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Application of arrangement theory to unfolding models

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Abstract.

Arrangement theory plays an essential role in the study of the unfolding model used in many fields. This paper describes how arrangement theory can be usefully employed in solving the problems of counting (i) the number of admissible rankings in an unfolding model and (ii) the number of ranking patterns generated by unfolding models. The paper is mostly expository but also contains some new results such as simple upper and lower bounds for the number of ranking patterns in the unidimensional case.

§1. Introduction

The unfolding model (Coombs [6], De Leeuw [8]) is a model for preference rankings in psychometrics. It is now widely applied not only in psychometrics (De Soete, Feger, and Klauer [10]) but also in other fields such as marketing science (DeSarbo and Hoffman [9]) and voting theory (Clinton, Jackman, and Rivers [5]). The model is also used as a submodel for more complex models, as in item response theory for unfolding (Andrich [1, 2]). Moreover, in the context of Voronoi diagrams, this model can be regarded as a higher-order Voronoi diagram (Okabe, Boots, Sugihara, and Chiu [22]).

The unfolding model describes the ranking process in which judges rank a set of objects in order of preference. In this model, judges and objects are assumed to be represented by points in the Euclidean space

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$\mathbb{R}^n$. Suppose a judge $y \in \mathbb{R}^n$ ranks m objects $x_1, \dots, x_m \in \mathbb{R}^n$. According to the unfolding model, y ranks $x_1, \dots, x_m$ in descending order of proximity in the usual Euclidean distance. Hence, y likes x_{i_1} best, x_{i_2} second best, and so on, iff $\|y - x_{i_1}\| < \|y - x_{i_2}\| < \cdots < \|y - x_{i_m}\|$. In this case, we will say y gives ranking $(i_1 i_2 \cdots i_m)$.

For a given m-tuple $(x_1, \dots, x_m)$ of objects, let $\mathrm{RP}^{\mathrm{UF}}(x_1, \dots, x_m)$ be the set of admissible rankings, i.e., $(i_1 \cdots i_m)$ such that $\|y - x_{i_1}\| < \cdots < \|y - x_{i_m}\|$ for some $y \in \mathbb{R}^n$. We call $\mathrm{RP}^{\mathrm{UF}}(x_1, \dots, x_m)$ the ranking pattern of the unfolding model with m-tuple $(x_1, \dots, x_m)$. In the psychometric literature, there has not been much study on the structure of the ranking pattern. In this paper, we investigate the ranking pattern by using the theory of hyperplane arrangements (Orlik and Terao [23]). Specifically, we consider the following two problems:

(i) Find the cardinality of $\mathrm{RP}^{\mathrm{UF}}(x_1, \dots, x_m)$ for a given generic m-tuple $(x_1, \dots, x_m)$;
(ii) Find the cardinality of

$$\{\mathrm{RP}^{\mathrm{UF}}(x_1, \dots, x_m) : (x_1, \dots, x_m) \text{ is a generic } m\text{-tuple}\}.$$

The first problem asks how many rankings are admissible in one unfolding model, and the second inquires how many ranking patterns are possible by using different unfolding models (that is, by taking different choices of m-tuples of objects). As we will see, these problems can be reduced to those of counting the numbers of chambers of some real arrangements; moreover, the latter problems can be solved by employing general results in the theory of hyperplane arragements (e.g., Zaslavsky's result on the number of chambers of a real arrangement, the finite field method, etc.). In this sense, arrangement theory plays an essential role in the study of the unfolding model.

This paper gives a survey of recent results ([13], [14], [15], [19]) on the problems stated above. It also contains new results on upper and lower bounds for the number of ranking patterns in the unidimensional case $n = 1$. In addition, the problem of counting inequivalent ranking patterns (i.e., those which cannot be obtained from one another by just the relabeling of the objects) when $n = 1$ was not dealt with specifically in [13] but is discussed fully in the present paper.

The organization of the paper is as follows. In Section 2, we define genericness of the unfolding model, and give the answer to problem (i) above, i.e., the number of admissible rankings of the unfolding model with generic objects. Next, in Section 3 we discuss the problem of counting the number of ranking patterns (problem (ii)). In Subsection 3.1, we deal with the unidimensional case, and give the number of ranking patterns in terms of the number of chambers of the mid-hyperplane arrangement. We also provide explicit upper and lower bounds for the number of ranking patterns. In Subsection 3.2, we treat the unfolding model of codimension one, where the restriction by dimension is weakest. In this case, we describe how the number of ranking patterns can be expressed by the number of chambers of an arrangement called the all-subset arrangement.

§2. Number of admissible rankings

In this section, we define genericness of the unfolding model, and discuss the problem of counting the number of admissible rankings generated by the unfolding model with generic objects.

Suppose we are given $x_1, \dots, x_m \in \mathbb{R}^n$ with $m \geq 3$ and $n \leq m-2$.

In general, for m distinct points $z_1, \dots, z_m \in \mathbb{R}^\nu$ $(m \geq \nu + 1)$, let $\overline{z_i z_j}$ denote the one-simplex connecting two points z_i and z_j $(i < j)$. Consider the following condition:

(A) The union of ν distinct one-simplices $\overline{z_{i_k} z_{j_k}}$ $(i_k < j_k,\ k = 1, \dots, \nu)$ contains no loop if and only if the corresponding vectors $z_{i_k} - z_{j_k}$ $(k = 1, \dots, \nu)$ are linearly independent.

We assume $x_1, \dots, x_m \in \mathbb{R}^n$ $(n \leq m-2)$ are generic in the sense that they satisfy the following two conditions:

(A1) The m points $x_1, \dots, x_m \in \mathbb{R}^n$ satisfy condition (A).
(A2) The m points $(x_1^T, \|x_1\|^2)^T, \dots, (x_m^T, \|x_m\|^2)^T \in \mathbb{R}^{n+1}$ satisfy condition (A).

