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Hypersphere arrangement and imaginary cycles for hypergeometric integrals

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Abstract.

We construct imaginary cycles as Lefschetz cycles for hypergeometric integrals associated with a hypersphere arrangement and discuss the relation between the twisted rational de Rham cohomology. We also discuss it in degenerate cases where several hyperspheres contact with each other. We pose two geometric problems involved in it.

§1. Introduction

First we want to illustrate in one-dimensional case the main objective discussed in this article. Let Q be the complex circle: $\{\xi = (\xi_1, \xi_2); \xi_1^2 + \xi_2^2 = 1\}$ in the complex affine plane $\mathbf{C}^2$. Q is isomorphic to $\mathbf{C}^*$ by taking $\xi_1 + \sqrt{-1}\xi_2 = \zeta$. Consider a family of m complex lines $H_j : f_j = 0$ where

$$f_j = u_{j,0} + u_{j,1}\xi_1 + u_{j,2}\xi_2$$

such that $u_{j,0}, u_{j,1}, u_{j,2} \in \mathbf{R}$ and that $u_{j,1}^2 + u_{j,2}^2 - u_{j,0}^2 = 1$. Denote

$$a_{i,j} = u_{i,1}u_{j,1} + u_{i,2}u_{j,2} - u_{i.0}u_{j,0},\ a_{i,0} = a_{0,i} = u_{i,0}, a_{0,0} = -1.$$

The intersection of Q and H_j consisits of two different points which we denote by ζ_j, ζ_j^* such that $|\zeta_j| = |\zeta_j^*| = 1$. Let R be the $\mathbf{C}[\xi_1, \xi_2]$

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module

$$R = \sum_{\nu_1 \geq 0, \ldots, \nu_m \geq 0} \mathbf{C}[\xi_1, \xi_2] \prod_{j=1}^{m} f_j^{-\nu_j}$$
$$= \sum_{\nu_1 \geq 0, \ldots, \nu_m \geq 0} \mathbf{C}[\zeta, \zeta^{-1}] \prod_{j=1}^{m} \{(\zeta - \zeta_j)(\zeta - \zeta_j^*)\}^{-\nu_j}$$

because f_j can be written as

$$f_j = \sqrt{-1} \frac{(\zeta - \zeta_j)(\zeta - \zeta_j^*)}{(\zeta_j^* - \zeta_j)\zeta}.$$

Consider the multiplicative function

$$\Phi_0(\xi) = \prod_{j=1}^{m} f_j^{\lambda_j} \quad (\lambda_j \in \mathbf{R}_{>\mathbf{0}})$$

and the associated rational de Rham cohomology on $Y = Q - \cup_{j=1}^{m} H_j$

$$H^1(Y, \nabla_0) \cong R\tau_Q / \nabla_0(R)$$

defined by the covariant differential $\nabla_0(\psi) = d\psi + d\log \Phi_0 \psi$, where we denote

$$\tau_Q = -\xi_1 d\xi_2 + \xi_2 d\xi_1 = \sqrt{-1} \frac{d\zeta}{\zeta}.$$

Suppose that $\zeta_1, \zeta_1^*, \ldots, \zeta_m, \zeta_m^*$ are different from each other. Then one can prove that for generic λ_j

$$H^1(Y, \nabla_0) \cong \mathbf{C}^{2m}$$

and it is spanned by

$$\varphi_Q(\emptyset)\tau_Q = \tau_Q, \varphi_Q(j)\tau_Q = \frac{\tau_Q}{f_j} = d\log \frac{\zeta - \zeta_j}{\zeta - \zeta_j^*} \ (1 \leq j \leq m),$$
$$\varphi_Q(j,k)\tau_Q = \frac{\tau_Q}{f_j f_k} \ (1 \leq j < k \leq m).$$

These one-forms are not linearly independent on Y. For any different i, j, k there exists the fundamental linear relation

$$(1.1) \quad c_i \varphi_Q(i) + c_j \varphi_Q(j) + c_k \varphi_Q(k) + c_{j,k} \varphi_Q(j,k) + c_{k,i} \varphi_Q(k,i) + c_{i,j} \varphi_Q(i,j) = 0$$

where $c_i, c_j, c_k, c_{j,k}, c_{k,i}, c_{i,j}$ can be written in terms of $a_{i,j}, a_{k,0}$ as

$$c_i = -\frac{A(0,i)}{A\begin{pmatrix} 0, & i, & k \\ 0, & i, & j \end{pmatrix}}, \quad c_{j,k} = \frac{A(j,k)}{A\begin{pmatrix} i, & j, & k \\ 0, & j, & k \end{pmatrix}}$$

$c_j, c_k, c_{k,i}, c_{i,j}$ being defined in the same way cyclically. Moreover $A(0,i)$ $= -1 - a_{i,0}^2, A(j,k) = 1 - a_{j,k}^2$ and $A\begin{pmatrix} i, & j, & k \\ i', & j', & k' \end{pmatrix}$ denotes the determinant of the matrix whose components are $a_{p,q}$ $(p = i,j,k; q = i', j', k')$.

The twisted homology $H_1(Y, \hat{\mathcal{L}}_0)$ dual to $H^1(Y, \nabla_0)$ is spanned by the linearly independent cycles which are expressed by the closures (arcs) of the connected components of $\Re Y$.

Suppose now that for a fixed pair i,j, one of ζ_i or ζ_i^* coincides with one of ζ_j or ζ_j^*. This occurs if and only if $A(i,j) = 0$, i.e.,$a_{i,j} = \pm 1$. If ζ_j tends to the point ζ_i^*, then the arc connecting the points ζ_i^*, ζ_j in $\Re Q$ reduces to a point. Hence if $\zeta_i^* = \zeta_j$, the dimension of $H_1(Y, \hat{\mathcal{L}}_0)$ decreases by one. On the other hand one can show that $\varphi_Q(i,j)$ can be described cohomologically as a linear combination of $\varphi_Q(k,i), \varphi_Q(k,j), \varphi_Q(k), \varphi_Q(\emptyset)$:

$$\begin{aligned}
(1.2) \quad 2(\lambda_i + \lambda_j - 1)\varphi_Q(i,j) \sim &- \sum_{k \neq i,j} \lambda_k \{ \frac{A(k,i)}{a_{k,i} + a_{k,j}} \varphi_Q(k,i) \\
&+ \frac{A(k,j)}{a_{k,i} + a_{k,j}} \varphi_Q(k,j) \} + \sum_{k=0}^{m} \lambda_k a_{k,0} \varphi_Q(k) \\
&- (\lambda_\infty - 1) \{ \frac{A(0,i)}{a_{i,0} + a_{j,0}} \varphi_Q(i) + \frac{A(0,j)}{a_{i,0} + a_{j,0}} \varphi_Q(j) \} - \lambda_\infty \varphi_Q(\emptyset),
\end{aligned}$$

where we denote $\lambda_\infty = \sum_{j=1}^m \lambda_j$.

In particular, consider the case where $m = 3$ and $A(1,2) = A(1,3) = A(2,3) = 0$, i.e., $\zeta_1^* = \zeta_2, \zeta_2^* = \zeta_3, \zeta_3^* = \zeta_1$. Then (1.1) reduces to the only one identity:

$$\varphi_Q(1) + \varphi_Q(2) + \varphi_Q(3) = 0$$

and there are three identities of type (1.2):

$$\begin{aligned}
&2(\lambda_i + \lambda_j - 1)\varphi_Q(i,j) \sim \sum_{k=1}^{3} \lambda_k a_{k,0} \varphi_Q(k) \\
&+ (\lambda_\infty - 1) \left\{ \frac{1 + a_{i,0}^2}{a_{i,0} + a_{j,0}} \varphi_Q(i) + \frac{1 + a_{j,0}^2}{a_{i,0} + a_{j,0}} \varphi_Q(j) \right\} - \lambda_\infty \varphi_Q(\emptyset).
\end{aligned}$$

Hence $H^1(Y,\nabla_0)$ is of dimension three and is spanned by a basis of representatives $\varphi_Q(\emptyset),\varphi_Q(1),\varphi_Q(2)$.

In this article we want to extend the above observation to the n-dimensional cases ($n\geq 1$). The twisted de Rham cohomology $H^n(Y,\nabla_0)$ can be formulated in the space Y, the complement of a union of ($n-1$)-dimensional hyperspheres in the n-dimensional fundamental complex hypersphere Q. There arise two problems.

