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Discrete topological methods for subspace arrangements

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§1. Introduction

A different (and relatively new) method to deal with the topology of Hyperplane Arrangements (and more generally Subspace Arrangements) is given by a discrete version of Morse Theory, namely the so called Discrete Morse Theory. Practically, this theory originated from the standard theory as a suitable adjustment to spaces usually produced by discrete data, like simplicial complexes and more generally CW-complexes (see [Fo98, Fo02, Ko08]).

In [SaSe07] we considered complement to *real* hyperplane arrangements applying Discrete Morse Theory to a well-known CW-complex with the same homotopy type ([Sa87]). We re-proved the *minimality* of the complement: the complement to a hyperplane arrangement is a *minimal space,* i.e. it has the homotopy type of a CW-complex with as many i-cells as its ith-Betti number ($i \geq 0$). This interesting result was proven independently in [DP03, Ra02] as an existence-like theorem; the explicit structure of the minimal complex was considered before us by [Yo05] and after us by [De08] (see also [DeSe]) .

The construction which uses Discrete Morse Theory is much more precise, even if superabundant in the description of the attaching maps of the cells (new "reduced" descriptions, at least in case of dimension two, were recently found by Yoshinaga himself and, by different method, by the two authors togheter with G. Gaiffi [GMS10]). This combinatorial method allows to produce algebraic complexes which calculate local

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systems in general (see formulas in [SaSe07, GaSa09]), which can be considered togheter with the many other well-known constructions which appear in the literature.

In this paper we outline an extension of the use of the combinatorial methods above to a certain class of *subspace arrangements,* which contains interesting spaces like $k(\pi, 1)$-spaces for Coxeter groups (see below).

Recall that classical Configuration Spaces in a manifold M are defined as the manifolds $F(M, n)$ of ordered n-tuples of *pairwise different* points in M $(n > 0)$. Here we are interested in the case $M = \mathbb{R}^d$, $d > 0$. By using coordinates in $(\mathbb{R}^d)^n = \mathbb{R}^{nd}$

$$x_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, d,$$

one has

$$F(\mathbb{R}^d, n) \ = \ \mathbb{R}^{nd} \setminus \cup_{i \neq j} H_{ij}^{(d)},$$

where $H_{ij}^{(d)}$ is the codimension d-subspace

$$\cap_{k=1,\ldots,d} \{x_{ik} = x_{jk}\}.$$

This latter subspace is the intersection of d hyperplanes in $\mathbb{R}^{nd}$, each obtained by the hyperplane $H_{ij} = \{x \in \mathbb{R}^n \ : \ x_i = x_j\}$, considered on the k-th component in $(\mathbb{R}^n)^d = \mathbb{R}^{nd}$, $k = 1, \ldots, d$.

We consider here a *Generalized Configuration Space* (for brevity, simply a Configuration Space). Such spaces start from any *Hyperplane Arrangement* $\mathcal{A}$ in $\mathbb{R}^n$. For each $d > 0$, one defines a *d-complexification* $\mathcal{A}^{(d)} \subset M^d$ of $\mathcal{A}$, which is given by the collection $\{H^{(d)},\ H \in \mathcal{A}\}$ of the *d-complexified* subspaces. There is an associated *configuration space,* namely the complement to the subspace arrangement

$$\mathcal{M}^{(d)} \ = \ \mathcal{M}(\mathcal{A})^{(d)} := \ (\mathbb{R}^n)^d \setminus \bigcup_{H \in \mathcal{A}} H^{(d)} \ .$$

Notice that for $d = 2$ one has the standard complexification of a real hyperplane arrangement. The natural inclusion $\mathcal{M}^{(d)} \hookrightarrow \mathcal{M}^{(d+1)}$ gives rise to a limit space which is contractible. In case of a reflection arrangement relative to a group W, the limit of the orbit space with respect to the action of W is a model of the classifying space of W; see [DS00].

In this paper we use the Discrete Morse Theory for CW-complexes to find an explicit *minimal* CW-complex for the configuration space $\mathcal{M}(\mathcal{A})^{(d)}$, for all $d \geq 1$. That is, we give a CW-complex having as many i-cells as the i-th Betti number B_i of $\mathcal{M}(\mathcal{A})^{(d)}$, $i \geq 0$.

For $d = 1$ the construction is trivial, since $\mathcal{M}^{(1)}$ is a disjoint union of convex sets (the *chambers*). Case $d = 2$ was that considered in [SaSe07]. Even if for $d > 2$ the minimality of configuration spaces can be theoretically deduced by their simply-connectness (see for ex. [Ha02], Prop. 4C.1), our construction is useful in order to produce geometric bases for the cohomology. In fact, we give explicit bases for the homology (and cohomology) of $\mathcal{M}^{(d+1)}$ which we call (d)-*polar bases* (see below). As far as we know, there is no other precise description of a geometric $\mathbb{Z}$-basis in the literature, except for some particular arrangements, even if the $\mathbb{Z}$-module structure derives from the well-known formulas in [GM88]: in fact, the intersection lattice of the d-complexification $\mathcal{A}^{(d)}$ is the same for all $d \geq 1$.

Notice also that the proof of minimality, in case $d > 2$, is straightforward from our construction because of the gaps among the dimensions of the critical cells.

We construct a discrete vector field, depending on a system of *polar coordinates* in $\mathbb{R}^n$, which is generic with respect to the arrangement. We use the coordinates to associate to such system a *total ordering* $\lhd$ (called *polar ordering*) on the set $\Phi := \{F\}$ of all the facets of the stratification of $\mathbb{R}^n$ induced by the arrangement. So, the philosophy of the paper is similar to that used for $d = 2$ in [SaSe07]; however, the extension to the case $d > 2$ is not trivial and technical so all details of the proof will be published elsewhere. For reader convenience, we prefer here to give detailed examples, writing down complete lists of cells of the *CW*-complexes involved, of the polar ordering introduced, of the pairs in the discrete fields and of the critical cells.

In some details, Section 2 is devoted to recall some notations and results from [DS00], including a description of the complex $\mathbf{S}^{(d)}$.

In Section 3 we introduce one example which will be reconsidered all over the paper: the full reflection arrangement A_3 in $\mathbb{R}^3$.

In Section 4 we briefly give the basic results from Discrete Morse Theory, using the original language from [Fo98],[Fo02], and some reference to the more combinatorial language in [Ko08].

In Section 5 we recall the main constructions of case $d = 2$, considered in [SaSe07].

In Section 7 we introduce the *degree-d discrete polar vector field* and we characterize its critical cells. We obtain that critical dk-cells in $\mathbf{S}^{(d)}$ correspond to chains

$$(C \prec F^k \prec \ldots \prec F^k)$$

(k is the codimension) if d is odd; to

$$(op_{F^k}(C) \prec F^k \prec \ldots \prec F^k)$$

if d is even, where $op_{F^k}(C)$ is the chamber opposite to C with respect to F^k. Here $(C \prec F^k)$ corresponds to a critical cell in $\mathbf{S}^{(1)}$, with respect to polar ordering $\lhd$, so it is characterized by

$$\begin{array}{lll} F \lhd G, & \forall\, G \text{ such that} & F \prec G; \\ H \lhd F, & \forall\, H \text{ such that} & C \prec H \prec F. \end{array}$$

As a final remark, we propose the following conjecture, related in some way to this paper: *torsion-free subspace arrangements are minimal*, that is, when the complement of the arrangement has torsion-free cohomology, then it is a minimal space (see the concluding part 9 for some more details).

§2. CW-structures for the configuration spaces

In this section we use some notations and results from [DS00].

Let $\mathcal{A} = \{H_j\}_{j \in J}$ be a finite arrangement of linear hyperplanes in $M := \mathbb{R}^n$. For brevity, we give all definitions in the case when $\mathcal{A}$ is a central arrangement, remarking that all we say can be generalized to the affine case with little changes.

We introduce a coordinate $x \in M$ and coordinates $(x_1, ..., x_d)$, $x_i \in M$, in M^d, $d > 0$. Each hyperplane is given by a linear equation $H_j = \{x \in M \;:\; a_j \cdot x = 0\}$, $a_j \in M \setminus \{0\}$. For each $d > 0$, one has the *d-complexification* $\mathcal{A}^{(d)} \subset M^d$ of $\mathcal{A}$, given by the collection of linear codimension-d *subspaces*

$$(H_j)^{(d)} := \{(x_1, \ldots, x_d) \;:\; a_j \cdot x_k = 0,\; k = 1, \ldots, d\}$$

(when $d = 2$ one has the standard complexification $\mathcal{A}_{\mathbb{C}} \subset \mathbb{C}^n$).

The *generalized configuration space* associated to $\mathcal{A}$ is the complement to the subspace arrangement

$$\mathcal{M}^{(d)} = \mathcal{M}(\mathcal{A})^{(d)} := M^d \setminus \bigcup_{H \in \mathcal{A}} H^{(d)} .$$

It is convenient to introduce the *intersection lattice* $\mathcal{L} := \mathcal{L}(\mathcal{A})$ of $\mathcal{A}$, whose elements are all the subspaces of M of the form

$$L = H_{j_1} \cap \cdots \cap H_{j_k}, \quad H_{j_l} \in \mathcal{A}.$$

The partial ordering in $\mathcal{L}$ is given by

$$L \prec L' \text{ iff } L' \subset L,$$

so there is a minimum element, corresponding to the empty intersection, which is the whole space M, and a maximum $L_0 := \bigcap_{H \in A} H$. The *rank* $rk(L)$ of a subspace $L \in \mathcal{L}$ is its codimension; the rank of $\mathcal{L}$ is the rank of L_0, and we also set $rk(\mathcal{A}) := rk(\mathcal{L}(\mathcal{A}))$. The arrangement is called *essential* when $rk(\mathcal{A}) = n = dim(M)$, i.e. when L_0 reduces to a single point.

In the present situation we can consider a finer poset $\Phi := \Phi(\mathcal{A}) := (\{F\}, \prec)$ whose elements are the strata (also called *facets*) of the stratification induced on M by $\mathcal{A}$, where, as usual:

$$F \prec F' \text{ iff } F' \subset cl(F).$$

The atoms (*chambers*) of $\Phi(\mathcal{A})$ are the connected components of $\mathcal{M}^{(1)}$.

We have a map $\Phi \to \mathcal{L}$ which associates to a facet F its *support* $|F|$, which is by definition the subspace generated by F (in a different language, this is the standard map between an oriented matroid and its underlying matroid). We define the rank function on Φ via this map:

$$rk(F) := rk(|F|) = codim(F).$$

Given $L \in \mathcal{L}(\mathcal{A})$, we will use the arrangements $\mathcal{A}_L := \{H \in \mathcal{A} : H \prec L\}$, $\mathcal{A}^L := \{L \cap H : H \in \mathcal{A},\ H \not\prec L\}$. The former is an arrangement in M of rank equal to $rk(L)$, the latter is an arrangement inside L itself (of rank $rk(\mathcal{A}) - rk(L)$). Let $\Phi_L := \Phi(\mathcal{A}_L)$, $\Phi^L := \Phi(\mathcal{A}^L)$ be the induced stratifications of M, L respectively. There is a map $pr_L : \Phi \to \Phi_L$, taking F' into the unique stratum containing it, and a map $j^L : \Phi^L \to \Phi$ just given by the inclusion.