Now, according to the unfolding model, judge $y \in \mathbb{R}^n$ prefers x_i to x_j $(i \neq j)$ iff $\|y - x_i\| < \|y - x_j\|$. This condition is equivalent to y being on the same side as x_i of the perpendicular bisector

$$\begin{aligned} H_{ij} &:= \{y \in \mathbb{R}^n : \|y - x_i\| = \|y - x_j\|\} \\ &= \{y \in \mathbb{R}^n : (x_i - x_j)^T (y - \frac{x_i + x_j}{2}) = 0\} \end{aligned}$$

of the line segment $\overline{x_i x_j}$ joining x_i and x_j. Let us define a hyperplane arrangement

$$\mathcal{A}_{m,n} = \mathcal{A}_{m,n}(x_1, \dots, x_m) := \{H_{ij} : 1 \leq i < j \leq m\}$$

in $\mathbb{R}^n$. We call $\mathcal{A}_{m,n}$ the unfolding arrangement.

Then $\mathcal{A}_{m,n}$, like any real hyperplane arrangement, cuts $\mathbb{R}^n$ into chambers, i.e., connected components of the complement $\mathbb{R}^n \setminus \bigcup \mathcal{A}_{m,n}$, where $\bigcup \mathcal{A}_{m,n} := \bigcup_{H \in \mathcal{A}_{m,n}} H$. Moreover, each of these chambers is of the form

$$C_{i_1 \cdots i_m} := \{\|y - x_{i_1}\| < \cdots < \|y - x_{i_m}\|\} \neq \emptyset$$

for some admissible ranking $(i_1 \cdots i_m) \in \mathbb{P}_m$, where $\mathbb{P}_m$ denotes the set of permutations of $[m] := \{1, \ldots, m\}$.

We observe that $y \in \mathbb{R}^n$ gives ranking $(i_1 \cdots i_m) \in \mathbb{P}_m$ if and only if $y \in C_{i_1 \cdots i_m} \neq \emptyset$. Thus there is a one-to-one correspondence between the set of admissible rankings and the set of chambers $\mathbf{Ch}(\mathcal{A}_{m,n})$ of $\mathcal{A}_{m,n}$:

$$(i_1 \cdots i_m) \leftrightarrow C_{i_1 \cdots i_m}$$

for $(i_1 \cdots i_m)$ such that $C_{i_1 \cdots i_m} \neq \emptyset$. This implies that the problem of counting the number of admissible rankings reduces to that of counting the number of chambers of $\mathcal{A}_{m,n}$. The answer to the latter problem is given by the theorem below. Let $\mathcal{S}_k^m$ ($k \in \mathbb{Z}$) be the signless Stirling numbers of the first kind: $t(t+1) \cdots (t+m-1) = \sum_k \mathcal{S}_k^m t^k$.

Theorem 1 (Good and Tideman [11], Kamiya and Takemura [14, 15], Zaslavsky [30]). *Suppose $x_1, \ldots, x_m \in \mathbb{R}^n$ $(n \leq m-2)$ are generic. Then, the number of chambers of $\mathcal{A}_{m,n} = \mathcal{A}_{m,n}(x_1, \ldots, x_m)$ is*

$$|\mathbf{Ch}(\mathcal{A}_{m,n})| = \mathcal{S}_{m-n}^m + \mathcal{S}_{m-n+1}^m + \cdots + \mathcal{S}_m^m.$$

Furthermore, the number of bounded chambers of $\mathcal{A}_{m,n}$ is

$$\mathcal{S}_{m-n}^m - \mathcal{S}_{m-n+1}^m + \mathcal{S}_{m-n+2}^m - \cdots + (-1)^n \mathcal{S}_m^m.$$

The proof of Theorem 1 is based on Zaslavsky's general result on the number of chambers of an arrangement (Zaslavsky [29]) and the following proposition. Denote by Π_m the partition lattice, consisting of partitions of $[m]$ and ordered by refinement. Further, let Π_m^n stand for the rank n truncation of Π_m, i.e., the subposet of Π_m comprising elements of rank ($= m -$ # of blocks) at most n.

Proposition 1 (Kamiya and Takemura [14, 15]). *The intersection poset $L(\mathcal{A}_{m,n})$ of the unfolding arrangement $\mathcal{A}_{m,n}$ is isomorphic to Π_m^n:*

$$L(\mathcal{A}_{m,n}) \cong \Pi_m^n.$$

The isomorphism is given by

$$L(\mathcal{A}_{m,n}) \ni X \mapsto I_X \in \Pi_m^n,$$

where I_X is the partition of $[m]$ into equivalence classes under the equivalence relation $\sim_X$ defined by $i \sim_X j \overset{\text{def}}{\iff} X \subseteq H_{ij}$ $(H_{ii} := \mathbb{R}^n)$.

Remark 1. *When $n \geq m-1$, and $x_1, \ldots, x_m \in \mathbb{R}^n$ satisfy condition (A1) with the $\nu = n$ in (A) replaced by $m-1$, we can easily see that $|\mathbf{Ch}(\mathcal{A}_{m,n})| = m!$ and that the number of bounded chambers of $\mathcal{A}_{m,n}$ is zero (so the results in Theorem 1 continue to be valid). Therefore, all $m!$ rankings arise as unbounded chambers of $\mathcal{A}_{m,n}$ in this case.*

§3. Number of ranking patterns

In this section, we consider the problem of counting the number of ranking patterns. We treat two extreme cases—the unidimensional unfolding model: $n = 1$ (Subsection 3.1) and the unfolding model of codimension one: $n = m - 2$ (Subsection 3.2).