In the first place, based on my preceding articles (see [2], [5]), we formulate the twisted de Rham cohomology presenting explicitly its basis in terms of the invariants $a_{i,j}$ obtained by the Lorentz inner products of the coefficients of two linear functions defining hyperspheres, together with $a_{k,0}$ which are the constant terms of a linear function. The differential structures of the hypergeometric integrals are described by the invariants $a_{i,j}, a_{k,0}$ under Lorentz groups or orthogonal groups. In the final section we show how they should be modified in some degenerate cases.

In the second place the basis of the twisted homology $H_n(Y,\hat{\mathcal{L}}_0)$ cannot always be realized by real domains in $\Re Y$ except in special domain of parameters. We must construct some of them as imaginary cycles. We want to show that this can be done by deforming real Lefschetz cycles from a special domain of parameters involved where all the cycles can be realized by real domains (see Theorem 3.11).

§2. Basic properties

Let $\mathcal{A}$ be an arrangement of m hyperplanes H_j $(1\leq j\leq m)$ defined over the real field of coefficients in the $(n+1)$-dimensional complex affine space $\mathbf{C}^{n+1}$. Each hyperplane can be described as

$$H_j : u_{j,0}+\sum_{\nu=1}^{n+1} u_{j,\nu}\xi_\nu = 0$$

for $\xi=(\xi_1,\ldots,\xi_{n+1})\in\mathbf{C}^{n+1}$.

We denote by $N(\mathcal{A})$ the union of hyperplanes: $=\bigcup_{H_j\in\mathcal{A}} H_j$ and by $X=M(\mathcal{A})$ the complement of $N(\mathcal{A}) := \mathbf{C}^{n+1}-N(\mathcal{A})$. Let Q be the complex hypersphere: $f_0=0$ defined by the quadratic polynomial$f_0 = 1-\sum_{\nu=1}^{n+1}\xi_\nu^2$. Each intersection $H_j\cap Q$ defines a hypersphere in Q provided it does not reduce to a point i.e., $-u_{j,0}^2+\sum_{\nu=1}^{n+1}u_{j,\nu}^2\neq 0$. Throughout this article we shall assume this condition and so may assume that

$$-u_{j,0}^2+\sum_{\nu=1}^{n+1}u_{j,\nu}^2 = 1.$$

The family $\mathcal{A}' = \{H_j \cap Q\}_{1\le j\le m}$ defines a hypersphere arrangement in Q. We denote by Y the intersection of X and $Q : Y = Q - \cup_{j=1}^m H_j \cap Q$. We denote by $\Re Q$ the real part of Q which is identified with the n-dimensional real hypersphere. The real part $S_j = H_j \cap \Re Q$ is a real $(n-1)$-dimensional hypersphere in $\Re Q$.

We define the $(m+1)\times(m+1)$ configuration matrix $A = (a_{i,j})_{0\le i,j\le m}$ associated with $\mathcal{A}'$, whose components are Lorentz inner products

$$a_{i,j} = -u_{i0}u_{j0} + \sum_{\nu=1}^{n+1} u_{i,\nu}u_{j,\nu} \ (1 \le i,j \le m); \ a_{i,0} = a_{0,i} = u_{i0}; \ a_{0,0} = -1$$

so that $a_{i,i} = 1$ for $1 \le i \le m$.

For a set of indices $I = \{i_1, \dots, i_p\} \subset \{0,1,2,\dots,m\}$ the size p will be denoted by $|I|$. We say that I is admissible if $I \subset \{1,2,\dots,m\}$. For two sets of indices $I = \{i_1, i_2, \dots, i_p\}$ and $J = \{j_1, \dots, j_p\}$ we define the subdeterminant

$$A\begin{pmatrix} I \\ J \end{pmatrix} = \begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_p} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_p} \\ \vdots & \vdots & & \vdots \\ a_{i_p,j_1} & a_{i_p,j_2} & \cdots & a_{i_p,j_p} \end{vmatrix}.$$

We abbreviate $A\begin{pmatrix} I \\ I \end{pmatrix}$ by $A(I)$.

For an admissible set $I = \{i_1, i_2, \dots, i_p\}$ and a set $J = \{j_1, j_2, \dots, j_p\}$ $\subset \{0,1,2,\dots,n+1\}$ $(p \le n+2)$ we denote the subdeterminant

$$U\begin{pmatrix} i_1, i_2, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix} = \begin{vmatrix} u_{i_1,j_1} & u_{i_1,j_2} & \cdots & u_{i_1,j_p} \\ u_{i_2,j_1} & u_{i_2,j_2} & \cdots & u_{i_2,j_p} \\ \vdots & \vdots & & \vdots \\ u_{i_p,j_1} & u_{i_p,j_2} & \cdots & u_{i_p,j_p} \end{vmatrix}.$$

Remark. *The matrix A has at most rank $n+2$. Assume that I is admissible. Then $A(I) = 0$ for $|I| \ge n+3$, and*

$$A(I) = -U^2\begin{pmatrix} i_1, i_2, \dots, i_{n+2} \\ 0, 1, \dots, n+1 \end{pmatrix}$$

for $|I| = n+2$. On the other hand $A(0,I) = 0$ for $|I| \ge n+2$ and

$$A(0,I) = -U^2\begin{pmatrix} i_1, i_2, \dots, i_{n+1} \\ 1, \dots, n+1 \end{pmatrix}$$

for $|I| = n+1$.

Lemma 2.1. *Fix an admissible set I and consider the intersection subspace $V = \cap_{j\in I} H_j$ in $\mathbf{C}^{n+1}$.*

(i) *In case where $|I| \le n$, $A(I) = 0$ if and only if V has contact with Q at one point.*

(ii) *In case where $|I| = n+1$, $A(I) = 0$ if and only if V has a common point with Q.*

(iii) *In case where $|I| = n+2$, $A(I) = 0$ if and only if V is not empty.*

Assume that

($\mathcal{H}1$) : $A(0, I) < 0$ for an arbitrary admissible set I such that $|I| \le n+1$, i.e., the homogeneous parts of $|I|$ linear functions f_j $(j \in I)$ are linearly independent.

Assume further that

($\mathcal{H}2$) : $A(I) \neq 0$ for an arbitrary admissible I $(2 \le |I| \le n+2)$.

Then for any $I \subset \{1, 2, \ldots, m\}$ with $|I| = n+2$ the $(n+2)\times(n+2)$ symmetric submatrix $(a_{i,j})_{i,j\in I}$ has the signature of $n+1$ (+)sign and one (−)sign so that $A(I) < 0$ for $|I| = n+2$. This is equivalent to say that for any sequence of increasing admissible sets of indices

$$I_1 \subset I_2 \subset \cdots \subset I_{n+1} \subset I_{n+2}$$

such that $|I_r| = r$, the signs of $A(I_r)A(I_{r+1})$ $(1 \le r \le n+1)$ are positive except for one. Hence $A(I) < 0$ implies $A(J) < 0$ if $I \subset J$, $|J| \le n+2$ (see [8]). In particular the following two cases are interesting:

($\mathcal{H}2$a): $A(I) > 0$ for all admissible I with $2 \le |I| \le n+1$ and $A(I) < 0$ for all admissible I with $|I| = n+2$.

($\mathcal{H}2$b): $A(I) < 0$ for all admissible I with $2 \le |I| \le n+2$.