Fixing a facet F, set also $\Phi_F = \{F' \in \Phi : F' \prec F\}$. It is easy to see that the restriction $\varphi_F := (pr_{|F|})_{|\Phi_F|} : \Phi_F \to \Phi_{|F|}$ is a dimension-preserving bijection of posets.

Let now Φ^d be the product of d copies of Φ, $d \geq 0$, and let

$$\Phi^{(d)} = \{(F_1, \dots, F_d) \in \Phi^d : F_1 \prec \cdots \prec F_d\}$$

be the set of d-chains in Φ (repetitions in the chain are allowed). Then $\Phi^{(d)}$ corresponds to a stratification of the space M^d as follows (see [DS00]): to each $\mathcal{F} = (F_1, \dots, F_d)$ in $\Phi^{(d)}$ it corresponds the stratum $\hat{\mathcal{F}}$ in M^d given by

$$\hat{\mathcal{F}} := \{(x_1, \dots, x_d) \in M^d : x_1 \in F_d\ ,\ x_k \in \varphi_{F_{d-k+2}}(F_{d-k+1})\ ,\ k = 2, \dots, d\}.$$

One has:

Proposition 2.1. *(i) Each $\hat{\mathcal{F}}$ is homeomorphic to an open cell*
(ii) $\bigcup_{\mathcal{F}\in\Phi^{(d)}} \hat{\mathcal{F}} = M^d$
(iii) $\hat{\mathcal{F}} \cap \hat{\mathcal{G}} = \emptyset$ *if* $\mathcal{F} \neq \mathcal{G}$
(iv) $cl(\hat{\mathcal{F}}) \cap \hat{\mathcal{G}} \neq \emptyset$ *iff* $cl(\hat{\mathcal{F}}) \supset \hat{\mathcal{G}}$
(v) $\mathcal{M}^{(d)} = \bigcup_{\{\mathcal{F}\in\Phi^{(d)}\ :\ F_1 \text{ is a chamber of } \Phi\}} \hat{\mathcal{F}}$.

For $\mathcal{F} = (F_1, \ldots, F_d)$, one has

$$codim(\mathcal{F}) := codim(\hat{\mathcal{F}}) = \sum_{i=1}^{d} codim(F_i).$$

The partial ordering on $\Phi^{(d)}$ is given by

$$\mathcal{F} \prec \mathcal{G} \quad \text{iff} \quad \hat{\mathcal{G}} \subset cl(\hat{\mathcal{F}}).$$

This has the following characterization.

Lemma 2.2. *For* $\mathcal{F} = (F_1, \ldots, F_d)$, $\mathcal{G} = (G_1, \ldots, G_d) \in \Phi^{(d)}$ *one has*

$$\mathcal{F} \prec \mathcal{G} \quad \textit{iff } F_d \prec G_d \quad \textit{and} \quad pr_{|F_{i+1}|}(F_i) \prec pr_{|F_{i+1}|}(G_i)$$

in the stratification $\Phi_{|F_{i+1}|}$, $i = d-1, \ldots, 1$.

Part (v) of Proposition 2.1 gives us the poset corresponding to the induced stratification of the generalized configuration space $\mathcal{M}^{(d)}$ which is

$$\Phi_0^{(d)} := \{\mathcal{F} = (F_1, \ldots, F_d) \in \Phi^{(d)} :\ rk(F_1) = 0\}$$

while the union $\bigcup_{H\in A} H^{(d)}$ of the d-complexified subspaces correspond to the poset

$$\Phi_+^{(d)} := \{\mathcal{F} = (F_1, \ldots, F_d) \in \Phi^{(d)} :\ rk(F_1) > 0\}.$$

Proposition 2.3. *The set*

$$\mathbf{Q}^{(d)} := \bigcup_{\mathcal{F}\in\Phi^{(d)}} e(\mathcal{F}),$$

where $e(\mathcal{F})$ is the dual *cell to the stratum $\mathcal{F}$, is a cellular $n'd$-ball in M^d (a regular cell complex) dual to the stratification, where $n' := rk(L_0)$.*

Remark. *It follows from Lemma 2.2 that if the first element F_1 of $\mathcal{F}$ is a chamber, then also the first element of any $\mathcal{G} \prec \mathcal{F}$ is a chamber.*

Definition 2.4. *We denote by* $\mathbf{S}^{(d)}$ *the subcomplex of* $\mathbf{Q}^{(d+1)}$ *whose cells correspond to* $\Phi_0^{(d+1)}$:

$$\mathbf{S}^{(d)} \; := \; \cup_{\mathcal{F}\in\Phi_0^{(d+1)}} e(\mathcal{F})$$

(case $d = 1$ was introduced in [Sa87]; see also [BZ92], [OT92]).

In general, given a chamber C and a facet F in $\Phi^{(d+1)}$ we will use the notation

$$C.F \; := \; \varphi_F^{-1}(pr_{|F|}(C))$$

which is a uniquely defined chamber containing F in its boundary.

One has:

Theorem 1. *(i)* $\mathbf{S}^{(d)}$ *is a deformation retract of* $\mathcal{M}^{(d+1)}$.
(ii) Writing a $d+1$*-chain* $\mathcal{F} \in \Phi_0^{(d+1)}$ *as* $\mathcal{F} = (C, \mathcal{F}')$, $\mathcal{F}' \in \Phi^{(d)}$, *we have:*

$$\partial(e(C,\mathcal{F}')) \; = \bigcup_{\mathcal{F}'' \prec \mathcal{F}',\; codim(\mathcal{F}'') \,=\, codim(\mathcal{F}')-1} e(C.F_1, \mathcal{F}'')$$

($\mathcal{F}'' = (F_1, \ldots)$.*)*
(ii) $dim\ e(C,\mathcal{F}') = dim\ e(\mathcal{F}') = codim_{M^d}(\mathcal{F}')$. *In particular,* $dim(\mathbf{S}^{(d)}) = dim(\mathbf{Q}^{(d)}) = n'd$.

§3. Examples (1)

We outline here a 3-dimensional example. We use it later in order to illustrate the results of next sections.

Example. The full reflection arrangement of type A_3 is an arrangement $\mathcal{A} \in \mathbb{R}^3 = (x_1, x_2, x_3)$ given by the section of the arrangement $\mathcal{B} = \{\{x_1 = x_2\}, \{x_1 = x_3\}, \{x_2 = x_3\}, \{x_1 = x_4\}, \{x_2 = x_4\}, \{x_3 = x_4\}\} \in \mathbb{R}^4$ with the 3-plane $\{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 1\}$ orthogonal to the line $l : x_1 = x_2 = x_3 = x_4$. Fig. 1 represents the section of $\mathcal{A}$ with the 2-plane $z = a,\ a > 0$, while Fig. 2 represents the section with the 2-plane $z = -a$. In Fig. 2 we indicate with $\overline{F}$ the opposite of a bounded facet F of Fig. 1. The center of this arrangement is a facet of dimension 0, not indicated in our figures and corresponding to the origin of $\mathbb{R}^3$, that we call P.

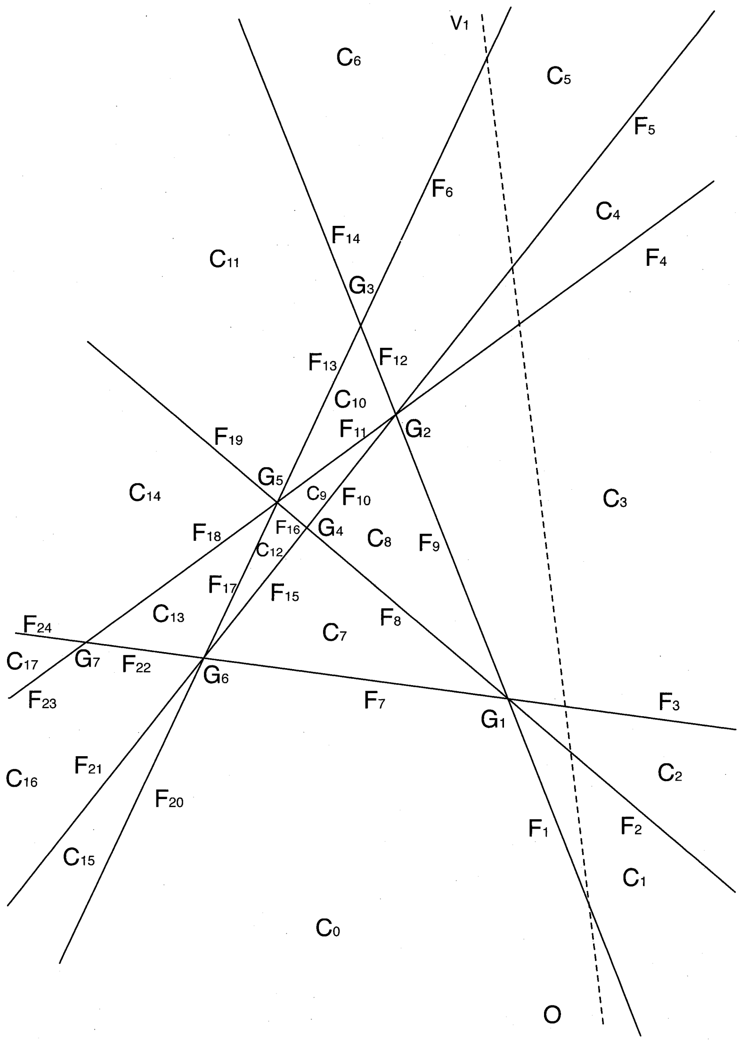

Fig. 1. An upper section of the reflection arrangement A_3

The complex $\mathbf{S}^{(1)}$ is composed by the following cells:

- 24 cells of dimension 0:

$$e(C_i, C_i),\ i = 0, \ldots, 17,$$

more

$$e(\overline{C}_7, \overline{C}_7), e(\overline{C}_8, \overline{C}_8), e(\overline{C}_9, \overline{C}_9), e(\overline{C}_{10}, \overline{C}_{10}), e(\overline{C}_{12}, \overline{C}_{12}), e(\overline{C}_{13}, \overline{C}_{13});$$

