{F_{p,i}\}$. By computing the Hilbert series, we have (note that $\deg F_{p,i} = p + 2i$)

$$H(\mathbb{C}[x_1, x_2]^{\mathfrak{S}_2}, t) = \frac{1}{(1-t)(1-t^2)} \leq H(\sum \mathbb{C} \cdot F_{p,i}, t) \leq \frac{1}{(1-t)(1-t^2)}.$$

We conclude that $\{F_{p,i}\}$ forms a basis. Q.E.D.

Let $S = \mathbb{C}[x_1, \ldots, x_n]$. We consider a subgroup $\mathfrak{S}_2 \times \mathfrak{S}_{n-2} \subset \mathfrak{S}_n$ which acts $\{x_1, x_2\}$ and $\{x_3, \ldots, x_n\}$ respectively.

Lemma 4.18.

$$S^{\mathfrak{S}_2 \times \mathfrak{S}_{n-2}} = \mathbb{C}[x_1, x_2]^{\mathfrak{S}_2} \cdot S^{\mathfrak{S}_n}.$$

Proof. The inclusion $\supseteq$ is clear. For the reverse inclusion, we first note that

$$S^{\mathfrak{S}_2 \times \mathfrak{S}_{n-2}} = \mathbb{C}[x_1, x_2]^{\mathfrak{S}_2} \otimes \mathbb{C}[x_3, \ldots, x_n]^{\mathfrak{S}_{n-2}}.$$

As is well known, $S^{\mathfrak{S}_n}$ is generated by $x_1^k + \cdots + x_n^k$ $(k \geq 0)$ as a $\mathbb{C}$-algebra. Thus

$$x_3^k + \cdots + x_n^k = (x_1^k + \cdots + x_n^k) - (x_1^k + x_2^k) \in \mathbb{C}[x_1, x_2]^{\mathfrak{S}_2} \cdot S^{\mathfrak{S}_n}.$$

Q.E.D.

Combining Lemma 4.17 and 4.18, we have.

Lemma 4.19.

$$S^{\mathfrak{S}_2\times\mathfrak{S}_{n-2}} = \sum_{p,i\geq 0} S^{\mathfrak{S}_n}\cdot F_{p,i}.$$

Now we prove Theorem 4.15. Denote $H_0 = \{z=0\}$ and $\alpha_{ij} = x_i - x_j$. The restriction $(\mathcal{A}^{H_0}, m^{H_0})$ is equal to $(A_{n-1}, 3)$. From the result of §3, $(\mathcal{A}^{H_0}, m^{H_0})$ is free with exponents $(0, n+1, \dots, 2n-1)$. We can choose basis $\eta_1, \dots, \eta_n \in D(\mathcal{A}^{H_0}, m^{H_0})^{\mathfrak{S}_n}$ from $\mathfrak{S}_n$-invariant vector fields. By definition, $\eta_i(x_1-x_2) = (x_1-x_2)^3 G_i$, and $G_i \in S^{\mathfrak{S}_2\times\mathfrak{S}_{n-2}}$. By Lemma 4.19, there exist symmetric polynomials $B_i^{p,r} \in S^{\mathfrak{S}_n}$ such that

$$G_i = \sum_{p,r\geq 0} B_i^{p,r}\cdot F_{p,r}. \tag{27}$$

With $B_i^{p,r}$ and $\mathfrak{S}_n$-invariant vector field $\delta_p = \sum_{i=1}^n x_i^p\partial_i$, let us define

$$\widetilde{\eta}_i := \eta_i - \sum_{p,r\geq 0} z^{2r+2}B_i^{p,r}\delta_{p+1}. \tag{28}$$