Let τ be the $(n+1)$-form $d\xi_1 \wedge \cdots \wedge d\xi_{n+1}$. on $\mathbf{C}^{n+1}$. We denote the n-form $-\tau_Q$ on $\mathbf{C}^{n+1}$ such that its restriction to Q is the standard volume form on $\Re Q$:

$$-\tau_Q = \sum_{\nu=1}^{n+1} (-1)^{\nu-1} \xi_\nu d\xi_1 \wedge \cdots \wedge d\xi_{\nu-1} \wedge d\xi_{\nu+1} \wedge \cdots \wedge d\xi_{n+1}$$

such that $df_0 \wedge \tau_Q \equiv \tau \mod (f_0)$. We consider the multiplicative function on X

$$\Phi_0(\xi) = \prod_{j=1}^{m} f_j(\xi)^{\lambda_j}$$

where we assume that every $\lambda_j \in \mathbf{R}$ is positive and generic. We denote by $H^r(X - Y, \nabla_0)$ and $H^r(Y, \nabla_0)$ the r-dimensional twisted rational de Rham cohomologies on $X-Y$ and Y associated with the covariant differentiation ∇_0 respectively:

$$\nabla_0(\psi) = d\psi + d\log\Phi_0 \wedge \psi.$$

These cohomologies are defined in a standard way by using differential forms ψ on $X-Y$ or Y with values in the $\mathbf{C}[\xi_1,\ldots,\xi_{n+1}]$-module

$$R = \sum_{\nu_1\geq 0,\ldots,\nu_m\geq 0} \mathbf{C}[\xi_1,\ldots,\xi_{n+1}]\prod_{k=1}^{m} f_k(\xi)^{-\nu_k}$$

$\mathcal{L}_0$ be the local systems on $X-Y$ and Y defined by $\Phi_0(\xi)$ respectively, and $\hat{\mathcal{L}}_0$ be their duals defined by $\Phi_0(\xi)^{-1}$. Then the $(n+1)$- and n-dimensional homologies $H_{n+1}(X-Y,\hat{\mathcal{L}}_0)$ and $H_n(Y,\hat{\mathcal{L}}_0)$ represented by twisted cycles are dual to the twisted rational de Rham cohomologies $H^{n+1}(X-Y,\nabla_0)$ and $H^n(Y,\nabla_0)$ through the pairs of integrals respectively

(2.1)
$$H^{n+1}(X-Y,\nabla_0)\times H_{n+1}(X-Y,\hat{\mathcal{L}}_0)\ni(\varphi,\mathfrak{c})\longrightarrow\langle\varphi,\mathfrak{c}\rangle=\int_{\mathfrak{c}}\Phi_0\varphi\tau,$$

$$H^n(Y,\nabla_0)\times H_n(Y,\hat{\mathcal{L}}_0)\ni(\varphi,\mathfrak{c})\longrightarrow\langle\varphi,\mathfrak{c}\rangle=\int_{\mathfrak{c}}\Phi_0\varphi\tau_Q. \tag{2.2}$$

The following two Propositions have been proved in [2] and [3] (see Proposition 3.2_p, 3.3_p and Lemma 4.2 in [2, I], and also [5]).

Proposition 2.2. *Under the conditions $(\mathcal{H}1), (\mathcal{H}2)$ we have the isomorphism*

$$H^n(Y,\nabla_0)\cong\mathbf{C}^{\kappa_n}$$

where $\kappa_n=\sum_{\nu=0}^{n}\binom{m}{\nu}+\binom{m-1}{n}$. $H^n(Y,\nabla_0)$ has a basis represented by the differential n-forms

$$\varphi_Q(I)\tau_Q=\frac{\tau_Q}{f_{i_1}\cdots f_{i_p}}$$

where I moves over the admissible sets I of indices such that $0\leq|I|\leq n+1$. We denote $\varphi_Q(\emptyset)=1$ for $|I|=0$. There exist the fundamental relations among them of the following type. For an arbitrary admissible *set of indices J with $|J| = n+2$ there exists the identity:*

$$(2.3)\quad \frac{1}{2}\sum_{\mu\neq\nu}(-1)^{\mu+\nu}\varphi_Q(\partial_\mu\partial_\nu J)\frac{A(0,\partial_\mu\partial_\nu J)}{A\begin{pmatrix}0,\partial_\mu J\\ 0,\partial_\nu J\end{pmatrix}} + \sum_{\mu=1}^{n+2}(-1)^{\mu-1}\varphi_Q(\partial_\mu J)\frac{A(\partial_\mu J)}{A\begin{pmatrix}0,\partial_\mu J\\ J\end{pmatrix}} = 0$$

where $\partial_\mu J$ denotes the subset of J deleted by the μth index j_μ. Further for $|I| = n+2$ a partial fraction gives

(2.4)

$$U\begin{pmatrix} I \\ 0,1,\ldots,n+1\end{pmatrix}\varphi_Q(I) = \sum_{\mu=1}^{n+2}(-1)^{\mu-1}U\begin{pmatrix}\partial_\mu I\\ 1,\ldots,n+1\end{pmatrix}\varphi_Q(\partial_\mu I).$$

We denote by $\mathcal{B}$ a linear space spanned by the representatives $\varphi_Q(I)$, $0 \le |I| \le n+1$.

Proposition 2.3. *Under the condition $(\mathcal{H}1), (\mathcal{H}2a)$ $H_n(Y, \hat{\mathcal{L}}_0)$ has a basis represented by the closures of all the connected components of $\Re Y = \Re Q \cap Y$. Their number is equal to κ_n. In other words, $H_n(Y, \hat{\mathcal{L}}_0)$ is spanned by only real twisted cycles defined by connected components of $\Re Y$.*

For example we have $\kappa_1 = 2m$, $\kappa_2 = m^2 - m + 2$, $\kappa_3 = \frac{1}{3}m^3 - m^2 + \frac{8}{3}m$.

Remark. *The number κ_n is also equal to the number of non-compact connected components of $\mathbf{R}^{n+1} - N(\mathcal{A})$.*

§3. Twisted imaginary cycles

We may assume without losing generality

$$(3.1)\qquad u_{j,0} \le 0 \quad \text{for all } j, 1 \le j \le m.$$

Define the set

$$S_{j,+} := \{\xi \in \Re Q; f_j(\xi) > 0\}$$

as the inside of the real hypersphere $S_j = \Re Q \cap H_j$. We denote by $\boldsymbol{\nu}_j$ the unit normal of $\Re H_j$:

$$(3.2)\qquad \boldsymbol{\nu}_j = \frac{(u_{j,1}, u_{j,2}, \ldots, u_{j,n+1})}{\sqrt{\sum_{\nu=1}^{n+1} u_{j,\nu}^2}}.$$

Remark that $\boldsymbol{\nu}_j \in S_{j,+}$.

First notice the following:

Lemma 3.1. *Suppose I is admissible. The real affine subspace $\bigcap_{j\in I} \Re H_j$ is disjoint with $\Re Q$ if and only if $A(I) < 0$.*

Proof. In fact the square of the distance between the subspace $\bigcap_{j\in I} \Re H_j$ and the origin is equal to $\{A(I) + A(0, I)\}/A(0, I)$. It is bigger than 1 if and only if $A(I) < 0$ because $A(0, I) < 0$. Q.E.D.

Corollary 3.2. *Suppose that $A(i,j) < 0$, i.e., $a_{i,j}^2 > 1$ for every pair $i, j \in \{1, 2, \dots, m\}, i \neq j$ then every S_j is disjoint with each other. In this case, $S_{i,+}, S_{j,+}$ are disjoint, or the one is included in the other, according as $a_{i,j} < -1$ or $a_{i,j} > 1$.*

Proposition 3.3. *Under the condition $(\mathcal{H}2)$ consider an admissible set I such that $2 \le |I| \le n+1$. Suppose further $A(I)$ is a positive number near 0 and that $A(J) < 0$ for any admissible $J \supset I$, $|J| > |I|$. Then the compact domain*

$$\mathfrak{l}(I) := \{\xi \in \Re Q; f_j(\xi) \ge 0 \quad (j \in I)\}$$

gives a twisted real cycle representing an element $H_n(Y, \hat{\mathcal{L}}_0)$. This cycle vanishes if it is deformed in an isotopic way by the matrix A as $A(I)$ tends to 0, any other $A(K)$ being never equal to 0.

Proof. Since $A(I) > 0$ and near 0, $\mathfrak{l}(I)$ is one of the compact components of $\Re Y$. This reduces to a point for $A(I) \to 0$ as is seen from Lemma 3.1. Q.E.D.

Definition 3.4. *The cycle $\mathfrak{l}(I)$ mentioned in* Proposition 3.3 *is called* **the twisted vanishing cycle (Lefschetz cycle) at the singularity** $A(I) = 0$.

Assume now the conditions $(\mathcal{H}2b)$ together with $(\mathcal{H}1)$. Then each $(n-1)$-dimensional hypersphere $\Re Q \cap H_j$ is disjoint with each other. This means that $\Re Y$ has only $m+1$ connected components which make only a part of the basis of $H_n(Y, \hat{\mathcal{L}}_0)$. We want to construct a basis of $H_n(Y, \hat{\mathcal{L}}_0)$ consisting of imaginary cycles in addition to real ones, using special cells $\Delta^*(I), \Delta^*_+(I; J)$ as follows.