- 72 cells of dimension 1: $e(C_0, F_1)$, $e(C_0, F_7)$, $e(C_0, F_{20})$, $e(C_1, F_1)$, $e(C_1, F_2)$, $e(C_1, \overline{F}_{13})$, $e(C_2, F_2)$, $e(C_2, F_3)$, $e(C_2, \overline{F}_{18})$, $e(C_3, F_3)$, $e(C_3, F_4)$, $e(C_3, F_9)$, $e(C_4, F_4)$, $e(C_4, F_5)$, $e(C_4, \overline{F}_{22})$, $e(C_5, F_5)$, $e(C_5, F_6)$, $e(C_5, F_{12})$, $e(C_6, F_6)$, $e(C_6, F_{14})$, $e(C_6, \overline{F}_7)$, $e(C_7, F_7)$, $e(C_7, F_8)$, $e(C_7, F_{15})$, $e(C_8, F_8)$, $e(C_8, F_9)$, $e(C_8, F_{10})$, $e(C_9, F_{10})$, $e(C_9, F_{11})$, $e(C_9, F_{16})$, $e(C_{10}, F_{11})$, $e(C_{10}, F_{12})$, $e(C_{10}, F_{13})$, $e(C_{11}, F_{13})$, $e(C_{11}, F_{14})$, $e(C_{11}, F_{19})$, $e(C_{12}, F_{15})$, $e(C_{12}, F_{16})$, $e(C_{12}, F_{17})$, $e(C_{13}, F_{17})$, $e(C_{13}, F_{18})$, $e(C_{13}, F_{22})$, $e(C_{14}, F_{18})$, $e(C_{14}, F_{19})$, $e(C_{14}, F_{24})$, $e(C_{15}, F_{20})$, $e(C_{15}, F_{21})$, $e(C_{15}, \overline{F}_{12})$, $e(C_{16}, F_{21})$, $e(C_{16}, F_{22})$, $e(C_{16}, F_{23})$, $e(C_{17}, F_{23})$, $e(C_{17}, F_{24})$, $e(C_{17}, \overline{F}_9)$, $e(\overline{C}_7, \overline{F}_{15})$, $e(\overline{C}_7, \overline{F}_7)$, $e(\overline{C}_7, \overline{F}_8)$, $e(\overline{C}_8, \overline{F}_{10})$, $e(\overline{C}_8, \overline{F}_9)$, $e(\overline{C}_8, \overline{F}_8)$, $e(\overline{C}_9, \overline{F}_{16})$, $e(\overline{C}_9, \overline{F}_{10})$, $e(\overline{C}_9, \overline{F}_{11})$, $e(\overline{C}_{10}, \overline{F}_{13})$, $e(\overline{C}_{10}, \overline{F}_{11})$, $e(\overline{C}_{10}, \overline{F}_{12})$, $e(\overline{C}_{12}, \overline{F}_{17})$, $e(\overline{C}_{12}, \overline{F}_{15})$, $e(\overline{C}_{12}, \overline{F}_{16})$, $e(\overline{C}_{13}, \overline{F}_{18})$, $e(\overline{C}_{13}, \overline{F}_{22})$, $e(\overline{C}_{13}, \overline{F}_{17})$;
- 72 cells of dimension 2: $e(C_0, G_1)$, $e(C_0, G_6)$, $e(C_0, \overline{G}_3)$, $e(C_1, G_1)$, $e(C_1, \overline{G}_3)$, $e(C_1, \overline{G}_5)$, $e(C_2, G_1)$, $e(C_2, \overline{G}_5)$, $e(C_2, \overline{G}_7)$, $e(C_3, G_1)$, $e(C_3, G_2)$, $e(C_3, \overline{G}_7)$, $e(C_4, G_2)$, $e(C_4, \overline{G}_7)$, $e(C_4, \overline{G}_6)$, $e(C_5, G_2)$, $e(C_5, G_3)$, $e(C_5, \overline{G}_6)$, $e(C_6, G_3)$, $e(C_6, \overline{G}_6)$, $e(C_6, \overline{G}_1)$, $e(C_7, G_1)$, $e(C_7, G_4)$, $e(C_7, G_6)$, $e(C_8, G_1)$, $e(C_8, G_2)$, $e(C_8, G_4)$, $e(C_9, G_2)$, $e(C_9, G_4)$, $e(C_9, G_5)$, $e(C_{10}, G_2)$, $e(C_{10}, G_3)$, $e(C_{10}, G_5)$, $e(C_{11}, G_3)$, $e(C_{11}, G_5)$, $e(C_{11}, \overline{G}_1)$, $e(C_{12}, G_4)$, $e(C_{12}, G_5)$, $e(C_{12}, G_6)$, $e(C_{13}, G_5)$, $e(C_{13}, G_6)$, $e(C_{13}, G_7)$, $e(C_{14}, G_5)$, $e(C_{14}, G_7)$, $e(C_{14}, \overline{G}_1)$, $e(C_{15}, G_6)$, $e(C_{15}, \overline{G}_2)$, $e(C_{15}, \overline{G}_3)$, $e(C_{16}, G_6)$, $e(C_{16}, G_7)$, $e(C_{16}, \overline{G}_2)$, $e(C_{17}, G_7)$, $e(C_{17}, \overline{G}_1)$, $e(C_{17}, \overline{G}_2)$, $e(\overline{C}_7, \overline{G}_6)$, $e(\overline{C}_7, \overline{G}_1)$, $e(\overline{C}_7, \overline{G}_4)$, $e(\overline{C}_8, \overline{G}_4)$, $e(\overline{C}_8, \overline{G}_1)$, $e(\overline{C}_8, \overline{G}_2)$, $e(\overline{C}_9, \overline{G}_5)$, $e(\overline{C}_9, \overline{G}_4)$, $e(\overline{C}_9, \overline{G}_2)$, $e(\overline{C}_{10}, \overline{G}_5)$, $e(\overline{C}_{10}, \overline{G}_2)$, $e(\overline{C}_{10}, \overline{G}_3)$, $e(\overline{C}_{12}, \overline{G}_6)$, $e(\overline{C}_{12}, \overline{G}_5)$, $e(\overline{C}_{12}, \overline{G}_4)$, $e(\overline{C}_{13}, \overline{G}_7)$, $e(\overline{C}_{13}, \overline{G}_6)$, $e(\overline{C}_{13}, \overline{G}_5)$;

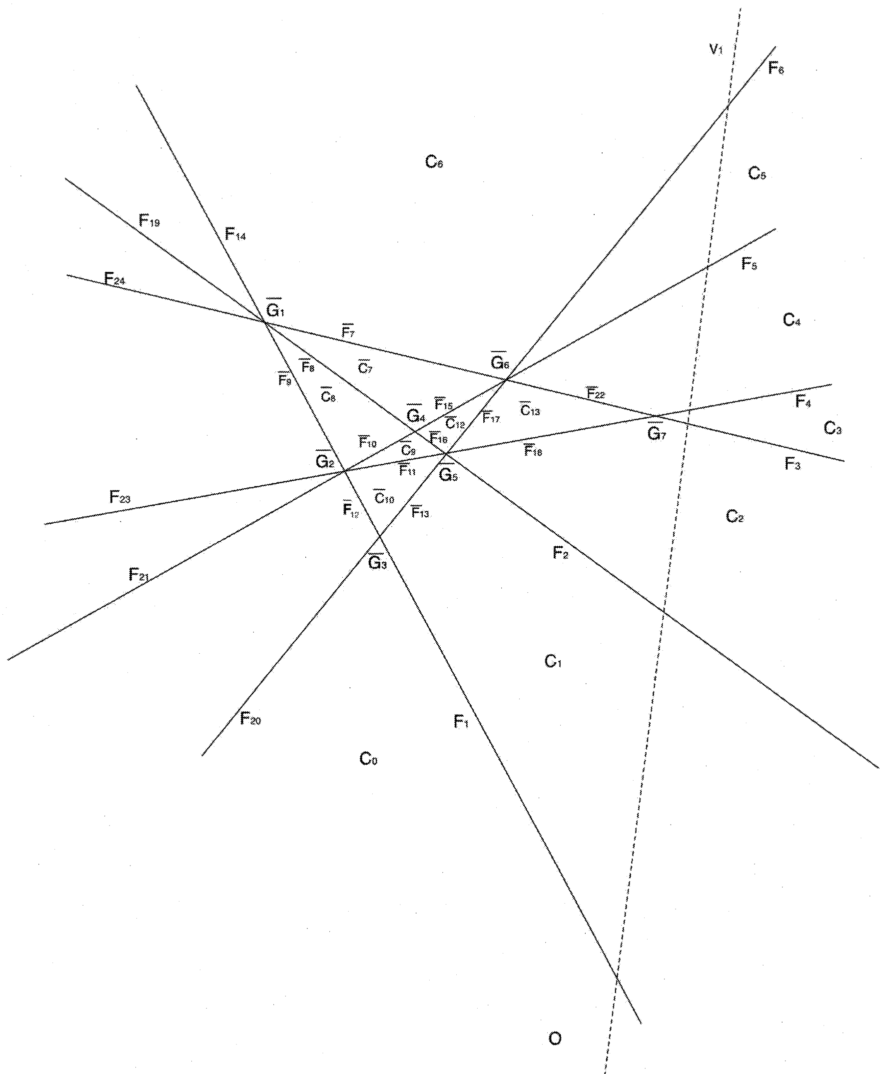

Fig. 2. A lower section of the reflection arrangement A_3

- 24 cells of dimension 3:

$$e(C_i, P),\ i = 0, \ldots, 17,$$

more

$$e(\overline{C}_7, P), e(\overline{C}_8, P), e(\overline{C}_9, P), e(\overline{C}_{10}, P), e(\overline{C}_{12}, P), e(\overline{C}_{13}, P).$$

The complex $\mathbf{S}^{(2)}$ is composed by the following cells:

- 24 cells of dimension 0: The cells of form $e(C_i, C_i, C_i)$ where $C_i \in \Phi(\mathcal{A})$ is a chamber, i.e. a facet of codimension 0.
- 72 cells of dimension 1: The cells of form $e(C_i, C_i, F_j)$ where C_i is a chamber, and $F_j \in \Phi(\mathcal{A})$ is a facet of codimension 1 in the boundary of C_i, i.e. $C_i \prec F_j$.
- 72+72 cells of dimension 2: 72 cells of form $e(C_i, C_i, G_k)$ where C_i is a chamber, and $G_k \in \Phi(\mathcal{A})$ is a facet of codimension 2 in the boundary of C_i, i.e. $C_i \prec G_k$; 72 cells of form $e(C_i, F_j, F_j)$ where C_i is a chamber, and F_j is a facet of codimension 1 in the boundary of C_i.
- 24+144 cells of dimension 3: 24 cells of form $e(C_i, C_i, P)$ where C_i is a chamber, and $P \in \Phi(\mathcal{A})$ is the center of $\mathcal{A}$, i.e. the only facet of codimension 3 in $\Phi(\mathcal{A})$; 144 cells of form $e(C_i, F_j, G_k)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i, and G_k is a facet of codimension 2 in the boundary of F_j, i.e. $F_j \prec G_k$.
- 72+72 cells of dimension 4: 72 cells of form $e(C_i, F_j, P)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i and P is the center of $\mathcal{A}$; 72 cells of form $e(C_i, G_k, G_k)$, where C_i is a chamber and G_k is a facet of codimension 2 in the boundary of C_i.
- 72 cells of dimension 5: The cells of form $e(C_i, G_k, P)$, where C_i is a chamber, G_k is a facet of codimension 2 in the boundary of C_i and P is the center of $\mathcal{A}$.
- 24 cells of dimension 6: The cells of the form $e(C_i, P, P)$ where C_i is a chamber, and P is the center of $\mathcal{A}$.