3.1. Unidimensional unfolding models

In this subsection, we look into the problem of counting the number of ranking patterns of unidimensional unfolding models: $n = 1$. A related problem is studied in Stanley [24].

In this case $n = 1$, objects are m points on the real line: $x_1, \ldots, x_m \in \mathbb{R}$. We assume $x_1, \ldots, x_m$ are generic, i.e., the midpoints $x_{ij} := (x_i + x_j)/2,\ 1 \leq i < j \leq m$, are all distinct. This condition can be written as

$$(x_1, \ldots, x_m) \in \mathbb{R}^m \setminus \bigcup \mathcal{M}_m,$$

where $\mathcal{M}_m := \mathcal{B}_m \cup \mathcal{N}_m$ is the mid-hyperplane arrangement (Kamiya, Orlik, Takemura, and Terao [13]) with

$$\begin{aligned}
\mathcal{B}_m &:= \{K_{ij} : 1 \leq i < j \leq m\}, \\
K_{ij} &:= \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i = x_j\}, \\
\mathcal{N}_m &:= \{H_{ijkl} : (i,j,k,l) \in I_4\}, \\
H_{ijkl} &:= \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i + x_j = x_k + x_l\}, \\
I_4 &:= \{(i,j,k,l) : i,j,k,l \text{ are all distinct}, \\
&\qquad\qquad 1 \leq i < j \leq m,\ i < k < l \leq m\}.
\end{aligned}$$

(In this subsection, we write elements of $\mathbb{R}^m$ as row vectors.) Note that $\mathcal{B}_m$ is the braid arrangement. We have $H_{ij} = \{x_{ij}\},\ 1 \leq i < j \leq m$, and $\mathcal{A}_{m,1} = \{\{x_{ij}\} : 1 \leq i < j \leq m\}$.

An m-tuple $\mathbf{x} := (x_1, \ldots, x_m)$ of objects gives the ranking pattern

$$\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) = \{(i_1 \cdots i_m) \in \mathbb{P}_m : |y - x_{i_1}| < \cdots < |y - x_{i_m}| \text{ for some } y \in \mathbb{R}\}.$$

We want to know

$$r(m) := |\{\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m \setminus \bigcup \mathcal{M}_m\}|. \tag{1}$$

The braid arrangement $\mathcal{B}_m$ has a chamber $C_0 \in \mathbf{Ch}(\mathcal{B}_m)$ defined by $x_1 < \cdots < x_m$:

$$C_0 := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 < \cdots < x_m\}.$$

Let us concentrate our attention on C_0. For $\mathbf{x} = (x_1, \ldots, x_m) \in C_0 \setminus \bigcup \mathcal{N}_m$ and $\mathbf{x}' = (x'_1, \ldots, x'_m) \in C_0 \setminus \bigcup \mathcal{N}_m$, we can easily see that $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) = \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')$ if and only if the order of the midpoints on $\mathbb{R}$ is the same for $\mathbf{x}$ and $\mathbf{x}'$ (i.e., $\forall (i,j,k,l) \in I_4 : x_{ij} < x_{kl} \iff x'_{ij} < x'_{kl}$). Noting that $x_{ij} < x_{kl}$ iff $(x_1, \ldots, x_m) \in H^-_{ijkl} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i + x_j < x_k + x_l\}$, we obtain the following lemma.

Lemma 1 (Kamiya, Orlik, Takemura, and Terao [13]). *For $\mathbf{x}, \mathbf{x}' \in C_0 \setminus \bigcup \mathcal{N}_m$, we have $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) = \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')$ if and only if $\mathbf{x}$ and $\mathbf{x}'$ are in the same chamber of $\mathcal{N}_m$.*

Put

$$r_0(m) := |\{\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) : \mathbf{x} \in C_0 \setminus \bigcup \mathcal{N}_m\}|,$$

i.e., the number of ranking patterns of unidimensional unfolding models with generic m-tuples such that $x_1 < \cdots < x_m$. Then, by Lemma 1 we have

$$r_0(m) = \frac{|\mathbf{Ch}(\mathcal{M}_m)|}{m!} \tag{2}$$

(Kamiya, Orlik, Takemura, and Terao [13]).

Now consider $r(m)$ in (1). For $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m \setminus \bigcup \mathcal{M}_m$, define $-\mathbf{x} := (-x_1, \ldots, -x_m) \in \mathbb{R}^m \setminus \bigcup \mathcal{M}_m$. Then, clearly we have $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) = \mathrm{RP}^{\mathrm{UF}}(-\mathbf{x})$. On the other hand, for $C, C' \in \mathbf{Ch}(\mathcal{M}_m)$ such that $C' \neq \pm C$ $(-C := \{-\mathbf{x} : \mathbf{x} \in C\})$, we can easily see that $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) \neq \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')$ for $\mathbf{x} \in C$ and $\mathbf{x}' \in C'$. These two facts, together with Lemma 1, yield the following theorem.

Theorem 2. *The number of ranking patterns of unidimensional unfolding models with generic m-tuples of objects is*

$$r(m) = \frac{m!}{2} r_0(m) = \frac{|\mathbf{Ch}(\mathcal{M}_m)|}{2}, \quad m \geq 3.$$

Let us define equivalence of ranking patterns by saying that two ranking patterns $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x})$ and $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')$ are equivalent iff

$$\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) = \sigma \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}') \quad \text{for some } \sigma \in \mathfrak{S}_m, \tag{3}$$

where $\mathfrak{S}_m$ is the symmetric group on m letters, consisting of all bijections: $[m] \to [m]$, and $\sigma \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}') := \{(\sigma(i_1) \cdots \sigma(i_m)) : (i_1 \cdots i_m) \in$

$\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')\}$. We want to find the number of inequivalent ranking patterns.