Let I be an admissible set such that $|I| = p, 2 \le p \le n$. We have $A(I) < 0$. We may assume without losing generality $I = \{1, 2, \dots, p\}$. Choose an orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ of $\mathbf{R}^{n+1}$ such that $\boldsymbol{\nu}_1 = e_1$ and the j-dimensional subspace $\langle \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_j \rangle$ is spanned by $e_1, \dots, e_j$

$(1 \leq j \leq p)$ where the coefficient of $\boldsymbol{\nu}_j$ with respect to e_j is positive. Hence $f_1, \dots, f_p$ can be represented as

$$(3.3) \quad f_j(\xi) = f_j(\xi_1, \dots, \xi_j) = u_{j,0} + \sum_{\nu=1}^{j} u_{j,\nu}\xi_\nu, \ (u_{j,j} > 0) \quad (1 \leq j \leq p).$$

We say that the basis $\{e_1, e_2, \dots, e_{n+1}\}$ is attached to I.

We denote by $\Delta^*(I)$ the n-dimensional cell consisting of all points

$$\xi = \sum_{k=1}^{p} \xi_k e_k + \sqrt{-1} \sum_{k=p+1}^{n+1} \xi_k^* e_k \in \{\mathbf{R}^p \times (\sqrt{-1}\mathbf{R})^{n+1-p}\} \cap Q$$

which is a piece of an ultra hyperboloid

$$(3.4) \qquad \xi_1^2 + \cdots + \xi_p^2 - {\xi_{p+1}^*}^2 - \cdots - {\xi_{n+1}^*}^2 = 1,$$

$$(3.5) \qquad f_j(\xi_1, \dots, \xi_j) \leq 0 \quad (1 \leq j \leq p).$$

Denote by $\Delta_k^*(\partial_k I)$ the $(n-1)$-dimensional face $\Delta^*(\partial_k I) \cap \Delta^*(I)$ (To define $\Delta_k^*(\partial_k I)$ one should take another orthonormal basis attached to $\partial_k I$). In particular one sees $\Delta_p^*(\partial_p I) = \Delta^*(I) \cap \{\xi_p = 0\}$.

Furthermore consider a pair of admissible sets of indices I, J ($|I| = p, |J| = q$) such that $I \supset J$ and $2 \leq q \leq p-1$. For simplicity assume that $I = \{1, 2, \dots, p\}$ and $J = \{1, 2, \dots, q\}$. There exist a basis $\{e_1, \dots, e_q, e_{q+1}, \dots, e_p, e_{p+1}, \dots, e_{n+1}\}$ of $\mathbf{R}^{n+1}$ attached to the pair (I, J) such that $||e_k|| = 1$ $(1 \leq k \leq n+1)$ ($||\ ||$ denotes the length), which satisfies the following properties $\mathcal{P}$:

(i) $\{e_1, \dots, e_q\}$ is an orthonormal system such that the subspace $\langle e_1, \dots, e_j \rangle$ spanned by $e_1, \dots, e_j$ coincides with the subspace $\langle \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_j \rangle$ $(1 \leq j \leq q)$.
(ii) The systems $\{e_1, \dots, e_q\}$ and $\{e_{q+1}, \dots, e_p\}$ are orthogonal to each other.
(iii) $\boldsymbol{\nu}_j$ lies in the subspace spanned by $e_1, \dots, e_q, e_j$ $(q+1 \leq j \leq p)$.
(iv) $\{e_{p+1}, \dots, e_{n+1}\}$ is an orthonormal system such that it is orthogonal to the system $\{e_1, \dots, e_p\}$.

We may assume that each coefficient of $\boldsymbol{\nu}_j$ with respect to e_j is positive $(1 \leq j \leq p)$.

Then in terms of the coordinates $\xi = \sum_{k=1}^{n+1} \xi_k e_k$ the fundamental hypersphere Q can be represented

$$(3.6) \qquad \sum_{k=1}^{n+1} \xi_k^2 + 2 \sum_{q+1 \leq j < k \leq p} \gamma_{j,k} \xi_j \xi_k = 1$$

where $\gamma_{j,k}$ denotes the inner product (e_j, e_k). Remark that the geodesic $(p-1)$-simplex $\langle \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_p \rangle$ with vertices $\boldsymbol{\nu}_1, \dots \boldsymbol{\nu}_p$ is contained in the real oriented cone

$$C(I;J) : \{\xi_{q+1} \geq 0, \dots, \xi_p \geq 0\} \subset \mathbf{R}^{n+1}$$

and the hyperplane $\xi_k = 0$ $(p \geq k \geq q+1)$ contains the totally geodesic face $\langle \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{k-1}, \boldsymbol{\nu}_{k+1}, \dots, \boldsymbol{\nu}_p \rangle$. Denote by $\Delta_+^*(I;J)$ the cell consisting of the points $\xi = \sum_{k=1}^q \xi_k e_k + \sqrt{-1} \sum_{k=q+1}^{n+1} \xi_k^* e_k \in \{\mathbf{R}^q \times (\sqrt{-1}\mathbf{R})^{n+1-q}\} \cap Q$ satisfying

$$\text{(3.7)} \qquad \sum_{k=1}^q \xi_k^2 - \sum_{k=q+1}^{n+1} \xi_k^{*2} - 2 \sum_{q+1 \leq j < k \leq p} \gamma_{j,k} \xi_j^* \xi_k^* = 1$$

$$\text{(3.8)} \qquad f_1(\xi_1) \leq 0, \dots, f_q(\xi_1, \dots, \xi_q) \leq 0, \xi_{q+1}^* \geq 0, \dots, \xi_p^* \geq 0.$$

In particular we put $\Delta_+^*(I;I) = \Delta^*(I)$ and $\Delta_+^*(I;J)$ for $q = 1$ to be an empty set. The above construction is also possible for the pairs $(\partial_k I; \partial_k J)$ or $(\partial_k I; J)$: $\Delta_+^*(\partial_k I; \partial_k J) \cap \Delta_+^*(I;J)$ and $\Delta_+^*(I;J) \cap \{\xi_k^* = 0\} = \Delta_+^*(\partial_k I; J) \cap \{\xi_k^* = 0\}$, being the $(n-1)$-dimensional k-th faces of $\Delta_+^*(I;J)$, are denoted by $\Delta_{k,+}^*(\partial_k I; \partial_k J)$ $(q \geq k \geq 1, q \geq 3)$ and by $\Delta_{k,+}^*(\partial_k I; J)$ $(p \geq k \geq q+1, q \geq 2)$ respectively. This is compatible because the coordinates $(\xi_1, \dots, \xi_q, \xi_{q+1}^*, \dots, \xi_{n+1}^*)$ attached to $(I;J)$ can be identified with the one attached to $(\partial_k I; J)$ or $(\partial_k I; \partial_k J)$ provided $\xi_k^* = 0$.

Remark 3.5. *$\Delta_+^*(I;J)$ is included in the oriented cone transformed from $C(I;J)$ by the rotations $(\xi_{q+1}, \cdots, \xi_{n+1}) \to e^{\sqrt{-1}\theta}(\xi_{q+1}, \dots, \xi_{n+1})$ $(0 \leq \theta \leq \frac{\pi}{2})$ and corresponds one-to-one to the real cell $\Delta_+(I;J) = C(I;J) \cap \Delta(I)$.*

Then

Lemma 3.6. *For $2 \leq p \leq n+1$ we have the boundary formulae modulo $\cup_{k \in I} H_k \cap Q$:*

$$\partial \Delta^*(I) \equiv 0 \quad (p = 2),$$

$$\partial \Delta^*(I) \equiv \sum_{k=1}^p (-1)^{k-1} \Delta_k^*(\partial_k I) \quad (p \geq 3),$$

$$\begin{aligned}\partial\Delta_+^*(I;\partial_K I) \equiv & \sum_{m\in I-K}(-1)^{m-1}\Delta_{m,+}^*(\partial_m I;\partial_m\partial_K I)\\ & + \sum_{m\in K}(-1)^{m-1}\Delta_{m,+}^*(\partial_m I;\partial_K I) \quad (l\geq 2,\ p-l\geq 3),\\ \equiv & \sum_{m\in K}(-1)^{m-1}\Delta_{m,+}^*(\partial_m I;\partial_K I) \quad (l\geq 2, p-l=2),\\ \equiv & \sum_{m\in I-K}(-1)^{m-1}\Delta_{m,+}^*(\partial_m I;\partial_m\partial_K I)+(-1)^{k-1}\Delta_k^*(\partial_k I)\\ & (l=1, K=\{k\}, p\geq 3)\end{aligned}$$

where K, $|K|=l$, denotes a subset of ordered indices $K=\{k_1,\ldots,k_l\}\subset I$ and ∂_K denotes $\partial_{k_1}\ldots\partial_{k_l}$.