The complex $\mathbf{S}^{(3)}$ is composed by the following cells:

- 24 cells of dimension 0: The cells of form $e(C_i, C_i, C_i, C_i)$ where C_i is a chamber.
- 72 cells of dimension 1: The cells of form $e(C_i, C_i, C_i, F_j)$ where C_i is a chamber, and F_j is a facet of codimension 1 in the boundary of C_i.
- 72+72 cells of dimension 2: 72 cells of form $e(C_i, C_i, C_i, G_k)$ where C_i is a chamber, and G_k is a facet of codimension 2 in

the boundary of C_i; 72 cells of form $e(C_i, C_i, F_j, F_j)$ where C_i is a chamber, and F_j is a facet of codimension 1 in the boundary of C_i.

- 24+144+72 cells of dimension 3: 24 cells of form $e(C_i, C_i, C_i, P)$ where C_i is a chamber and P is the center of $\mathcal{A}$; 144 cells of form $e(C_i, C_i, F_j, G_k)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i, and G_k is a facet of codimension 2 in the boundary of F_j; 72 cells of form $e(C_i, F_j, F_j, F_j)$ where C_i is a chamber, and F_j is a facet of codimension 1 in the boundary of C_i.
- 72+72+144 cells of dimension 4: 72 cells of form $e(C_i, C_i, F_j, P)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i and P is the center of $\mathcal{A}$; 72 cells of form $e(C_i, C_i, G_k, G_k)$, where C_i is a chamber and G_k is a facet of codimension 2 in the boundary of C_i; 144 cells of form $e(C_i, F_j, F_j, G_k)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i, and G_k is a facet of codimension 2 in the boundary of F_j.
- 72+72+144 cells of dimension 5: 72 cells of form $e(C_i, C_i, G_k, P)$, where C_i is a chamber, G_k is a facet of codimension 2 in the boundary of C_i and P is the center of $\mathcal{A}$; 72 cells of form $e(C_i, F_j, F_j, P)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i and P is the center of $\mathcal{A}$; 144 cells of form $e(C_i, F_j, G_k, G_k)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i, and G_k is a facet of codimension 2 in the boundary of F_j.
- 24+144+72 cells of dimension 6: 24 cells of form $e(C_i, C_i, P, P)$ where C_i is a chamber, and P is the center of $\mathcal{A}$; 144 cells of form $e(C_i, F_j, G_k, P)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i, and G_k is a facet of codimension 2 in the boundary of F_j and P is the center of $\mathcal{A}$; 72 cells of form $e(C_i, G_k, G_k, G_k)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i.
- 72+72 cells of dimension 7: 72 cells of form $e(C_i, G_k, G_k, P)$, where C_i is a chamber, G_k is a facet of codimension 2 in the boundary of C_i and P is the center of $\mathcal{A}$; 72 cells of form $e(C_i, F_j, P, P)$, where C_i is a chamber, F_j is a facet of codimension 1 in the boundary of C_i and P is the center of $\mathcal{A}$.
- 72 cells of dimension 8: The cells of form $e(C_i, G_k, P, P)$, where C_i is a chamber, G_k is a facet of codimension 2 in the boundary of C_i and P is the center of $\mathcal{A}$.

- 24 cells of dimension 9: The cells of form $e(C_i, P, P, P)$ where C_i is a chamber, and P is the center of $\mathcal{A}$.

§4. Essentials on discrete Morse theory for CW-complexes

We recall here some of the main definitions and results from [Fo98], [Fo02], where Morse theory from a combinatorial viewpoint was first developed. See also [Ko08], where the same theory is re-formulated from a more combinatorial point of view.

Let $\mathcal{C}$ be a finite *regular* CW-complex. Let K be the collection of cells of $\mathcal{C}$, partially ordered by

$$\sigma < \tau \quad \Leftrightarrow \quad \sigma \subset \tau.$$

As usual, denote by K_p the p-skeleton of K.

Definition 4.1. *A discrete Morse function on $\mathcal{C}$ is a function*

$$f : K \longrightarrow \mathbb{R}$$

satisfying for all $\sigma^{(p)} \in K_p$ the following two conditions

$$\begin{array}{lll} (i) & \sharp\{\tau^{(p+1)} > \sigma^{(p)} \mid f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\} & \leq \;\; 1 \\ (ii) & \sharp\{v^{(p-1)} < \sigma^{(p)} \mid f(\sigma^{(p)}) \leq f(v^{(p-1)})\} & \leq \;\; 1. \end{array}$$

Actually, one shows that, if f satisfies (i) and (ii) above, then for any given cell of K at least one between (i), (ii) is a strict inequality.
The analog of a critical point of index p in standard Morse theory is here a *critical cell* of dimension p: a p-cell $\sigma^{(p)}$ is critical iff both the cardinalities in i), ii) are zero.

Let $m_p(f)$ denote the number of critical p-cells of f. Then one has

Proposition 4.2. *$\mathcal{C}$ is homotopy equivalent to a CW-complex which has exactly $m_p(f)$ cells of dimension p.*

The *discrete gradient vector field* Γ_f of a Morse function f over K is the set of all pairs of cells for which the exception in Definition 4.1 happens:

$$\Gamma_f \;=\; \{(\sigma^{(p)}, \tau^{(p+1)}) |\; \sigma^{(p)} < \tau^{(p+1)},\; f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\}.$$

Since, for any given cell, at most one between (i), (ii) in 4.1 is an equality, it follows that each cell belongs to at most one pair of Γ_f.

A general definition of discrete vector field is the following.

Definition 4.3. *A discrete vector field Γ on $\mathcal{C}$ is a collection of pairs of cells $(\sigma^{(p)}, \tau^{(p+1)}) \in \mathcal{C} \times \mathcal{C}$ such that $\sigma^{(p)} < \tau^{(p+1)}$ and such that each cell of $\mathcal{C}$ belongs to at most one pair of Γ.*

Remark 4.4. *A discrete vector field over a CW-complex corresponds to a* matching *of the associated poset (see* [Ko08]*).*

For Γ as above, define a Γ-path as a sequence of cells

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \cdots, \tau_r^{(p+1)}, \sigma_{r+1}^{(p)} \tag{1}$$

such that for each $i = 0, \cdots, r$ one has $(\sigma_r^{(p)}, \tau_r^{(p+1)}) \in \Gamma$ and $\sigma_i^{(p)} \neq \sigma_{i+1}^{(p)} < \tau_i^{(p+1)}$.
The Γ-path is closed (and non-trivial) iff $\sigma_0^{(p)} = \sigma_{r+1}^{(p)},\ r \geq 0$. One has:

Theorem 2. *A discrete vector field Γ is the gradient vector field of a discrete Morse function on $\mathcal{C}$ iff there are no non-trivial closed Γ-paths.*

Remark 4.5. *A gradient vector field over a CW complex $\mathcal{C}$ corresponds to an* acyclic matching *of the associated poset, i.e. a matching without closed loops. In many cases it is convenient to find directly a gradient vector field, or equivalent an acyclic matching, without passing through an effective discrete Morse function. The critical cells in this case are simply all the cells which do not belong to pairs of the field. The datum of an acyclic matching produces a* collapsing ordering *on the non singular cells of $\mathcal{C}$, producing a constructive reduction of $\mathcal{C}$ in proposition 4.2 (see* [Ko08]*).*

§5. The standard case of complexified real arrangements

In this part we consider the case of standard complexified arrangements.

The last part of Section 5 is devoted to recall (skipping some details) some of the results in [SaSe07], where we applied the theory to Hyperplane Arrangements. We set in this section $\mathbf{S} := \mathbf{S}^{(1)}$, the case of standard complexification. A k-cell of $\mathbf{S}$ is written as $e(\mathcal{F}),\ \mathcal{F} = (C, F)$, with $codim(F) = k$. We write for brevity also $e(\mathcal{F}) = e(C, F)$. The boundary condition given in Lemma 2.2 specializes here to: $e(D, G)$ is in the boundary of $e(C, F)$ iff

i) $G \prec F$

ii) the chambers C and D are contained in the same chamber of $\mathcal{A}_G$, that is $D = C.G$ in the notations of part 2.

The following constructions are based on a *generic* system of *polar coordinates* in M, which is associated to a *generic* flag of subspaces

$$V_i \;=< \mathbf{e}_1, ..., \mathbf{e}_i >, \quad i = 0, ..., n \; (dim(V_i) = i).$$

Consider the pencil of $n-1$-dimensional subspaces of M, $V_{n-1}(\theta)$, $\theta \in \mathbb{R}$, with base V_{n-2} : so $V_{n-1}(0) = V_{n-1}$ and θ grows according to some positive orientation of V/V_{n-2}. By recurrence, let $V_i(\theta, \theta_{i+1}, \dots, \theta_{n-1})$, $\theta \in \mathbb{R}$, be the pencil of i-dimensional subspaces in $V_{i+1}(\theta_{i+1}, \dots, \theta_{n-1})$, with base V_{i-1}, $i = 1, \dots, n-2$. Let also $V_0(\theta, \theta_1, \dots, \theta_{n-1})$ be the point with distance θ from the origin V_0 inside the line $V_1(\theta_1, \dots, \theta_{n-1})$.

Each point $P \neq V_0$ is written uniquely as $P = V_0(\theta_0, \theta_1, \dots, \theta_{n-1})$, so we associate to P the polar coordinates $(\theta_0, \dots, \theta_{n-1})$.

Definition 5.1. *We say that the above flag is* generic *with respect to the arrangement $\mathcal{A}$ if it satisfies the following conditions:*

i) the origin V_0 is contained in a chamber C_0 of $\mathcal{A}$;

ii) there exists a $\delta > 0$, $\delta << 1$, such that the set of bounded facets of $\mathcal{A}$ is contained into $\{P \in M : \; 0 < \theta_i(P) < \delta, \; i = 0, \dots, n-1\}$;

iii) subspaces $V_i(\bar\theta) = V_i(\bar\theta_i, ..., \bar\theta_{n-1})$ which intersect the union of bounded facets are generic with respect to $\mathcal{A}$, in the sense that, for each codim-k subspace $L \in L(\mathcal{A})$,

$$i \geq k \;\Rightarrow\; V_i(\bar\theta) \cap L \cap clos(\tilde B) \neq \emptyset \text{ and } dim(|V_i(\bar\theta)| \cap L) \;=\; i - k.$$

One proves that a generic system of polar coordinates always exists.