Then $\widetilde{\eta}_1, \dots, \widetilde{\eta}_n$ and $\theta_E = \sum_{i=1}^n x_i\partial_i + z\partial_z$ form a basis of $D(\mathrm{Cat}_n)$. Indeed, they are linearly independent over $\mathbb{C}[x_1, \dots, x_n, z]$ (since η_i's are independent), and

$$\begin{aligned}
\widetilde{\eta}_i(\alpha_{12}\pm z) &= \widetilde{\eta}_i\alpha_{12}\\
&= \eta(x_1-x_2) - \sum_{p,r\geq 0} z^{2r+2}B_i^{p,r}(x_1^{p+1}-x_2^{p+1})\\
&= (x_1-x_2)^3G_i - \sum_{p,r\geq 0} z^{2r+2}B_i^{p,r}(x_1^{p+1}-x_2^{p+1}).
\end{aligned}$$

The last polynomial is divisible by $z^2-\alpha_{12}^2$. Indeed, put $z=\pm\alpha_{12}$ in the last formula, we have from (27)

$$\begin{aligned}
&= (x_1-x_2)^3G_i - \sum_{p,r\geq 0}(x_1-x_2)^{2r+2}B_i^{p,r}(x_1^{p+1}-x_2^{p+1})\\
&= (x_1-x_2)^3\left(G_i - \sum_{p,r\geq 0}(x_1-x_2)^{2r}B_i^{p,r}\frac{x_1^{p+1}-x_2^{p+1}}{x_1-x_2}\right)\\
&= (x_1-x_2)^3\left(G_i - \sum_{p,r\geq 0} B_i^{p,r}F_{p,r}\right)\\
&= 0.
\end{aligned}$$

Since $\widetilde{\eta}_i$ is $\mathfrak{S}_n$-invariant, $\widetilde{\eta}_i \alpha_{jk}$ is divisible by $\alpha_{jk}(\alpha_{jk}^2 - z^2)$ for any j, k. This completes the proof.

§5. Concluding remarks and open problems

One of the central problems in the theory of hyperplane arrangements is to decide to what extent the structure of an arrangement is determined by the combinatorics of the arrangement.

Problem 5.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be central arrangements in $\mathbb{K}^\ell$. Assume that $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$. Does the freeness of $\mathcal{A}_1$ imply the freeness of $\mathcal{A}_2$?*

The above (by Terao [39]) is a long standing problem, even for the case $\ell = 3$, since the beginning of this area. Note that several variants of this problem are known to have counter examples:

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are multiarrangements ([55]),
- $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined over different fields ([56]),
- $\mathcal{A}_1$ and $\mathcal{A}_2$ are line-conic arrangements ([30]).

There are several characterizations for 3-arrangements to be free via Ziegler's restriction map (§2.3 and see [2] for recent developments). However the author does not know the answer to the following.

Problem 5.2. *Does the converse to Theorem 2.22 hold?*

As in §4.1 the modules of multiderivations naturally appear in the study of relative de Rham cohomology groups for the adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{h}/W$. However the idea of the proof of Theorem 4.2 works only for the type ADE and the cohomology of degree 2.

Problem 5.3. *Study the structure of the relative de Rham cohomology group $H^k(\Omega_\chi)$ as S^W-module.*

The structures of $H^2(\Omega_\chi)$ for non simply laced cases are expected to be related with the module of derivations with a non constant multiplicity. Similarly higher degree cases are expected to be related with the module of higher derivations $D^k(\mathcal{A}, m)$ defined in [3].

Postnikov and Stanley [24] observed curious properties of the characteristic polynomials that for some truncated affine Weyl arrangements, the all roots of the characteristic polynomial have the same real part ("Riemann hypothesis"). By Solomon–Terao's formula (Theorem 2.9), the characteristic polynomial is determined by the module $D(\mathcal{A})$. It would be natural to expect curious behaviours of the characteristic polynomial reflect the structure of $D(\mathcal{A})$. The following question posed by Athanasiadis ([7, Question 6.5]) is still challenging.

Problem 5.4. *Are there any natural algebraic structures of $D(\mathcal{A})$ which cause Riemann hypothesis?*

For instance, is the "functional equation" ([24, (9.12)]) deduced from the self duality (up to degree shift) $D_0(\mathcal{A}) \simeq D_0(\mathcal{A})[-d]^\vee$ of certain module?

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