The following Lemma follows immediately from the preceding Lemma:

Lemma 3.7. *Suppose $2\leq p\leq n+1$. Then the n-chain*

$$\mathfrak{l}^*(I)=\Delta^*(I)+\sum_{l=1}^{p-2}\sum_{K,|K|=l}\Delta_+^*(I;\partial_K I)(-1)^l \tag{3.9}$$

defines an n-cycle in $H_n(Y,\hat{\mathcal{L}}_0)$. Here K moves over the family of subsets of ordered indices $K=\{k_1,\ldots,k_l\}\subset I$.

Definition 3.8. *$\mathfrak{l}^*(I)$ is called* **twisted Lefschetz cycle** *associated with I. If $A(I)$ is near 0, this is a deformation of $\mathfrak{l}(I)$ as the matrix A moves from the part $A(I)>0$ to the one $A(I)<0$ being detoured from the singularity $A(I)=0$.*

Note that $\mathfrak{l}^*(I)$ coincides with $\Delta^*(I)$ if $|I|=2$ and with $\Delta^*(I)-\sum_{k=1}^3\Delta_+^*(I;\partial_k I)$ if $|I|=3$.

Assume in particular that $p=n+1$ in case $m\geq n+1$. First consider the case where $n=1$, $m\geq 2$. By hypothesis every $a_{i,j}^2>1$. The $2m$ points $\cup_{j=1}^m \Re Q\cap H_j$ are different from each other. Therefore $\Re Y$ consists of $2m$ connected components which make a basis of $H_1(Y,\hat{\mathcal{L}}_0)$. None of imaginary cycles occur as stated in the Introduction. Next consider the case where $n=2$, $m\geq 3$. Suppose $I=\{1,2,3\}$ is admissible with $|I|=3$. Let $\Delta(I)$ be the geodesic triangle in $\Re Q$ with vertices $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2,\boldsymbol{\nu}_3$. Then

$$\mathfrak{l}^*(I)=\Delta^*(I)-\sum_{k=1}^3\Delta_+^*(I;\partial_k I)$$

where $\Delta^*(I)=\overline{\Delta(I)-S_{1,+}-S_{2,+}-S_{3,+}}$ (the overline $\overline{\Delta-\cdots}$ denotes the closure).

In general there exist $\binom{m}{n+1}$ cycles $\mathfrak{l}^*(I)$ with $|I| = n+1$. All the cycles thus constructed are not necessarily linearly independent in $H_n(Y, \hat{\mathcal{L}}_0)$. In fact we have

Lemma 3.9. *For an admissible I with $|I| = n+2$, the following identity holds modulo $\cup_{k\in I} H_k \cap Q$:*

$$\sum_{j=1}^{n+2} (-1)^{j-1} \mathfrak{l}^*(\partial_j I) \equiv 0. \tag{3.10}$$

We may choose as linearly independent ones the cycles $\mathfrak{l}^*(I), |I| = n+1$ such that $1 \in I$ so that their number is equal to $\binom{m-1}{n}$.

Proof. We may assume that $I = \{1, 2, \ldots, n+2\}$. Then

$$\begin{aligned}\sum_{j=1}^{n+2} (-1)^{j-1} \mathfrak{l}^*(\partial_j I) &= \sum_{j=1}^{n+2} (-1)^{j-1} \Delta^*(\partial_j I) \\ &+ \sum_{j=1}^{n+2} (-1)^{j-1} \sum_{l=1}^{n-1} \sum_{K\subset \partial_j I, |K|=l} \Delta_+^*(\partial_j I; \partial_j \partial_K I).\end{aligned} \tag{3.11}$$

On the other hand we see

$$\sum_{j=1}^{n+2} (-1)^{j-1} \Delta^*(\partial_j I) \equiv 0, \ \sum_{j\in L} (-1)^{j-1} \Delta_+^*(\partial_j I; \partial_L I) \equiv 0$$

for $L \subset I$, $|L| = l+1 \geq 2$ being fixed. In view of Remark 3.5 the last identity is derived from the one

$$\sum_{j\in L} (-1)^{j-1} C(\partial_j I; \partial_L I) = 0.$$

Hence the RHS of (3.11) vanishes modulo $\cup_{k\in I} H_k \cap Q$. Q.E.D.

As an immediate consequence we have

Proposition 3.10. *The number of Lefschetz cycles is equal to κ_n.*

Summing up the above we have proved the following:

Theorem 3.11. *Under the conditions $(\mathcal{H}1)$ and $(\mathcal{H}2b)$, as a basis of $H_n(Y, \hat{\mathcal{L}}_0)$, one can choose the representatives of twisted cycles of the following kinds:*

(i) *Real cycles. This can be realized by the real chambers which are the closures of the connected components of $\Re Y$. Their number is equal to $1+m$.*

(ii) *Imaginary Lefschetz cycles* $\mathfrak{l}^*(I)$ *such that* $2 \leq |I| \leq n$. *Their number is equal to* $\sum_{\nu=2}^{n} \binom{m}{\nu}$.

(iii) *Imaginary Lefschetz cycles* $\mathfrak{l}^*(I), |I| = n+1$ *such that* $1 \in I$. *Their number is equal to* $\binom{m-1}{n}$.

§4. Stereographic projection

The cycles defined in the previous section can also be described in the n-dimensional Euclidean space as below. The complement of the south pole, $\Re Q - \{(-1, 0, \ldots, 0)\}$, is isomorphic to $\mathbf{R}^n$ through the stereopgraphic projection

$$\eta_1 = \frac{\xi_2}{1+\xi_1}, \ldots, \eta_n = \frac{\xi_{n+1}}{1+\xi_1} \tag{4.1}$$

which is a conformal transformation. Then a hypersphere S in $\Re Q$ corresponds to a hypersphere or a hyperplane $\tilde{S}$ in $\mathbf{R}^n$:

$$\sum_{\nu=1}^{n} (\eta_\nu - v_\nu)^2 = r^2 \quad (r > 0)$$

where a hyperplane can be regarded as a limiting case for $r = \infty$. Denote the center of $\tilde{S}$ by $\mathbf{v} = (v_1, \ldots, v_n)$ and its length by $||\mathbf{v}|| = \sqrt{\sum_{\nu=1}^{n} v_\nu^2}$. Then we have

$$S : u_0 + \sum_{\nu=0}^{n+1} u_\nu \xi_\nu = 0, \ (u_0 \leq 0)$$

$$u_0 = \frac{r^2 - 1 - ||\mathbf{v}||^2}{2r}, u_1 = \frac{r^2 + 1 - ||\mathbf{v}||^2}{2r}, u_{\nu+1} = \frac{v_\nu}{r} \ (1 \leq \nu \leq n)$$

or

$$-u_0 = \frac{r^2 - 1 - ||\mathbf{v}||^2}{2r}, -u_1 = \frac{r^2 + 1 - ||\mathbf{v}||^2}{2r}, -u_{\nu+1} = \frac{v_\nu}{r} \ (1 \leq \nu \leq n)$$

according as $r^2 - 1 - ||\mathbf{v}||^2 \leq 0$ or > 0, namely

Lemma 4.1. S_+ *corresponds to the inside or the outside of* $\tilde{S}$ *according as* $r^2 - 1 - ||\mathbf{v}||^2 < 0$ *or* > 0.

As for $a_{i,j}$

$$a_{i,j} = \frac{r_i^2 + r_j^2 - ||\mathbf{v}^{(i)} - \mathbf{v}^{(j)}||^2}{2r_i r_j}$$

where $r_i, r_j, \mathbf{v}^{(i)}, \mathbf{v}^{(j)}$ denote the radii and the centers of $\tilde{S}_i, \tilde{S}_j$ respectively. Hence

Lemma 4.2. *We have*

$$A(i,j) = \frac{(r_i - r_j + a)(-r_i + r_j + a)(r_i + r_j + a)(r_i + r_j - a)}{4r_i^2 r_j^2}$$

where we put $a = ||\mathbf{v}^{(i)} - \mathbf{v}^{(j)}||$. *This implies* $a_{i,j} > 1$ *if and only if* $|r_i - r_j| > a$. $a_{i,j} < -1$ *if and only if* $r_i + r_j < a$.