Let $r_{\mathrm{IE}}(m)$ be the number of inequivalent ranking patterns of unidimensional unfolding models with generic m-tuples of objects:

$$r_{\mathrm{IE}}(m) := |\{[\mathrm{RP}^{\mathrm{UF}}(\mathbf{x})] : \mathbf{x} \in \mathbb{R}^m \setminus \bigcup \mathcal{M}_m\}|,$$

where $[\,\cdot\,]$ stands for the equivalence class under the equivalence relation defined by (3). We will see that $r_{\mathrm{IE}}(m)$ is half of $r_0(m)$ for $m \geq 4$. Suppose we are given $\mathbf{x} = (x_1, \ldots, x_m) \in C_0 \setminus \bigcup \mathcal{N}_m$ with $m \geq 4$. Then $\mathbf{x}' = (x'_1, \ldots, x'_m) := (-x_m, \ldots, -x_1)$ also lies in $C_0 \setminus \bigcup \mathcal{N}_m$: $\mathbf{x}' \in C_0 \setminus \bigcup \mathcal{N}_m$. Moreover, since $m \geq 4$, four indices $1, 2, m-1, m$ are all distinct and we have $x_{1m} < x_{2,m-1}$ iff $x'_{1m} > x'_{2,m-1}$. This means $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) \neq \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')$ by Lemma 1. However, $[\mathrm{RP}^{\mathrm{UF}}(\mathbf{x})] = [\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')]$ since $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}) = \mathrm{RP}^{\mathrm{UF}}(-\mathbf{x})$. Next, it can be seen that any $\mathbf{x}'' \in C_0 \setminus \bigcup \mathcal{N}_m$ such that $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}'') \neq \mathrm{RP}^{\mathrm{UF}}(\mathbf{x})$ and $[\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}'')] = [\mathrm{RP}^{\mathrm{UF}}(\mathbf{x})]$ satisfies $\mathrm{RP}^{\mathrm{UF}}(\mathbf{x}'') = \mathrm{RP}^{\mathrm{UF}}(\mathbf{x}')$. These arguments lead to the following theorem.

Theorem 3. *The number of inequivalent ranking patterns of unidimensional unfolding models with generic m-tuples of objects is*

$$r_{\mathrm{IE}}(m) = \begin{cases} r_0(3) = \frac{|\mathbf{Ch}(\mathcal{B}_3)|}{3!} = 1 & \text{if } m = 3, \\ \frac{r_0(m)}{2} = \frac{|\mathbf{Ch}(\mathcal{M}_m)|}{2 \cdot m!} & \text{if } m \geq 4. \end{cases}$$

So far, we have expressed the number of ranking patterns in terms of the number of chambers of an arrangement. We can use the finite field method (Athanasiadis [3, 4], Crapo and Rota [7], Kamiya, Takemura, and Terao [16, 17, 18], Stanley [25, Lecture 5]) to calculate specific values of $r_0(m)$, $m \leq 10$:

$$r_0(4) = 2,\ r_0(5) = 12,\ r_0(6) = 168,\ r_0(7) = 4680,$$
$$r_0(8) = 229386,\ r_0(9) = 18330206,\ r_0(10) = 2241662282.$$

The values of $r(m)$ for $m \leq 8$ are given in Kamiya, Orlik, Takemura, and Terao [13] along with the characteristic polynomials $\chi(\mathcal{M}_m, t)$ of $\mathcal{M}_m$, $m \leq 8$. After [13], the second author of the present paper, Takemura [26], improved on Lemma 3.3 of [13] and calculated $\chi(\mathcal{M}_9, t)$ and

$r_0(9)$; later Ishiwata [12] obtained $\chi(\mathcal{M}_{10},t)$ and $r_0(10)$ after an extensive computation. The characteristic polynomials found by them are:

$$\begin{aligned}\chi(\mathcal{M}_9,t) &= t(t-1)(t^7-413t^6+73780t^5\\ &\qquad -7387310t^4+447514669t^3\\ &\qquad -16393719797t^2+336081719070t\\ &\qquad -2972902161600),\\ \chi(\mathcal{M}_{10},t) &= t(t-1)(t^8-674t^7+201481t^6-34896134t^5\\ &\qquad +3830348179t^4-272839984046t^3\\ &\qquad +12315189583899t^2-321989533359786t\\ &\qquad +3732690616086600).\end{aligned}$$

However, for large values of m, the finite field method is not feasible. We will provide simple upper and lower bounds for $r_0(m)$.

Theorem 4. *For all $m\ge 4$, we have*

$$2\left(\frac{3}{4}\right)^{m-4}\{(m-3)!\}^2 \le r_0(m) < \frac{2}{m!}\left\{\frac{em(m-1)^2}{8}\right\}^{m-2}.$$

Proof. First, we derive the upper bound in the theorem.

Define $H_0 := \{(x_1,\dots,x_m)\in\mathbb{R}^m : x_1+\cdots+x_m=0\}$, and consider the essentialization (Stanley [25, p.392]) $\mathcal{M}_m^0 := \{H\cap H_0 : H\in\mathcal{M}_m\}$ of $\mathcal{M}_m$. Since $L(\mathcal{M}_m^0)\cong L(\mathcal{M}_m)$, we may consider the essential, central arrangement $\mathcal{M}_m^0$ in H_0 ($\dim H_0 = m-1$) instead of $\mathcal{M}_m$.