Fix a system of polar coordinates associated to a generic flag.

We consider the subspace $V_i(\bar\theta)$, $\bar\theta = (\bar\theta_i, ..., \bar\theta_{n-1})$. The arrangement $\mathcal{A}$ induces a stratification $\Phi(\bar\theta)$ in $V_i(\bar\theta)$: given a codimension-k facet $F \in \mathcal{S}$, let us denote by

$$F(\bar\theta) := F \cap V_i(\bar\theta), \quad \theta_j \in [0, \delta], \; j = i, ..., n-1.$$

By genericity conditions, if $i \geq k$ then $F(\bar\theta)$ is either empty or it is a codimension $k + n - i$ facet contained in $V_i(\bar\theta)$.

For each such $\bar\theta$ we want to give a *total ordering* on $\Phi(\bar\theta)$.

Definition 5.2. *Given any facet $F(\bar\theta)$ let us denote by*

$$P_{F(\bar\theta)} \in clos(F(\bar\theta))$$

the "minimum" vertex of $clos(F(\bar\theta)) \cap \{\theta_i \geq 0, \; i = 0, \dots\}$: this is the 0-dimensional facet which has minimum polar coordinates, with respect to the anti-lexicographic ordering of the coordinates (one says that $(\theta_0, \dots, \theta_{n-1})$ is less than $(\theta'_0, \dots, \theta'_{n-1})$ iff, being k the last index such that $\theta_h \neq \theta'_h$, one has $\theta_k < \theta'_k$).

Remark that the minimum vertex is well defined by genericity conditions.

Definition 5.3. *We define the* polar ordering *by recurrence on the dimension as follows: given $F,\ G\ \in \Phi$, and given $\bar{\theta} = (\bar{\theta}_i, ..., \bar{\theta}_{n-1})$, $0 \leq i \leq n$, $\bar{\theta}_j \in [0,\delta]$ for $j \in i,..,n-1$, ($\bar{\theta} = \emptyset$ for $i = n$) such that $F(\bar{\theta}), G(\bar{\theta}) \neq \emptyset$, we set*

$$F(\bar{\theta}) \ \lhd\ G(\bar{\theta})$$

iff one of the following cases occurs:

i) $P_{F(\bar{\theta})} \neq P_{G(\bar{\theta})}$ and the coordinates of the former point are lower (in the anti-lexicographical ordering) than the coordinates of the latter point.

ii) $P_{F(\bar{\theta})} = P_{G(\bar{\theta})}$. Then either

iia) $dim(F(\bar{\theta})) = 0$ (so $P_{F(\bar{\theta})} = F(\bar{\theta})$) and $F(\bar{\theta}) \neq G(\bar{\theta})$ (so $dim(G(\bar{\theta})) > 0$)

or

iib) $dim(F(\bar{\theta})) > 0,\ dim(G(\bar{\theta})) > 0$.

Let $i_0 := max\{j|\ \theta_j(P_{F(\bar{\theta})}) \neq 0\}+1$, and let $i_1 := min\{i_0, i\}$. If the coordinates of the minimal point are $P \equiv (\bar{\theta}_0, \ldots, \bar{\theta}_{n-1})$ then $\forall \epsilon,\ 0 < \epsilon << \delta$, it must happen

$$F(\bar{\theta}_{i_1-1}+\epsilon, \bar{\theta}_{i_1}, ...)\ \lhd\ G(\tilde{\theta}_{i_1-1}+\epsilon, \bar{\theta}_{i_1}, ...).$$

The idea in (iib) is to intersect F and G with a lower dimensional subspace contained in $V_i(\bar{\theta})$, very close to the minimum point (see also [DeSe] for a purely combinatorial version of polar ordering).

One shows that polar ordering $\lhd$ is a total ordering on the facets of $V_i(\bar{\theta})$, for any given $\bar{\theta} = (\bar{\theta}_i, ..., \bar{\theta}_{n-1})$. In particular (taking $\bar{\theta} = \emptyset$, which corresponds to all $V_n = M$) it gives a total ordering on Φ.

We consider now the regular CW-complex $\mathbf{S} = \mathbf{S}^{(1)}$ and we define a combinatorial *gradient vector field* Γ over $\mathbf{S}$. One can describe Γ as a collection of pairs of cells

$$\Gamma\ \subset\ \{(e,f) \in \mathbf{S}\times\mathbf{S} \mid dim(f) = dim(e)+1,\ e \in \partial(f)\}$$

so that Γ decomposes into its dimension-p components

$$\Gamma\ =\ \bigsqcup_{p=1}^{n} \Gamma^p, \qquad \Gamma^p \subset \mathbf{S}_{p-1}\times\mathbf{S}_p$$

($\mathbf{S}_p$ being the p-skeleton of $\mathbf{S}$).

We give the following recursive definition:

Definition 5.4 (Polar Gradient)**.** *We define a combinatorial gradient field* Γ *over* $\mathbf{S}$ *in the following way:*

the $(j+1)$*-th component* Γ^{j+1} *of* Γ, $j = 0, ..., n-1$, *is given by all pairs*

$$(e(C, F^j), e(C, F^{j+1})), \quad F^j \prec F^{j+1}$$

(same chamber C*) such that*

1. $F^{j+1} \lhd F^j$
2. $\forall F^{j-1} \prec F^j$ *such that* $C \prec F^{j-1}$ *the pair*

$$(e(C \prec F^{j-1}), e(C \prec F^j)) \notin \Gamma^j.$$

Theorem 3. *One has:*

1. Γ *is a combinatorial vector field on* $\mathbf{S}$ *which is the gradient of a discrete Morse function.*
2. *The pair*

$$(e(C \prec F^j), e(C \prec F^{j+1})), \quad F^j \prec F^{j+1}$$

belongs to Γ *iff the following conditions hold:*
(a) $F^{j+1} \lhd F^j$
(b) $\forall\ F^{j-1}$ *such that* $C \prec F^{j-1} \prec F^j$, *one has* $F^{j-1} \lhd F^j$.

3. *Given* $F^j \in \Phi$, *there exists a chamber* C *such that the cell* $e(C \prec F^j)$ *is the second factor of a pair in* Γ *iff there exists* $F^{j-1} \prec F^j$ *with* $F^j \lhd F^{j-1}$. *More precisely, for each chamber* C *such that there exists* F^{j-1} *with*

$$C \prec F^{j-1} \prec F^j, \quad F^j \lhd F^{j-1} \tag{$*$}$$

the pair $(e(C \prec \bar{F}^{j-1}), e(C \prec F^j)) \in \Gamma$, *where* $\bar{F}^{j-1}$ *is the maximum* $(j-1)$*-facet (with respect to polar ordering) satisfying conditions (*).*

4. *The set of* k*-dimensional critical cells is given by*

$$\begin{aligned} &Sing_k(\mathbf{S}) \\ &= \{e(C \prec F^k)\ :\ F^k \cap V_k \neq \emptyset,\ F^j \lhd F^k,\ \forall\ C \prec F^j \precneqq F^k\}. \end{aligned} \tag{2}$$

Equivalently, $F^k \cap V_k$ *is the maximum (in polar ordering) among all facets of* $C \cap V_k$.

In Section 7 we will need some more specific results which were used in the Proof of part (1) of Theorem 3.

Take a Γ-path in $\mathbf{S}$ (according to [Fo02])

(3)
$$e(C_1, F_1^k), e(C_1, F_1^{k+1}), \ldots, e(C_m, F_m^k), e(C_m, F_m^{k+1}), e(C_{m+1}, F_{m+1}^k).$$

Here the pair $(e(C_i, F_i^k), e(C_i, F_i^{k+1}))$ is an element of Γ, and $e(C_i, F_i^k)$ is in the boundary of $e(C_{i-1}, F_{i-1}^{k+1})$.

According to a standard result in Discrete Morse Theory ([Fo98]) we have to show that, if the path (3) is closed, (i.e. if $e(C_{m+1}, F_{m+1}^k)$ equals to $e(C_1, F_1^k)$), then it is trivial, i.e. $F_i^k = F_{i+1}^k$, $F_i^{k+1} = F_{i+1}^{k+1}$, and $C_i = C_{i+1}$ $(i = 1, \ldots, m-1)$.

The proof directly follows from the following two claims.

Claim 1. *Given a triple of consecutive cells in (3) of the form:*

$$e(C_i, F_i^{k+1}), e(C_{i+1}, F_{i+1}^k), e(C_{i+1}, F_{i+1}^{k+1}), \tag{4}$$

we have that $F_{i+1}^{k+1} \trianglelefteq F_i^{k+1}$.

Claim 2. *Given a quadruple of consecutive cells in (3) of the form:*

$$e(C_i, F_i^k), e(C_i, F^{k+1}), e(C_{i+1}, F_{i+1}^k), e(C_{i+1}, F^{k+1}), \tag{5}$$

we have $F_i^k \trianglelefteq F_{i+1}^k$.

Remark 5.5. *1) Once a polar ordering is assigned, the set of singular cells is described only in terms of it by*
$Sing_k(\mathbf{S}) := \{e(C \prec F^k) :$

a) $F^k \triangleleft F^{k+1}, \quad \forall F^{k+1} \quad s.t. \quad F^k \prec F^{k+1}$
b) $F' \triangleleft F^k, \quad \forall F' \quad s.t. \quad C \prec F' \prec F^k\ \}.$

2) The construction of Theorem 3 gives an explicit additive basis for the homology and for the cohomology in terms of the singular cells in $\mathbf{S}$. *We can call it a* polar basis *(relative to a given system of generic polar coordinates).*

3) The minimality of the associated Morse complex is obtained by the one-to-one correspondence between singular cells and the set of all the chambers of Φ, *and the well-known formula* $\sum B_i = |\{chambers\}|$ *(see* [Za75]*). Here* B_i *means the ith-Betti number of the complement.*

§6. Examples (2)

Consider now the arrangement $\mathcal{A} \in \mathbb{R}^3$ of Example 1 (see Fig. 1 and 2). The system of polar coordinates here is given by O, V_1 and V_2. Here

V_2 is the 2-plane in $\mathbb{R}^3$ which corresponds to the plane $z = a,\ a > 0$, i.e. the plane of Fig. 1, while V_1 is the line indicated in Fig. 1.