In the same way

Lemma 4.3. *Suppose that* $|a_{i,j}| < 1$ *for an admissible* $I = \{i, j, k\}$ *and put* $-\cos\alpha_{i,j} = a_{i,j}$ *such that* $0 < \alpha_{i,j} < \pi$. *Then*

$$A(i,j,k) = -4\cos\frac{\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k}}{2} \cdot \cos\frac{-\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k}}{2}$$
$$\cdot \cos\frac{\alpha_{i,j} - \alpha_{j,k} + \alpha_{i,k}}{2} \cdot \cos\frac{\alpha_{i,j} + \alpha_{j,k} - \alpha_{i,k}}{2}.$$

The three hyperspheres $\tilde{S}_i, \tilde{S}_j, \tilde{S}_k$ *intersect each other and* $\pi - \alpha_{i,j}$ *is equal to the angle subtended by the tangents of* $\tilde{S}_i, \tilde{S}_j$ *at an intersection point of* $\tilde{S}_i \cap \tilde{S}_j$. $A(1,2,3) = 0$ *if and only if* $\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k} = \pi$, *or* $-\alpha_{i,j} + \alpha_{j,k} + \alpha_{i,k} = \pi$, *or* $\alpha_{i,j} - \alpha_{j,k} + \alpha_{i,k} = \pi$, *or* $\alpha_{i,j} + \alpha_{j,k} - \alpha_{i,k} = \pi$.

Lemma 4.4. *For an arbitrary admissible* $I, |I| \leq n+1$ *there exist the new coordinates* $\eta = (\eta_1, \dots, \eta_n)$ *of* $\mathbf{R}^n$ *such that* $\mathbf{v}^{(i_1)} = 0$ *and* $\mathbf{v}^{(i_{l+1})}$ *lies in the* $\eta_1, \dots, \eta_l$*-subspace* $2 \leq l \leq |I| - 1$.

Proof. We may assume that $I = \{1, 2, \dots, p\}$. By the change of coordinates (4.1), there exist the coordinates $\xi_1, \dots, \xi_{n+1}$ such that $\boldsymbol{\nu}_{l+1}$ lies in the $\xi_1, \dots, \xi_{l+1}$-subspace $(1 \leq l \leq p)$. Since $u_{1,\nu} = 0$ for $\nu \geq 2$, $\mathbf{v}^{(1)} = 0$. And $u_{l,\nu} = 0$ for $\nu \geq l+1$, $\mathbf{v}^{(l+1)}$ lies in the $\eta_1, \dots, \eta_l$-subspace.

The cycles equivalent to the one constructed in Section 3 are described as follows:

Consider the case where $m = 2$. Let $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ be the centers of $\tilde{S}_1, \tilde{S}_2$ and the insides of $\tilde{S}_1, \tilde{S}_2$ be denoted by $\tilde{S}_{1,+}, \tilde{S}_{2,+}$ respectively. Suppose first that $|a_{1,2}| < 1$. Then $\tilde{S}_{1,+} \cap \tilde{S}_{2,+}$ is a non-empty domain so that $\mathbf{R}^n - \tilde{S}_1 \cup \tilde{S}_2$ consists of 4 connected components:

$$\mathbf{R}^n - \tilde{S}_{1+} \cup \tilde{S}_{2,+}, \tilde{S}_{1,+} - \tilde{S}_{2,+}, \tilde{S}_{2,+} - \tilde{S}_{1,+}, \tilde{S}_{1,+} \cap \tilde{S}_{2,+}.$$

Their closures make the representatives of a basis of $H_n(Y, \hat{\mathcal{L}}_0)$.

Suppose that $a_{1,2} < -1$. Then $\tilde{S}_{1,+}$ is disjoint with $\tilde{S}_{2,+}$. We have three real domains $\tilde{S}_{1,+}, \tilde{S}_{2,+}, \mathbf{R}^n - \tilde{S}_{1+} \cup \tilde{S}_{2,+}$. On the other hand suppose that $a_{1,2} > 1$. Then $\tilde{S}_{1,+}$ includes or is included in $\tilde{S}_{2,+}$. Assume for example that $\tilde{S}_{1,+} \supset \tilde{S}_{2,+}$. Then there are three real domains $\mathbf{R}^n - \tilde{S}_{1,+}, \tilde{S}_{1,+} - \tilde{S}_{2+}, \tilde{S}_{2,+}$.

There is the Lefschetz cycle enclosed by two pieces of hyperboloids

$$\tilde{\mathfrak{l}}(\{1,2\}) : \{\eta = (\eta_1, \sqrt{-1}\eta_2^*, \dots, \sqrt{-1}\eta_n^*) \in \mathbf{R} \times (\sqrt{-1}\mathbf{R})^{n-1};$$
$$\eta_1^2 - \sum_{\nu=2}^n \eta_\nu^{*2} \geq r_1^2, (\eta_1 - v_1^{(2)})^2 - \sum_{\nu=2}^n \eta_\nu^{*2} \geq r_2^2\}.$$

More generally suppose that $A(I) < 0$ for $|I| = p\,(2 \leq p \leq n)$. We may assume that $I = \{1, 2, \dots, p\}$.

There exists an orthonormal basis $(\tilde{e}_k)_k$ of $\mathbf{R}^n$ such that the new coordinates $\eta = \sum_{k=1}^n \eta_k \tilde{e}_k \in \mathbf{R}^n$ define $\tilde{S}_j\,(1 \leq j \leq p)$ by

$$\sum_{\nu=1}^{j-1} (\eta_\nu - v_\nu^{(j)})^2 + \sum_{\nu=j}^n \eta_\nu^2 = r_j^2.$$

Corresponding to the $(p-1)$-simplex $\tilde{\Delta}(I) = \langle \mathbf{v}^{(1)}, \dots, \mathbf{v}^{(p)} \rangle$ with vertices $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(p)}$, define the cell $\tilde{\Delta}^*(I)$ enclosed by p pieces of ultra hyperboloids

$$\tilde{\Delta}^*(I) = \{\eta = (\eta_1, \dots, \eta_{p-1}, \sqrt{-1}\eta_p^*, \dots, \sqrt{-1}\eta_n^*) \in \mathbf{R}^{p-1}$$
$$\times (\sqrt{-1}\mathbf{R})^{n-p+1}; \sum_{\nu=1}^{j-1} (\eta_\nu - v_\nu^{(j)})^2 + \sum_{\nu=j}^{p-1} \eta_\nu^2 - \sum_{\nu=p}^n \eta_\nu^{*2} \geq r_j^2$$
$$(1 \leq j \leq p)\}.$$

Further we put

$$\tilde{\Delta}_+^*(I; \partial_p I) = \{\eta = (\eta_1, \dots, \eta_{p-2}, \sqrt{-1}\eta_{p-1}^*, \dots, \sqrt{-1}\eta_n^*) \in \mathbf{R}^{p-2}$$
$$\times (\sqrt{-1}\mathbf{R})^{n-p+2}; \eta_{p-1}^* \geq 0, \sum_{\nu=1}^{j-1} (\eta_\nu - v_\nu^{(j)})^2 + \sum_{\nu=j}^{p-2} \eta_\nu^2 - \sum_{\nu=p-1}^n \eta_\nu^{*2}$$
$$\geq r_j^2, (1 \leq j \leq p-1)\}.$$

which is the n-cell enclosed by $p-1$ pieces of ultra hyperboloids and the hyperplane $\eta_{p-1}^* = 0$. By exchange of coordinates one can similarly define the cells $\tilde{\Delta}_+^*(I; \partial_k I)\,(1 \leq k \leq p-1)$. More generally for a pair $I, J\,(I \supset J, |I| = p, |J| = q)$ such that $2 \leq q \leq p-1$ there exists a basis $\tilde{e}_k\,(1 \leq k \leq n)$ of $\mathbf{R}^n$ attached to the pair satisfying the properties $\mathcal{P}$

in §3, q, p being replaced by $q-1, p-1$ respectively, such that one can define the n-cell

$$\tilde{\Delta}^*_+(I;J) := \{\eta = \sum_{k=1}^{q-1} \eta_k e_k + \sqrt{-1}\sum_{k=q}^{n} \eta^*_k e_k \in \mathbf{R}^{q-1} \times (\sqrt{-1}\mathbf{R})^{n-q+1};$$

$$\sum_{k=1}^{j-1} (\eta_k - v_k^{(j)})^2 + \sum_{k=j}^{q-1} \eta_k^2 - \sum_{k=q}^{n} \eta_k^{*2} - 2 \sum_{q \le j < k \le p-1} \tilde{\gamma}_{j,k} \eta^*_j \eta^*_k \ge r_j^2,$$

$$(1 \le j \le q), \eta^*_q \ge 0, \dots, \eta^*_{p-1} \ge 0\}$$

where $\tilde{\gamma}_{j,k}$ denotes the inner product $(\tilde{e}_j, \tilde{e}_k)$. Then according to (3.9) the Lefschetz cycle $\tilde{\mathfrak{l}}^*(I)$ $(2 \le p \le n+1)$ is defined to be

$$\tilde{\mathfrak{l}}^*(I) = \tilde{\Delta}^*(I) + \sum_{j=1}^{p-2} (-1)^j \sum_{K \subset I, |K|=j} \tilde{\Delta}^*_+(I; \partial_K I).$$