Recall, in general, that h hyperplanes divide $\mathbb{R}^d$ into at most $\sum_{i=0}^d \binom{h}{i} \le (eh/d)^d =: c(h,d)$ chambers (see, e.g., [20, Proposition 6.1.1] and [21, Theorem 3.6.1]). Thus, $\tilde h$ linear hyperplanes divide $\mathbb{R}^{\tilde d}$ into at most $2c(\tilde h-1,\tilde d-1)$ chambers.

In our case, $\mathcal{M}_m^0$ is central, so we can take $\tilde h = |\mathcal{M}_m| = |\mathcal{B}_m| + |\mathcal{N}_m| = \binom{m}{2}+3\binom{m}{4} \le m(m-1)^2(m-2)/8$ ($m\ge 4$) and $\tilde d = m-1$. Hence, we have

$$\begin{aligned}|\mathbf{Ch}(\mathcal{M}_m^0)| &\le 2c(\tilde h-1,\tilde d-1)\\ &\le 2\times\left\{\frac{e\left(\frac{m(m-1)^2(m-2)}{8}-1\right)}{m-2}\right\}^{m-2}\\ &< 2\times\left\{\frac{em(m-1)^2}{8}\right\}^{m-2}.\end{aligned}$$

This together with (2) and $|\mathbf{Ch}(\mathcal{M}_m)| = |\mathbf{Ch}(\mathcal{M}_m^0)|$ gives the upper bound of $r_0(m)$ in the theorem.

Next, we will obtain the lower bound in the theorem.

Let $\mathbf{x} = (x_1, \ldots, x_m)$, $x_1 < \cdots < x_m$ be fixed. We add one more object $y = x_m + 2t$ $(t > 0)$ to $\mathbf{x}$, and we will count the number of ranking patterns arising from $\mathbf{y}_t = (\mathbf{x}, y)$, $t > 0$. Let $M = \{x_{ij} : 1 \le i < j \le m\}$ be the set of midpoints for $\mathbf{x}$, and $Y_t = \{x_{im} + t : 1 \le i \le m\}$ the set of midpoints of x_i $(1 \le i \le m)$ and y. Then $M \cup Y_t$ is the set of midpoints for $\mathbf{y}_t$. To guarantee all these midpoints are distinct, we require the following. First, by perturbing each x_i without changing the ranking pattern of $\mathbf{x}$, we may assume that $x_1, \ldots, x_m$ are independent over $\mathbb{Q}$. Then we have $|M \cap Y_t| \le 1$ for all $t > 0$. Next, let $T_0 = \{t > 0 : |M \cap Y_t| = 1\}$, $T_1 = (0, \infty) \setminus T_0$, and we only consider $t \in T_1$. Then $M \cup Y_t$ is legal, i.e., all midpoints are distinct.

Now the crucial observation is as follows: $|\{\mathrm{RP}^{\mathrm{UF}}(\mathbf{y}_t) : t \in T_1\}| = 1 + |T_0|$. Moreover, we have $|T_0| = \sum_{i=1}^{m-1} |V_i|$, where $V_i = \{v \in M : x_{im} < v\}$. Using $|V_i| \ge m - 1 - i$ obtained by $V_i \supset \{x_{jm} : i < j < m\}$, we have

$$|\{\mathrm{RP}^{\mathrm{UF}}(\mathbf{y}_t) : t \in T_1\}| = 1 + \sum_{i=1}^{m-1} |V_i| \ge 1 + |V_1| + \frac{(m-3)(m-2)}{2} =: N.$$

Namely, N is a lower bound for the number of ranking patterns arising from $\mathbf{y}_t$, $t \in T_1$.

Applying exactly the same argument to $\mathbf{x}' = (-x_m, \ldots, -x_1)$ instead of $\mathbf{x}$, we see that the number of ranking patterns arising from $(\mathbf{x}', -x_1 + 2t)$, $t > 0$ (or equivalently, $(x_1 - 2t, \mathbf{x})$, $t > 0$) is at least $N' = 1 + |V_1'| + (m-3)(m-2)/2$, where $|V_1'| = |\{u \in M : u < x_{1m}\}| = \binom{m}{2} - |V_1| - 1$. Notice that $N + N' = 1 + \binom{m}{2} + (m-3)(m-2) > (3/2)(m-2)^2$. Therefore, by the averaging argument, we have

$$r_0(m+1) \ge r_0(m) \times \tfrac{1}{2}(N + N') > \tfrac{3}{4}(m-2)^2\, r_0(m).$$

So the induction starting from $r_0(4) = 2$ gives the desired lower bound. Q.E.D.

Let $\ell(m)$ and $u(m)$ be the lower and upper bounds in the theorem, respectively. A computation shows $\{u(m)\}^{1/m}/m^2 \to e^2/8 \approx 0.92$ and $\{\ell(m)\}^{1/m}/m^2 \to 3/(4e^2) \approx 0.1$ as $m \to \infty$. It would be interesting to prove (or disprove) the existence of $\lim\{r_0(m)\}^{1/m}/m^2$.

Strangely enough, $r_0(m) = a(m)$ holds for $4 \le m \le 7$, where

$$a(m) := \frac{(m-2)\{(m-2)^{m-3} - 1\} \cdot (m-4)!}{m-3},$$

but $r_0(8) > a(8)$, $r_0(9) > a(9)$, $r_0(10) > a(10)$. Also, $a(m)$ satisfies $\{a(m)\}^{1/m}/m^2 \to 1/e \approx 0.37$. We mention that $a(m)/\{(m-3)!\} =$

m	$r_0(m)$	$a(m)$	$\ell(m)$	$u(m)$	$f(m)$
4	2	2	2	12	2
5	12	12	6	334	12
6	168	168	41	18,744	286
7	4,680	4,680	486	1.82×10^6	33,592
8	229,386	223,920	9,113	2.76×10^8	23,178,480
9	18,330,206	16,470,720	246,038	6.06×10^{10}	108,995,910,720
10	2,241,662,282	1,725,655,680	9.05×10^6	1.81×10^{13}	3,973,186,258,569,120

Table 1. $r_0(m)$, $a(m)$, $\ell(m)$, $u(m)$, $f(m)$, $4 \le m \le 10$.