To give the total ordering in $\Phi(\mathcal{A})$ we proceed as follows: the cells intersecting the plane V_2 comes before all the others cells, and they are ordered as in the case of a line arrangement with O, V_1 as polar coordinates system. The center P comes after these cells and before of the other ones. For ordering the other ones we have to move the plane V_2 fixing the line V_1, until meeting the center P, and then we have to pass to the other side and ordering all the cells comparing in Fig. 2 as in the case of a line arrangement with O, V_1 as polar coordinates system. So the polar ordering in A_3 is given by:

$$C_0 \lhd F_1 \lhd C_1 \lhd F_2 \lhd C_2 \lhd F_3 \lhd C_3 \lhd F_4 \lhd C_4 \lhd F_5 \lhd C_5 \lhd F_6 \lhd C_6 \lhd G_1 \lhd F_7 \lhd C_7 \lhd F_8 \lhd C_8 \lhd F_9 \lhd G_2 \lhd F_{10} \lhd C_9 \lhd F_{11} \lhd C_{10} \lhd F_{12} \lhd G_3 \lhd F_{13} \lhd C_{11} \lhd F_{14} \lhd G_4 \lhd F_{15} \lhd C_{12} \lhd F_{16} \lhd G_5 \lhd F_{17} \lhd C_{13} \lhd F_{18} \lhd C_{14} \lhd F_{19} \lhd G_6 \lhd F_{20} \lhd C_{15} \lhd F_{21} \lhd C_{16} \lhd F_{22} \lhd G_7 \lhd F_{23} \lhd C_{17} \lhd F_{24} \lhd P \lhd \overline{G}_7 \lhd \overline{F}_{18} \lhd \overline{C}_{13} \lhd \overline{F}_{22} \lhd \overline{G}_6 \lhd \overline{F}_{17} \lhd \overline{C}_{12} \lhd \overline{F}_{15} \lhd \overline{C}_7 \lhd \overline{F}_7 \lhd \overline{G}_5 \lhd \overline{F}_{13} \lhd \overline{C}_{10} \lhd \overline{F}_{11} \lhd \overline{C}_9 \lhd \overline{F}_{16} \lhd \overline{G}_4 \lhd \overline{F}_{10} \lhd \overline{C}_8 \lhd \overline{F}_8 \lhd \overline{G}_3 \lhd \overline{F}_{12} \lhd \overline{G}_2 \lhd \overline{F}_9 \lhd \overline{G}_1.$$

Recall that the complex $\mathbf{S}^{(1)}$ has 192 cells. The polar gradient has components: $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3$. Following Definition 5.4 we can see that:

- Γ_1 is composed by 23 pairs of type $(e(C_i, C_i), e(C_i, F_j))$, with $F_j \lhd C_i$;
- Γ_2 is composed by 43 pairs of type $(e(C_i, F_j), e(C_i, G_k))$, with $G_k \lhd F_j$ and $e(C_i, F_j)$ does not belong to any pair of Γ_1.
- Γ_3 is composed by 18 pairs of type $(e(C_i, G_k), e(C_i, P))$, with $P \lhd G_k$ and $e(C_i, G_k)$ does not belong to any pair of Γ_2.

So the set of critical cells $Sing(\mathbf{S})$ is composed by the cells not appearing in any pair of Γ, i.e. by the following cells:

- 1 critical 0-cell $e(C_0, C_0)$;
- 6 critical 1-cells $e(C_0, F_1), e(C_1, F_2), e(C_2, F_3), e(C_3, F_4), e(C_4, F_5), e(C_5, F_6)$, which correspond to those 1-cells $e(C_i, F_j)$ with $F_j \cap V_1 \neq \emptyset$ and $F_j \cap V_1$ is the maximum (in polar ordering) among all facets of $C \cap V_1$;
- 11 critical 2-cells $e(C_1, G_1)$, $e(C_2, G_1)$, $e(C_3, G_2)$, $e(C_4, G_2)$, $e(C_5, G_3)$, $e(C_7, G_6)$, $e(C_8, G_4)$, $e(C_9, G_5)$, $e(C_{10}, G_5)$, $e(C_{12}, G_6)$, $e(C_{13}, G_7)$, which correspond to those 2-cells $e(C_i, G_k)$ with $G_k \cap V_2 \neq \emptyset$ and $G_k \cap V_2$ is the maximum (in polar ordering) among all facets of $C \cap V_2$.
- 6 critical 3-cells $e(C_7, P), e(C_8, P), e(C_9, P), e(C_{10}, P), e(C_{12}, P), e(C_{13}, P)$, which correspond to those 2-cells $e(C_i, P)$ with P is the maximum (in polar ordering) among all facets of C.

Remark 6.1. *The number of cells is* $1+6+11+6+2*(23+43+18) = 192$.

§7. Discrete methods on configuration spaces

We now consider general configuration spaces as in part 2, and we generalize the theory of part 5. We fix a polar ordering $\lhd$ induced by a generic system of polar coordinates $V_0, V_1, \ldots, V_n$ as in part 5.

Lemma 7.1. *Let $L \in \mathcal{L}(\mathcal{A})$ be a codimension k subspace. Then the system $V_0, \ldots, V_k$ gives a generic system of polar coordinates for the arrangement $\mathcal{A}_L \cap V_k := \{H \cap V_k|\ H \in \mathcal{A}_L\}$ in V_k, so it induces a polar ordering $\lhd_L$ on Φ_L. The system $V_k \cap L, \ldots, V_n \cap L$ gives a generic system of polar coordinates for the arrangement $\mathcal{A}^L$ on L, inducing a polar ordering $\lhd^L$ on Φ^L.*

One has that $\lhd^L$ coincides with the restriction $\lhd_{|\Phi^L}$ of the polar ordering $\lhd$ to $\Phi^L \subset \Phi$.

Definition 7.2. *Given $G \in \Phi$, we set also $\lhd_G := \lhd_{|G|}$.*

Definition 7.3. *Given $G \in \Phi$, and being Φ_G as in Section 2, we define an involution:*

$$\begin{aligned} op_G : \Phi_G &\to \Phi_G \\ F &\mapsto op_G(F) \end{aligned}$$

where $op_G(F)$ is the unique facet which is symmetric to F with respect to supp(G). In other terms (using the maps φ_G, $pr_{|G|}$ defined in Section 2):

$$op_G(F) := \varphi_G^{-1}(-(pr_{|G|}(F))).$$

Here we notice that, for a central arrangement, every facet F has a unique opposite $\overline{F}$ with respect to the center.

Definition 7.4. *Define the* opposite *polar ordering $\lhd_G^{op}$ in Φ_G as:*

$$F \lhd_G^{op} F' \Leftrightarrow op_G(F) \lhd_G op_G(F')$$

$F, F' \in \Phi_G$.

Definition 7.5. *For all arrangements $\mathcal{A}$, all polar orderings $\lhd$ on $\mathcal{A}$, and all $d \geq 1$, we define the degree-d discrete field*

$$\Gamma_{(d)} := \Gamma_{(d)}(\mathcal{A}, \lhd)$$

on the complex $\mathbf{S}^{(d)}(\mathcal{A})$. Assume by recurrence that $\Gamma_{(d')}(\mathcal{A}^L)$ has been defined for $d' < d$, for any $\mathcal{A}^L \subset \mathcal{A}$, $L \in \mathcal{L}(\mathcal{A})$, for the induced polar ordering $\lhd^L$ (see Lemma 7.1). Then the k-dimensional part $\Gamma^k_{(d)}(\mathcal{A})$ is given by the set of pairs of cells in $\mathbf{S}^{(d)}(\mathcal{A})$

$$(e(\mathcal{F}), e(\mathcal{F}'))$$

where $dim(e(\mathcal{F})) = k-1$, $dim(e(\mathcal{F}')) = k$, $\mathcal{F}' \prec \mathcal{F}$ (so $e(\mathcal{F}) \subset \partial(e(\mathcal{F}')))$, and the two flags differ only in a single position:

$$\mathcal{F} = (C, F_1, \ldots, F_{i-1}, F_i^j, F_{i+1}, \ldots, F_d),$$

$$\mathcal{F}' = (C, F_1, \ldots, F_{i-1}, F_i^{j+1}, F_{i+1}, \ldots, F_d)$$

with $F_i^j \prec F_i^{j+1}$ (j, $j+1$ denote codimensions). Moreover, such pairs satisfy the following conditions (6),(7):

(6) *for $i < d$, $e(F_i^j, F_{i+1}, \ldots, F_d)$ is a critical cell in the complex $\mathbf{S}^{(d-i)}(\mathcal{A}^L)$, endowed with the discrete $(d-i)$-vector field $\Gamma_{(d-i)}(\mathcal{A}^L)$, with $L := |F_i^j|$ (i.e., $e(F_i^j, F_{i+1}, \ldots, F_d) \notin \Gamma_{(d-i)}(\mathcal{A}^L)$, see Remark (4.5))*

Set $l = d - i$; then:

(7)
for even l
$$F_i^{j+1} \lhd_{F_{i+1}} F_i^j \quad \text{and} \quad F_i^j = max_{\lhd_{F_{i+1}}}\{F \mid F_{i-1} \prec F \prec F_i^j\};$$
for odd l
$$F_i^{j+1} \lhd^{op}_{F_{i+1}} F_i^j \quad \text{and} \quad F_i^j = max_{\lhd^{op}_{F_{i+1}}}\{F \mid F_{i-1} \prec F \prec F_i^j\}.$$

For $i = d$ there is no F_{i+1} in the first condition of (7), which is to be considered in this case as defined by using the given polar ordering $\lhd$.

We have $\Gamma_{(d)} = \oplus_{k=1}^{n'd}\Gamma^k_{(d)}$, $n' = rk(\mathcal{A})$.

Definition 7.6. *Let $L \in \mathcal{L}(\mathcal{A})$ be a codimension k subspace.*

Set $\Gamma^L_{(d)}$ as the degree-d discrete field of the arrangement $\mathcal{A}^L$ with respect to the polar ordering $\lhd^L = \lhd_{|_L}$.

Set $\Gamma_{L,(d)}$ as the degree-d discrete field of the arrangement $\mathcal{A}_L \cap V^k$ with respect to the polar ordering $\lhd_L$ (see Lemma (7.1)).

The following main theorem describes the minimal complex in terms of the field $\Gamma_{(d)}$, exhibiting its critical cells.

Theorem 4. *One has:*

1. $\Gamma_{(d)}$ *is a discrete vector field;*
2. $\Gamma_{(d)}$ *is a gradient field of a discrete Morse function;*
3. *the critical cells of* $\Gamma_{(d)}$ *are the following ones, depending on the parity of* d*:*

$$e(C, F^k, \ldots, F^k) \tag{8}$$

with $e(C, F^k) \in \mathbf{S}^{(1)}$ *critical cell for* $(\Gamma_{(1)}, \lhd)$, *if* d *is odd;*

$$e(op_{F^k}(C), F^k, \ldots, F^k) \tag{9}$$

with $e(C, F^k) \in \mathbf{S}^{(1)}$ *critical cell for* $(\Gamma_{(1)}, \lhd)$, *if* d *is even.*

As an immediate consequence of Theorem 4 we have

Theorem 5. *1. The configuration space* $\mathcal{M}^{(d)}(\mathcal{A})$ *is a minimal space* ($d \geq 1$).