In the same way as Lemma 3.9 we have

Lemma 4.5. *For any admissible I such that $|I| = n+2$, the identity holds*

$$\sum_{j=1}^{n+2} (-1)^{j-1} \tilde{\mathfrak{l}}^*(\partial_j I) \equiv 0$$

modulo the complexification of $\cup_{j \in I} \tilde{S}_j$ in $\mathbf{C}^n$.

In conclusion we may choose admissible I with $|I| = n+1$, such that $1 \in I$ so that any other can be a linear combination of them. We have the same conclusion as Theorem 3.11.

§5. Degenerate cases

In Section 3, and 4 we have assumed $(\mathcal{H}1)$ and $(\mathcal{H}2)$. In this section we discuss the cases where these conditions are not necessarily satisfied.

First note the following (for example, see [2, I], Lemma 4.2):

Lemma 5.1. *We have the commutative diagram:*

$$\begin{array}{ccc} H^{n+1}(X-Y;\nabla_0) & \xrightarrow{\mathrm{Res}} & H^n(Y;\nabla_0) \\ \updownarrow & & \updownarrow \\ H_{n+1}(X-Y;\hat{\mathcal{L}}_0) & \xleftarrow{\delta} & H_n(Y;\hat{\mathcal{L}}_0) \end{array}$$

where Res *denotes the Residue along Y, and δ means the boundary operation (Leray map) into a tubular neighborhood of Y in $X-Y$.*

For an arbitrary $\varphi(\xi)\tau \in R\tau$ such that its representative $\in H^{n+1}(X-Y, \nabla_0)$, denote

$$\varphi^{(1)}\tau_Q = \mathrm{Res}\left(\frac{\varphi}{f_0}\tau\right) = \left[\frac{\varphi\tau}{df_0}\right]_Y, \quad \varphi^{(2)}\tau_Q = \mathrm{Res}\left(\frac{\varphi}{f_0^2}\tau\right).$$

Then $\varphi^{(1)}(I)$ is equal to the restriction of

$$\varphi(I) = \frac{1}{f_{i_1}\cdots f_{i_p}}$$

to Q. As for $\varphi^{(2)}(I)$ the following two recurrence relations play an important role in the sequel:

Lemma 5.2. *For an admissible I with $|I| = p\,(0 \le p \le n+1)$*

$$\text{(5.1)}\quad A(I)\varphi^{(2)}(I) \sim \sum_{k\notin I} \lambda_k A\begin{pmatrix} 0, & I \\ k, & I \end{pmatrix} \varphi_Q(k,I)$$
$$+(\lambda_\infty + n - p - 1)A(0,I)\varphi_Q(I) - \sum_{\nu=1}^{p}(-1)^{\nu-1} A\begin{pmatrix} I \\ 0, \partial_\nu I \end{pmatrix} \varphi^{(2)}(\partial_\nu I).$$

In particular

$$\varphi^{(2)}(\emptyset) \sim \sum_{k=1}^{m} \lambda_k a_{k,0}\varphi_Q(k) - (\lambda_\infty + n - 1)\varphi(\emptyset),$$
$$\varphi^{(2)}(j) \sim \sum_{k\ne j} \lambda_k A\begin{pmatrix} 0, & j \\ k, & j \end{pmatrix} \varphi_Q(k,j) - \sum_{k=1}^{m} \lambda_k a_{j,0}a_{k,0}\varphi_Q(k)$$
$$+(\lambda_\infty + n - 2)A(0,j)\varphi_Q(j) + (\lambda_\infty + n - 1)a_{j,0}\varphi_Q(\emptyset).$$

Therefore $\varphi^{(2)}(I)$ can be described as a linear combination of $\varphi_Q(J)$ such that $|J - J\cap I| \le 1$ with the coefficients of rational functions of $a_{i,j}$, $a_{k,0}$ whose denominators are products of $A(K)$ for $K \subset I$.

For the proof see [2, I], Proposition 4.2.

Lemma 5.3. *Fix an admissible I with $p = |I| \leq n+1$. Then an arbitrary $\mu, 1 \leq \mu \leq p$*

$$
\begin{aligned}
(5.2)\quad (-1)^{\mu-1}(\lambda_{i_\mu} - 1)A(I)\frac{\varphi_Q(I)}{f_{i_\mu}} &\sim -\sum_{k \notin I} \lambda_k A\begin{pmatrix} I \\ k, \partial_\mu I \end{pmatrix} \varphi_Q(k, I) \\
&- (\lambda_\infty + n - p - 1)A\begin{pmatrix} I \\ 0, \partial_\mu I \end{pmatrix} \varphi_Q(I) \\
&+ \sum_{\nu=1}^{p} (-1)^{\nu-1} A\begin{pmatrix} \partial_\mu I \\ \partial_\nu I \end{pmatrix} \varphi^{(2)}(\partial_\nu I).
\end{aligned}
$$

For the proof see [2, I], Proposition 4.2.

Owing to Lemma 5.2 and 5.3 an arbitray form $\frac{\tau_Q}{\prod_{k=1}^m f_k^{\nu_k}}$ $(\nu_k \geq 0)$ can be described explicitly as a linear combination of the representatives of admissible forms $\varphi_Q(I)\tau_Q$.

Proposition 5.4. *In addition to $(\mathcal{H}1)$ suppose the following condition:*

$(\mathcal{H}IV(p))$ *For a fixed admissible I with $p = |I| \leq n$, $A(I) = 0$. But for any other admissible J such that $|J| \leq n+2$ $A(J) \neq 0$.*

Then $\mathfrak{l}^*(I)$ vanishes. The dimension of $H_n(Y, \hat{\mathcal{L}}_0)$ decreases by one and is equal to $\kappa_n - 1$. On the other hand the representatives $\varphi_Q(I)$ in Proposition 2.2 *does not make a basis of $H^n(Y, \nabla_0)$. We have a linear relation*

(5.3)

$$
\begin{aligned}
&(\sum_{j \in I} \lambda_j + \lambda_\infty + n - p - 1)A(0, I)\varphi_Q(I) + \sum_{k \notin I} \lambda_k A\begin{pmatrix} 0, & I \\ k, & I \end{pmatrix} \varphi_Q(k, I) \\
&- \sum_{k \notin I} \lambda_k \sum_{\nu=1}^{p} (-1)^{\nu-1} \frac{A\begin{pmatrix} I \\ 0, & \partial_\nu I \end{pmatrix} A\begin{pmatrix} k, & \partial_\nu I \\ 0, & \partial_\nu I \end{pmatrix}}{A(\partial_\nu I)} \varphi_Q(k, \partial_\nu I) \\
&\equiv 0 \mod \mathcal{B}(I)
\end{aligned}
$$

where $\mathcal{B}(I)$ denotes a linear space spanned by $\varphi_Q(J)$ such that $|J - J \cap I| \leq 1$ and $|J| < |I|$. $H^n(Y, \nabla_0)$ is of dimension $\kappa_n - 1$ and is spanned by $\varphi_Q(J)$ such that $J \neq I$ and $|J| \leq n+1$ with the fundamental relations (2.3), (2.4).