$(m-2)\{(m-2)^{m-3}-1\}/(m-3)^2$ $(m \ge 4)$ is the number of acyclic-function digraphs on $m-2$ vertices (Walsh [28], OEIS id:A058128).

Thrall [27] gave an upper bound $f(m)$ for $r_0(m)$:

$$f(m) := \frac{\{\frac{m(m-1)}{2}\}! \prod_{i=1}^{m-2} i!}{\prod_{i=1}^{m-1}(2i-1)!}.$$

Here, $f(m)$ is the number of mappings $\{(i,j) : 1 \le i < j \le m\} \ni (i,j) \mapsto d(i,j) \in \{1,2,\dots,m(m-1)/2\}$ satisfying the condition that $d(i,j)$ be increasing in i for each fixed j as well as increasing in j for each fixed i. He obtained this number by considering a problem similar to that of counting the number of standard Young tableaux. Since for $\mathbf{x} = (x_1,\dots,x_m) \in C_0 \setminus \bigcup \mathcal{N}_m$, the ranks $d_{\mathbf{x}}(i,j)$ of the midpoints $x_{ij} = (x_i+x_j)/2$ from left to right on the real line $\mathbb{R}$ meet this condition, $f(m)$ is an upper bound for $r_0(m)$. We can see our $u(m)$ satisfies $f(m) < u(m)$ for $m \le 8$, $f(m) > u(m)$ for $m \ge 9$, and $u(m) = o(f(m))$. For m such that $f(m) < u(m)$, we know the exact values $r_0(m)$ anyway, so the upper bound $u(m)$ based on arrangement theory may be said to be better than $f(m)$.

We list the values of $r_0(m)$, $a(m)$, $f(m)$ and approximate values of $\ell(m)$, $u(m)$ for $m = 4,\dots,10$ in Table 1. (For $\ell(m)$, $m \le 9$, and $u(m)$, $m \le 6$, we exhibit $\lceil \ell(m) \rceil$ and $\lfloor u(m) \rfloor$, respectively. For $\ell(10)$, we display $\lceil \ell(m) \times 10^{-4} \rceil \times 10^4$, and similarly using $\lfloor \cdot \rfloor$ for $u(m)$, $m \ge 7$.)

3.2. Unfolding models of codimension one

In this subsection, we deal with the problem of counting the number of ranking patterns of unfolding models of codimension one: $n = m-2$ (i.e., when the restriction by dimension is weakest).

First, let us forget the unfolding model for a while and consider the ranking patterns of braid slices.

We begin by defining the ranking pattern of a braid slice. For

$$H_0 = \{x = (x_1, \dots, x_m)^T \in \mathbb{R}^m : x_1 + \cdots + x_m = 0\},$$

consider the essential arrangement

$$\mathcal{B}_m^0 := \{H \cap H_0 : H \in \mathcal{B}_m\}$$

in H_0, and write its chambers as

$$B_{i_1 \cdots i_m} := \{x = (x_1, \dots, x_m)^T \in H_0 : x_{i_1} > \cdots > x_{i_m}\} \in \mathbf{Ch}(\mathcal{B}_m^0)$$

for $(i_1 \cdots i_m) \in \mathbb{P}_m$. Moreover, define a hyperplane

$$K_v := \{x \in H_0 : v^T x = 1\}$$

in H_0 for each $v \in \mathbb{S}^{m-2} := \{x \in H_0 : \|x\| = 1\}$. Now we call the subset

$$\mathrm{RP}(v) := \{(i_1 \cdots i_m) \in \mathbb{P}_m : K_v \cap B_{i_1 \cdots i_m} \neq \emptyset\}, \quad v \in \mathbb{S}^{m-2},$$

of $\mathbb{P}_m$ the ranking pattern of the braid slice by K_v.

Next, let us define genericness of the braid slice as follows. For the all-subset arrangement (Kamiya, Takemura, and Terao [19])

$$\mathcal{A}_m := \{H_I : I \subseteq [m], \ |I| \geq 1\}$$

with $H_I := \{x = (x_1, \dots, x_m)^T \in \mathbb{R}^m : \sum_{i \in I} x_i = 0\}$, $\emptyset \neq I \subseteq [m]$, consider its restriction to $H_0 = H_{[m]}$:

$$\mathcal{A}_m^0 := \mathcal{A}_m^{H_0} = \{H_I^0 : I \subset [m], \ 1 \leq |I| \leq m-1\},$$
$$H_I^0 := H_I \cap H_0 \quad (1 \leq |I| \leq m-1).$$

Then define

$$\mathcal{V} := (H_0 \setminus \bigcup \mathcal{A}_m^0) \cap \mathbb{S}^{m-2}.$$

We will say $v \in \mathbb{S}^{m-2}$, or the braid slice by K_v, is generic if $v \in \mathcal{V}$.