2. The cohomology of $\mathbf{S}^{(d)}$ *(or of* $\mathcal{M}^{(d+1)}$*),* $d \geq 1$, *is concentrated in dimension*

$$id\ ,\ i = 0 \ldots n\ .$$

The Betti numbers are given by

$$B_{id}(\mathbf{S}^{(d)}) = B_i(\mathbf{S}^{(1)}).$$

Proof of Theorem 5. Case $d = 1$ is considered in Section 5. For $d > 1$ minimality follows immediately from the gap between the dimensions of the critical cells. Q.E.D.

The Proof of Theorem 4 is very technical and will appear elsewhere.

Remark. *For* $d > 1$ *the configuration space is simply connected, therefore its minimality follows by general means. For* $d = 1$ *there is also a non-trivial Morse complex for the cohomology with local coefficients (see* [SaSe07],[GaSa09]*).*

Remark. *Notice the essential difference between the case when* d *is even and the case when* d *is odd, while the shape of the flags is essentially the same when the parity is the same. This fact reflects similar well known phenomena about the representation of a reflection group* W *onto the cohomology of the corresponding configuration spaces: for odd complexifications one has the regular representation of* W, *while in the even case one obtain an induced representation (see* [Ma96],[Le00]*).*

Q.E.D.

§8. Examples (3)

We consider again now the arrangement $\mathcal{A} \subset \mathbb{R}^3$ of Example 1 (see Fig. 3 and 2).

Case $d = 2$.

Recall that the complex $\mathbf{S}^{(2)}$ has 648 cells.

The polar gradient $\Gamma_{(2)}$ has components: $\Gamma_{(2)} = \bigsqcup_{i=1}^{6} \Gamma^i_{(2)}$. By Definition 7.5 we have:

- $\Gamma^1_{(2)}$ is composed by 23 pairs of type $(e(C_i, C_i, C_i), e(C_i, C_i, F_j))$, with $F_j \lhd C_i$, i.e. $(e(C_i, C_i), e(C_i, F_j)) \in \Gamma^1_{(1)}$;
- $\Gamma^2_{(2)}$ is composed by two types of pairs: 43 pairs of type $(e(C_i, C_i, F_j), e(C_i, C_i, G_k))$, with $G_k \lhd F_j$ and $e(C_i, C_i, F_j)$ does not belong to any pair of $\Gamma^1_{(2)}$, i.e. $(e(C_i, F_j), e(C_i, G_k)) \in \Gamma^2_{(1)}$; 6 pairs of type $(e(C_i, C_i, F_j), e(C_i, F_j, F_j))$ with $e(C_i, F_j)$ critical for $\Gamma_{(1)}$;
- $\Gamma^3_{(2)}$ is composed by three types of pairs: 60 pairs of type $(e(C_i, F_j, F_j), e(C_i, F_j, G_k))$, with $G_k \lhd F_j$; 11 pairs of type $(e(C_i, C_i, G_k), e(C_i, F_j, G_k))$ with $F_j \lhd^{op}_{G_k} C_i$, and $e(C_i, G_k)$ is critical for $\Gamma_{(1)}$; 18 pairs of type $(e(C_i, C_i, G_k), e(C_i, C_i, P))$, with $P \lhd G_k$ and $e(C_i, C_i, G_k)$ does not belong to any pair of $\Gamma^2_{(2)}$, i.e. $(e(C_i, G_k), e(C_i, P)) \in \Gamma^3_{(1)}$;
- $\Gamma^4_{(2)}$ is composed by three types of pairs: 48 pairs of type $(e(C_i, F_j, G_k), e(C_i, F_j, P))$ with $P \lhd G_k$ and $G_k = max_{\lhd}\{F | C_i \prec F \prec G_k\}$; 25 pairs of type $(e(C_i, F_j, G_k), e(C_i, G_k, G_k))$ with $G_k \lhd^{op}_{G_k} F_j$ and $e(F_j, G_k)$ critical for $\Gamma^{|F_j|}_{(1)}$, i.e. $G_k \lhd P$; 6 pairs of type $(e(C_i, C_i, P), e(C_i, F_j, P))$ with $F_j \lhd^{op}_P C_i$ and $e(C_i, P)$ critical for $\Gamma_{(1)}$, i.e. $C_i \cap V_2 \neq \emptyset$ and bounded;
- $\Gamma^5_{(2)}$ is composed by two types of pairs: 18 pairs of type $(e(C_i, F_j, P), e(C_i, G_k, P))$ with $e(F_j, P)$ critical for $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$, i.e. $F_j \cap V_2 \neq \emptyset$ and bounded, and $G_k \lhd^{op}_P F_j$; 36 pairs of type $(e(C_i, G_k, G_k), e(C_i, G_k, P))$ with $P \lhd G_k$;
- $\Gamma^6_{(2)}$ is composed by 18 pairs of type $(e(C_i, G_k, P), e(C_i, P, P))$, with $P \lhd^{op}_P G_k$, such that $e(C_i, G_k, P)$ is not in a pair of $\Gamma^3_{(2)}$;

So the set of critical cells $Sing(\mathbf{S})$ is given by the cells not appearing in any pair of $\Gamma_{(3)}$, i.e. by the following cells:

- 1 critical 0-cell $e(C_0, C_0, C_0)$;
- 6 critical 2-cells $e(C_1, F_1, F_1)$, $e(C_2, F_2, F_2)$, $e(C_3, F_3, F_3)$, $e(C_4, F_4, F_4)$, $e(C_5, F_5, F_5)$, $e(C_6, F_6, F_6)$;

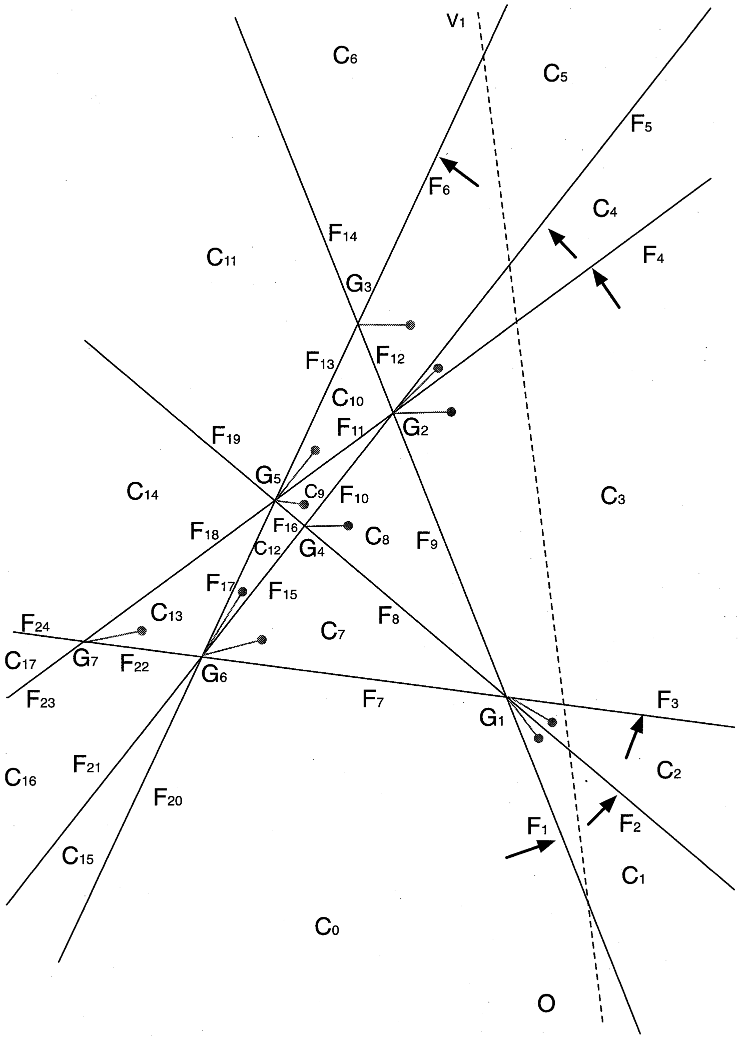

Fig. 3. An upper section of A_3 and critical cells in case d odd

- 11 critical 4-cells $e(C_7, G_1, G_1)$, $e(C_8, G_1, G_1)$, $e(C_9, G_2, G_2)$, $e(C_{10}, G_2, G_2)$, $e(C_{11}, G_3, G_3)$, $e(C_{12}, G_4, G_4)$, $e(C_{13}, G_5, G_5)$, $e(C_{14}, G_5, G_5)$, $e(C_{15}, G_6, G_6)$, $e(C_{16}, G_6, G_6)$, $e(C_{17}, G_7, G_7)$;

- 6 critical 6-cells $e(\overline{C}_7, P, P)$, $e(\overline{C}_8, P, P)$, $e(\overline{C}_9, P, P)$, $e(\overline{C}_{10}, P, P)$, $e(\overline{C}_{12}, P, P)$, $e(\overline{C}_{13}, P, P)$.

Remark 8.1. *As in previous examples we can calculate the number of cells as* $1 + 6 + 11 + 6 + 2 * (23 + 43 + 6 + 60 + 11 + 18 + 48 + 25 + 6 + 18 + 36 + 18) = 648$.

Case $d = 3$.

Recall that the complex $\mathbf{S}^{(3)}$ has 1536 cells.