Now, we will see that the set of ranking patterns $\mathrm{RP}(v)$ for generic v's is in one-to-one correspondence with the set of chambers of $\mathcal{A}_m^0$. Write $\mathcal{V}$ as $\mathcal{V} = \bigsqcup_{D \in \mathbf{D}(\mathcal{A}_m^0)} D$ (disjoint union), where

$$\mathbf{D}(\mathcal{A}_m^0) := \{D = \tilde{D} \cap \mathbb{S}^{m-2} : \tilde{D} \in \mathbf{Ch}(\mathcal{A}_m^0)\},$$

which clearly is in one-to-one correspondence with $\mathbf{Ch}(\mathcal{A}_m^0)$. Then, we can prove (Kamiya, Takemura, and Terao [19]) that there is a bijection from $\mathbf{D}(\mathcal{A}_m^0)$ to $\{\mathrm{RP}(v) : v \in \mathcal{V}\}$ given by

$$\mathbf{D}(\mathcal{A}_m^0) \ni D \mapsto \mathrm{RP}(v), \ v \in D. \tag{4}$$

Hence,

$$\mathrm{RP}_D := \mathrm{RP}(v) \text{ for } v \in D \in \mathbf{D}(\mathcal{A}_m^0)$$

is well-defined, and the mapping $\mathbf{D}(\mathcal{A}_m^0) \to \{\mathrm{RP}_D : D \in \mathbf{D}(\mathcal{A}_m^0)\} = \{\mathrm{RP}(v) : v \in \mathcal{V}\} : D \mapsto \mathrm{RP}_D$ is bijective.

Let us get back to the unfolding model and consider the ranking pattern of the unfolding model of codimension one.

Suppose we are given $x_1, \ldots, x_m \in \mathbb{R}^n$ with $n = m - 2 \geq 1$. We assume $x_1, \ldots, x_m$ are generic in the sense that they satisfy (A1) and (A2) in Section 2. We call the unfolding model with such $x_1, \ldots, x_m \in \mathbb{R}^{m-2}$ the unfolding model of codimension one (for the reason stated below). In addition, we will assume without loss of generality that $x_1, \ldots, x_m$ are taken so that $\sum_{i=1}^m x_i = 0$, $\sum_{i=1}^m \|x_i\|^2/m = 1$.

We will see that the ranking pattern of the unfolding model of codimension one with m-tuple $(x_1, \ldots, x_m)$:

$$\begin{aligned} \mathrm{RP}^{\mathrm{UF}}(x_1, \ldots, x_m) \quad &= \quad \{(i_1 \cdots i_m) \in \mathbb{P}_m : \|y - x_{i_1}\| < \cdots < \|y - x_{i_m}\| \\ (5) \qquad & \qquad\qquad\qquad\qquad \text{for some } y \in \mathbb{R}^{m-2}\} \end{aligned}$$

can be expressed as the ranking pattern of a braid slice.

Define

$$W \quad = \quad \mathrm{W}(x_1, \ldots, x_m) = (w_1, \ldots, w_{m-2}) := \begin{pmatrix} x_1^T \\ \vdots \\ x_m^T \end{pmatrix} \in \mathrm{Mat}_{m \times (m-2)}(\mathbb{R}),$$

$$u \quad = \quad \mathrm{u}(x_1, \ldots, x_m) := -\frac{1}{2} \begin{pmatrix} \|x_1\|^2 - 1 \\ \vdots \\ \|x_m\|^2 - 1 \end{pmatrix} \in \mathbb{R}^m,$$

where $\mathrm{Mat}_{m \times (m-2)}(\mathbb{R})$ denotes the set of $m \times (m-2)$ matrices with real entries. For the affine map $\kappa : \mathbb{R}^{m-2} \to \mathbb{R}^m$ defined by $\kappa(y) := Wy + u$, $y \in \mathbb{R}^{m-2}$, consider the image $K := \operatorname{im} \kappa = \{k(y) : y \in \mathbb{R}^{m-2}\}$ of κ. Then we have

$$K = u + \operatorname{col} W \subset H_0,$$

where $\operatorname{col} W$ stands for the column space of W. Using this K, we can easily see that $\mathrm{RP}^{\mathrm{UF}}(x_1, \ldots, x_m)$ in (5) can be expressed as

$$(6) \qquad \mathrm{RP}^{\mathrm{UF}}(x_1, \ldots, x_m) = \{(i_1 \cdots i_m) \in \mathbb{P}_m : K \cap B_{i_1 \cdots i_m} \neq \emptyset\}.$$

We have $\dim K = \dim H_0 - 1$ and $u \notin \operatorname{col} W$ by (A1) and (A2), respectively. That is, K is an affine hyperplane of H_0. For this reason, we called the unfolding model with generic $x_1, \ldots, x_m \in \mathbb{R}^{m-2}$ the unfolding model of codimension one.

Write the affine hyperplane $K \subset H_0$ as

$$K = K_{\tilde{v}} := \{x \in H_0 : \tilde{v}^T x = \|\tilde{v}\|^2\}$$

using the orthogonal projection of $u \in H_0$ on $(\operatorname{col} W)^\perp := \{x \in H_0 : x^T W = 0\}$:

$$\tilde{v} := \tilde{\mathrm{v}}(x_1, \ldots, x_m) = u - \operatorname{proj}_{\operatorname{col} W}(u), \quad u = \mathrm{u}(x_1, \ldots, x_m),$$

where $\operatorname{proj}_{\operatorname{col} W}$ denotes the orthogonal projection on $\operatorname{col} W$. Noting $\tilde{v} \neq 0$, we can represent (6) as

(7)
$$\mathrm{RP}^{\mathrm{UF}}(x_1, \ldots, x_m) = \{(i_1 \cdots i_m) \in \mathbb{P}_m : K_{\mathrm{v}(x_1,\ldots,x_m)} \cap B_{i_1 \cdots i_m} \neq \emptyset\},$$
$$\mathrm{v}(x_1, \ldots, x_m) := \frac{1}{\|\tilde{v}\|}\tilde{v} \in \mathbb{S}^{m-2},$$

in terms of $K_{\mathrm{v}(x_1,\ldots,x_m)} = \{x \in H_0 : \mathrm{v}(x_1, \ldots, x_m)^T x = 1\}$ instead of $K = K_{\tilde{v}}$. The right-hand side of (7) is the ranking pattern of the braid slice by $K_{\mathrm{v}(x_1,\ldots,x_m)}$: $\mathrm{RP}(\mathrm{v}(x_1, \ldots, x_m))$. Besides, it can be seen that $\mathrm{v}(x_1, \ldots, x_m) \in \mathcal{V}$.

Proposition 2 (Kamiya, Takemura, and Terao [19]). *For generic $x_1, \ldots, x_m \in \mathbb{R}^{m-2}$, we have $\mathrm{v}(x_1, \ldots, x_m) \in \mathcal{V}$ and*

$$\mathrm{RP}^{\mathrm{UF}}(x_1, \ldots, x_m) = \mathrm{RP}(\mathrm{v}(x_1, \ldots, x_m)).$$

Proposition 2 and bijection (4) tell us that in order to find the number of ranking patterns of unfolding models of codimension one, we need to study the image of the mapping $\mathrm{v} : \{(x_1, \ldots, x_m) : x_1, \ldots, x_m \in \mathbb{R}^{m-2}$ are generic$\} \to \mathcal{V} = \bigsqcup_{D \in \mathbf{D}(\mathcal{A}_m^0)} D$, $(x_1, \ldots, x_m) \mapsto \mathrm{v}(x_1, \ldots, x_m)$. In their main theorem (Theorem 4.1), Kamiya, Takemura, and Terao [19] proved that the image $\operatorname{im} \mathrm{v}$ is given by

$$\operatorname{im} \mathrm{v} = \mathcal{V}_2 \sqcup D_1 \sqcup \cdots \sqcup D_m = \mathcal{V} \setminus ((-D_1) \sqcup \cdots \sqcup (-D_m)), \tag{8}$$

where

$$\mathcal{V}_2 := \{v = (v_1, \ldots, v_m)^T \in \mathcal{V} : v_j > 0 \text{ for at least two } j \in [m] \text{ and } v_k < 0 \text{ for at least two } k \in [m]\}$$

and

$$\begin{aligned} D_i &:= \{v = (v_1, \dots, v_m)^T \in \mathcal{V} : v_i > 0,\ v_j < 0\ (j \neq i)\} \in \mathbf{D}(\mathcal{A}_m^0), \\ -D_i &:= \{-v : v \in D_i\} \\ &= \{v = (v_1, \dots, v_m)^T \in \mathcal{V} : v_i < 0,\ v_j > 0\ (j \neq i)\} \in \mathbf{D}(\mathcal{A}_m^0) \end{aligned}$$

for $i \in [m]$.

By Proposition 2 and im v in (8), we obtain the number of ranking patterns of unfolding models of codimension one, which is denoted by

$$q(m) := |\{\mathrm{RP}^{\mathrm{UF}}(x_1, \dots, x_m) : \text{ generic } x_1, \dots, x_m \in \mathbb{R}^{m-2}\}|.$$

Theorem 5 (Kamiya, Takemura, and Terao [19]). *The number $q(m)$ of ranking patterns of unfolding models of codimension one is given by*

$$q(m) = |\mathbf{Ch}(\mathcal{A}_m^0)| - m.$$

Kamiya, Takemura, and Terao [19, Lemma 5.3] obtained the characteristic polynomials $\chi(\mathcal{A}_m^0, t)$ of $\mathcal{A}_m^0$ for $m \leq 8$ by the finite field method. Then $q(m)$ can be calculated by $q(m) = (-1)^{m-1}\chi(\mathcal{A}_m^0, -1) - m$:

$$q(3) = 3,\ q(4) = 28,\ q(5) = 365,$$
$$q(6) = 11286,\ q(7) = 1066037,\ q(8) = 347326344$$

([19, Corollary 5.5]).

We end this subsection by looking at the problem of finding the number of inequivalent ranking patterns of unfolding models of codimension one.

In (3), we defined equivalence of ranking patterns of unidimensional unfolding models. We define equivalence of ranking patterns of unfolding models of codimension one in an obvious similar manner. At the moment, we can only give an upper bound for the number $q_{\mathrm{IE}}(m)$ of inequivalent ranking patterns of unfolding models of codimension one:

(9)
$$q_{\mathrm{IE}}(m) \leq \frac{|\mathbf{Ch}(\mathcal{A}_m^0 \cup \mathcal{B}_m^0)|}{m!} - 1 = |\mathbf{D}^{1\cdots m}(\mathcal{A}_m^0)| - 1 = |\mathbf{D}_2^{1\cdots m}(\mathcal{A}_m^0)| + 1$$

for $m \geq 3$ (Kamiya, Takemura, and Terao [19]), where $\mathbf{D}^{1\cdots m}(\mathcal{A}_m^0) := \{D \in \mathbf{D}(\mathcal{A}_m^0) : D \cap B_{1\cdots m} \neq \emptyset\}$ and $\mathbf{D}_2^{1\cdots m}(\mathcal{A}_m^0) := \{D \in \mathbf{D}(\mathcal{A}_m^0) : D \subset \mathcal{V}_2,\ D \cap B_{1\cdots m} \neq \emptyset\} = \mathbf{D}^{1\cdots m}(\mathcal{A}_m^0) \setminus \{D_1, -D_m\}$. It is shown in [19], however, that the upper bound in (9) is actually the exact number for $m \leq 6$. The specific values are

$$q_{\mathrm{IE}}(3) = 1,\ q_{\mathrm{IE}}(4) = 3,\ q_{\mathrm{IE}}(5) = 11,\ q_{\mathrm{IE}}(6) = 55$$

([19, Subsection 6.2]).

Open problem: Does the upper bound in (9) agree with the exact number $q_{\mathrm{IE}}(m)$ for all m?

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