The polar gradient $\Gamma_{(3)}$ has components: $\Gamma_{(3)} = \bigsqcup_{i=1}^{9} \Gamma^i_{(3)}$. Following Definition 7.5 we can see that:

- $\Gamma^1_{(3)}$ is composed by 23 pairs of type $(e(C_i, C_i, C_i, C_i), e(C_i, C_i, C_i, F_j))$, with $F_j \lhd C_i$, i.e. $(e(C_i, C_i, C_i), e(C_i, C_i, F_j)) \in \Gamma^1_{(2)}$;
- $\Gamma^2_{(3)}$ is composed by two types of pairs: 43 pairs of type $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, C_i, G_k))$, with $G_k \lhd F_j$ and $e(C_i, C_i, C_i, F_j)$ does not belong to any pair of $\Gamma^1_{(3)}$, i.e. $(e(C_i, C_i, F_j), e(C_i, C_i, G_k)) \in \Gamma^2_{(2)}$; 6 pairs of type $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, F_j, F_j))$ with $e(C_i, F_j)$ critical for $\Gamma_{(1)}$, i.e. $(e(C_i, C_i, F_j), e(C_i, F_j, F_j)) \in \Gamma^2_{(2)}$;
- $\Gamma^3_{(3)}$ is composed by four types of pairs: 60 pairs of type $(e(C_i, C_i, F_j, F_j), e(C_i, C_i, F_j, G_k))$, with $G_k \lhd F_j$, i.e. $(e(C_i, F_j, F_j), e(C_i, F_j, G_k)) \in \Gamma^3_{(2)}$; 11 pairs of type $(e(C_i, C_i, C_i, G_k), e(C_i, C_i, F_j, G_k))$ with $F_j \lhd^{op}_{G_k} C_i$, and $e(C_i, G_k)$ is critical for $\Gamma^1_{(1)}$, i.e. $(e(C_i, C_i, G_k), e(C_i, F_j, G_k)) \in \Gamma^3_{(2)}$; 18 pairs of type $(e(C_i, C_i, C_i, G_k), e(C_i, C_i, C_i, P))$, with $P \lhd G_k$ and $e(C_i, C_i, C_i, G_k)$ does not belong to any pair of $\Gamma^2_{(3)}$, i.e. $(e(C_i, C_i, G_k), e(C_i, C_i, P)) \in \Gamma^3_{(2)}$; 6 pairs of type $(e(C_i, C_i, F_j, F_j), e(C_i, F_j, F_j, F_j))$ with $e(C_i, F_j, F_j)$ critical for $\Gamma_{(2)}$;
- $\Gamma^4_{(3)}$ is composed by four types of pairs: 48 pairs of type $(e(C_i, C_i, F_j, G_k), e(C_i, C_i, F_j, P))$ with $P \lhd G_k$ and $G_k = max_{\lhd}\{F | C_i \prec F \prec G_k\}$, i.e. $(e(C_i, F_j, G_k), e(C_i, F_j, P)) \in \Gamma^4_{(2)}$; 25 pairs of type $(e(C_i, C_i, F_j, G_k), e(C_i, C_i, G_k, G_k))$ with $G_k \lhd^{op}_{G_k} F_j$ and $e(F_j, G_k)$ critical for $\Gamma^{|F_j|}_{(1)}$, i.e. $(e(C_i, F_j, G_k), e(C_i, G_k, G_k)) \in \Gamma^4_{(2)}$; 6 pairs of type $(e(C_i, C_i, C_i, P), e(C_i, C_i, F_j, P))$ with $F_j \lhd^{op}_P C_i$ and $e(C_i, P)$ critical for $\Gamma_{(1)}$, i.e. $(e(C_i, C_i, P), e(C_i, F_j, P)) \in \Gamma^4_{(2)}$; 60 pairs of type $(e(C_i, F_j, F_j, F_j), e(C_i, F_j, F_j, G_k))$, with $G_k \lhd F_j$. Notice that in $\Gamma_{(3)}$ there are no pair of type $(e(C_i, C_i, F_j, G_k), e(C_i, F_j, F_j, G_k))$ because no cell of the form $e(C_i, F_j, G_k)$ is critical for $\Gamma_{(2)}$;

- $\Gamma^5_{(3)}$ is composed by five types of pairs: 18 pairs of type $(e(C_i, C_i, F_j, P),\ e(C_i, C_i, G_k, P))$ with $e(F_j, P)$ critical for $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$, and $G_k \lhd^{op}_P F_j$, i.e. $(e(C_i, F_j, P),\ e(C_i, G_k, P)) \in \Gamma^5_{(2)}$; 36 pairs of type $(e(C_i, C_i, G_k, G_k),\ e(C_i, C_i, G_k, P))$ with $P \lhd G_k$, i.e. $(e(C_i, G_k, G_k),\ e(C_i, G_k, P)) \in \Gamma^5_{(2)}$; 11 pairs of type $(e(C_i, C_i, G_k, G_k),\ e(C_i, F_j, G_k, G_k))$ with $F_j \lhd C_i$, and $e(C_i, G_k, G_k)$ is critical for $\Gamma_{(2)}$; 48 pairs of type $(e(C_i, F_j, F_j, G_k),\ e(C_i, F_j, F_j, P))$ with $P \lhd G_k$ and $G_k = max_{\lhd}\{F|F_j \prec F \prec G_k\}$; 36 pairs of type $(e(C_i, F_j, F_j, G_k),\ e(C_i, F_j, G_k, G_k))$ with $e(F_j, G_k)$ critical for $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$, $G_k \lhd^{op}_{G_k} F_j$. Notice that in $\Gamma_{(3)}$ there are no pair of type $(e(C_i, C_i, F_j, P),\ e(C_i, F_j, F_j, P))$ because no cell of the form $e(C_i, F_j, P)$ is critical for $\Gamma_{(2)}$;
- $\Gamma^6_{(3)}$ is composed by four types of pairs: 18 pairs of type $(e(C_i, C_i, G_k, P), e(C_i, C_i, P, P))$, with $P \lhd^{op}_P G_k$, such that $e(C_i, C_i, G_k, P)$ is not in $\Gamma^5_{(3)}$, i.e. $(e(C_i, G_k, P), e(C_i, P, P)) \in \Gamma^6_{(2)}$; 72 pairs of type $(e(C_i, F_j, G_k, G_k),\ e(C_i, F_j, G_k, P))$ with $P \lhd G_k$; 25 pairs of type $(e(C_i, F_j, G_k, G_k), e(C_i, G_k, G_k, G_k))$ with $G_k \lhd F_j$ and $e(F_j, G_k, G_k)$ critical for $\Gamma^{|F_j|}_{(2)}$, i.e. $G_k \lhd P$; 24 pairs of type $(e(C_i, F_j, F_j, P), e(C_i, F_j, G_k, P))$ with $e(F_j, P)$ critical for $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$, i.e. $F_j \cap V_2 \neq \emptyset$ and bounded, and $G_k \lhd^{op}_P F_j$. Notice that in $\Gamma_{(3)}$ there are no pair of type $(e(C_i, C_i, G_k, P),\ e(C_i, F_j, G_k, P))$ because no cell of the form $e(C_i, G_k, P)$ is critical for $\Gamma_{(2)}$;
- $\Gamma^7_{(3)}$ is composed by three types of pairs: 6 pairs of type $(e(C_i, C_i, P, P),\ e(C_i, F_j, P, P))$ with $F_j \lhd C_i$, and $e(C_i, P, P)$ is critical for $\Gamma_{(2)}$; 36 pairs of type $(e(C_i, G_k, G_k, G_k), e(C_i, G_k, G_k, P))$ with $P \lhd G_k$; 48 pairs of type $(e(C_i, F_j, G_k, P),\ e(C_i, F_j, P, P))$, with $P \lhd^{op}_P G_k$, and $G_k = max_{\lhd^{op}_P}\{F|F_j \prec F \prec G_k\}$. Notice that in $\Gamma_{(3)}$ there are no pair of type $(e(C_i, F_j, G_k, P),\ e(C_i, G_k, G_k, P))$ because no cell of the form $e(F_j, G_k, P)$ is critical for $\Gamma^{|F_j|}_{(2)}$;
- $\Gamma^8_{(3)}$ is composed by two types of pairs: 18 pairs of type $(e(C_i, F_j, P, P),\ e(C_i, G_k, P, P))$ with $e(F_j, P, P)$ critical for $\Gamma_{(2)}(\mathcal{A}^{|F_j|})$, i.e. $F_j \cap V_2 = \emptyset$, and $G_k \lhd F_j$; 36 pairs of type $(e(C_i, G_k, G_k, P),\ e(C_i, G_k, P, P))$ with $P \lhd^{op}_P G_k$;
- $\Gamma^9_{(3)}$ is composed by 18 pairs of type $(e(C_i, G_k, P, P), e(C_i, P, P, P))$, with $P \lhd G_k$, such that $e(C_i, G_k, P, P)$ is not in a pair of $\Gamma^8_{(3)}$;

So the set of critical cells $Sing(\mathbf{S})$ is composed by the cells not appearing in any pair of Γ, i.e. by the following cells:

- 1 critical 0-cell $e(C_0, C_0, C_0, C_0)$;
- 6 critical 3-cells $e(C_0, F_1, F_1, F_1), e(C_1, F_2, F_2, F_2), e(C_2, F_3, F_3, F_3)$, $e(C_3, F_4, F_4, F_4)$, $e(C_4, F_5, F_5, F_5), e(C_5, F_6, F_6, F_6)$;
- 11 critical 6-cells $e(C_1, G_1, G_1, G_1)$, $e(C_2, G_1, G_1, G_1)$, $e(C_3, G_2, G_2, G_2)$, $e(C_4, G_2, G_2, G_2)$, $e(C_5, G_3, G_3, G_3)$, $e(C_8, G_4, G_4, G_4)$, $e(C_9, G_5, G_5, G_5)$, $e(C_{10}, G_5, G_5, G_5)$, $e(C_7, G_6, G_6, G_6)$, $e(C_{12}, G_6, G_6, G_6)$, $e(C_{13}, G_7, G_7, G_7)$;
- 6 critical 9-cells $e(C_7, P, P, P), e(C_8, P, P, P), e(C_9, P, P, P), e(C_{10}, P, P, P)$, $e(C_{12}, P, P, P)$, $e(C_{13}, P, P, P)$.

Remark 8.2. *The number of cells is* $1+6+11+6+2*(23+43+6+60+11+18+6+48+25+6+60+18+36+11+48+36+18+72+25+24+6+36+48+18+36+18) = 1536$.

§9. Concluding remarks

As we mentioned in the introduction, it would be interesting to analyze the situation for general subspace arrangements. Of course, minimality condition requires that the cohomology is torsion-free. For Hyperplane Arrangements, such property was known much before than the minimality condition.

For general subspace arrangements, it is well known that the complement can have torsion. An explicit example can be found in [J94]. Let $V = \mathbb{R}^{10}$. By using coordinates $(x_1, \ldots, x_{10})$ in V, the arrangement is given by $\mathcal{A} = \{A_i\}_{i=1\cdots 6}$, where $A_1, \ldots, A_6$ are the coordinate-subspaces:

$A_1 = \{x_1 = x_2 = x_3 = x_4 = x_5 = 0\}$
$A_2 = \{x_1 = x_2 = x_6 = x_7 = x_8 = 0\}$
$A_3 = \{x_1 = x_3 = x_6 = x_9 = x_{10} = 0\}$
$A_4 = \{x_2 = x_4 = x_7 = x_9 = x_{10} = 0\}$
$A_5 = \{x_3 = x_5 = x_7 = x_8 = x_9 = 0\}$
$A_6 = \{x_4 = x_5 = x_6 = x_8 = x_{10} = 0\}$.

One can verify (for example by using the Goreski–MacPherson formula) that the complement has 2-torsion in dimension 7.

The previous example can be generalized by using the recent theory of *moment-angle complexes*, of the special type $Z(K, (D^2, S^1))$. For every simplicial complex K, such complexes realize the complement of certain coordinate-subspace arrangements ([BP00]).

The non-trivial part of the conjecture stated in the introduction deals with non-torsion and non-simply connected subspace arrangements.

The previous examples in general produce simply-connected subspace arrangements.

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