Proof. In fact since $A(I) = 0$, the LHS of (5.1) vanishes. A repeated application of (5.1) to $\varphi^{(2)}(\partial_\nu I)$ shows the RHS of (5.1) equals the RHS of (5.3) in view of the Jacobi identities

$$A^2\begin{pmatrix} & I & \\ 0, & & \partial_\nu I \end{pmatrix} = -A(0,I)A(\partial_\nu I).$$

Q.E.D.

Proposition 5.5. *In addition to* ($\mathcal{H}1$) *suppose the following condition:*

$\mathcal{H}IV(n+1)$ *For a fixed admissible I with $|I| = n+1$, $A(I) = 0$. But for any other admissible J such that $|J| \leq n+2$ $A(J) \neq 0$. Then $\mathfrak{l}^*(I)$ vanishes and* $\dim H_n(Y, \hat{L}_0) = \kappa_n - 1$. *We have a linear relation*

$$(5.4) \qquad 2\Big(\sum_{j\in I}\lambda_j - 1\Big)A(0,I)\varphi_Q(I) \equiv \sum_{k\notin I}\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\lambda_k \cdot \frac{A(0,I)A\begin{pmatrix} I \\ k, \partial_\nu I \end{pmatrix}A(k,\partial_\nu I)}{A(k,I)A(\partial_\nu I)}\varphi_Q(k,\partial_\nu I) \mod \mathcal{B}(I).$$

$H^n(Y, \nabla_0)$ is of dimension $\kappa_n - 1$ and is spanned by $\varphi_Q(J)$ such that $J \neq I$ and $|J| \leq n+1$ with the fundamental relations (2.3), (2.4).

Proof. Since $A(I) = 0$ the LHS of (5.1) vanishes. Applying repeatedly (2.4) to $\varphi_Q(k,I)$ and (5.1) to $\varphi^{(2)}(\partial_\nu I)$ one sees that the RHS of (5.1) equals

$$-2\Big(\sum_{j\in I}\lambda_j - 1\Big)A(0,I)\varphi_Q(I)$$
$$+\sum_{k\notin I}\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\lambda_k \frac{A\begin{pmatrix} 0, I \\ k, I \end{pmatrix}A\begin{pmatrix} I \\ 0, \partial_\nu I \end{pmatrix}A(k,\partial_\nu I)}{A(k,I)A(\partial_\nu I)}\varphi_Q(k,\partial_\nu I)$$
$$\mod \mathcal{B}(I).$$

Hence (5.4) follows owing to the identities

$$A(k,I) = -U^2\begin{pmatrix} k, I \\ 0, 1, \dots, n+1 \end{pmatrix},$$
$$A\begin{pmatrix} 0, I \\ k, I \end{pmatrix}A\begin{pmatrix} 0, \partial_\nu I \\ I \end{pmatrix} = A(0,I)A\begin{pmatrix} I \\ k, \partial_\nu I \end{pmatrix}.$$

Q.E.D.

Corollary 5.6. *Suppose that $m = n+2, n \geq 1$ and that $A(I) = 0$ for all admissible I with $|I| = n+1$. Then $H^n(Y, \nabla_0)$ is of dimension $\kappa_n - (n+2) = 2^{n+2} - n - 4$ and is spanned by the representatives $\varphi_Q(I)$ with $|I| \leq n$ with the one fundamental relation: For $J = \{1, 2, \ldots, n+2\}$.*

$$(5.5) \qquad \sum_{\mu \neq \nu} (-1)^{\mu+\nu} \varphi_Q(\partial_\nu \partial_\nu J) \frac{A(0, \partial_\mu \partial_\nu J)}{A\begin{pmatrix} 0, & \partial_\mu J \\ 0, & \partial_\nu J \end{pmatrix}} = 0.$$

$\varphi_Q(I)\,(|I| = n+1)$ can be expressed as

$$(5.6) \qquad 2(\sum_{j \in I} \lambda_j - 1) A(0, I) \varphi_Q(I) \equiv 0 \mod \mathcal{B}(I)$$

over the coefficients of rational functions of $a_{i,j}, a_{k,0}$ with the denominators $A(K)\,(|K| \leq n)$. This identity is just an n-dimensional version of (1.2).

Proof. (5.5) is a special case of (2.3) since $A(\partial_\mu J) = 0$. On the other hand (5.6) is a special case of (5.4) since $A(k, \partial_\nu I) = 0$. Q.E.D.

Proposition 5.7. *In addition to ($\mathcal{H}1$) suppose the following condition:*

($\mathcal{H}IV(n+2)$) *For a fixed admissible I with $|I| = n+2$, $A(I) = 0$. But for any other admissible J such that $|J| \leq n+2$ $A(J) \neq 0$. Then there is no vanishing of Lefschetz cycles and* $\dim H_n(Y, \hat{\mathcal{L}}_0) = \kappa_n$. *On the other hand, for any fixed μ*

$$(5.7) \quad (\sum_{\nu \in I} \lambda_\nu - 1) U\begin{pmatrix} \partial_\mu I \\ 1, \ldots, n+1 \end{pmatrix} \varphi_Q(I) \equiv 0 \mod \mathcal{B}(I).$$

$H^n(Y, \nabla_0)$ is of dimension κ_n and is spanned by $\varphi_Q(J)$ such that $|J| \leq n+1$ with the fundamental relations (2.3), (2.4).

Proof. By hypothesis the LHS of (2.4) vanishes. By a multiplication by f_{i_μ} of both sides of (2.4) one sees that $\varphi_Q(I)$ is linearly dependent on $\frac{\varphi_Q(\partial_\nu I)}{f_{i_\mu}}$ $(\nu \neq \mu)$. On the other hand due to (5.2) each $\frac{\varphi_Q(\partial_\nu I)}{f_{i_\mu}}$ is linearly dependent on admissible $\varphi_Q(J)$ with $|J| \leq n+1$. Hence the Proposition follows. Q.E.D.

Finally we consider the special case where $n \geq 2, m = n+2$ and $A(i,j) = 0$, i.e., $a_{i,j} = \pm 1$ for all $i, j \in \{1, 2, \ldots, n+2\}$ $(i \neq j)$. Since the signature of A is of type $(n+1, 1)$, we have $A(1,2) = 0, A(1, 2, \ldots, p) < 0$ if $3 \leq p \leq n+2$.

By a suitable Lorentz transformation we may assume that $a_{i,j} = -1$ for all i, j $(i \neq j)$. In fact

Lemma 5.8. *There exist a diagonal matrix P with diagonal elements equal to ± 1 such that $B = P \cdot A \cdot {}^tP$ is the matrix with diagonal elements 1 and off-diagonal elements -1.*

Proof. Denote by B_r the matrix of size $r+2$ with diagonal elements 1 and off-diagonal elements -1. Let A_r be the matrix with the (i,j) elements $a_{i,j}$ $(1 \le i, j \le r+2)$. For $r = 0$ the Lemma is trivial. Suppose that the Lemma is true for A_{r-1}. There exists a diagonal matrix P_{r-1} with diagonal elements ± 1 such that $B_{r-1} = P_{r-1} \cdot A_{r-1} \cdot {}^t P_{r-1}$. Let $\tilde{P}_r$ be the diagonal matrix of size $r+2$ such that the first $r+1$ diagonal elements coincides with the ones of P_{r-1} and the last one equal to 1. Then $\tilde{P}_r \cdot A_r \cdot {}^t\tilde{P}_r$ has the same components as B_r except for the off-diagonal components in the last column or row. Denote these components by $\varepsilon_1, \dots, \varepsilon_{r+1}$. Then we have

$$\det(\tilde{P}_r \cdot A_r \cdot^t \tilde{P}_r) = (1-r)2^r + (r-2)2^{r-1} \sum_{k=1}^{r+1} \varepsilon_k^2$$
$$-2^r \sum_{1 \le i < j \le r+1} \varepsilon_i \varepsilon_j < 0.$$

But this inequality goes to a contradiction except for the case where all ε_j equal 1 or all ε_j equal -1. One sees that the first case is equivalent to B_r, while the last one coincides with B_r. Q.E.D.

Lemma 5.9. *Suppose $I = \{1, 2, \dots, n+2\}$. The matrix A for all off diagonal elements $a_{i,j} = -1$ defines the hypersphere arrangement $\mathcal{A}'$ if and only if $\{a_{j,0}\}_j$ satisfy the quadratic relation*

$$(n-1) \sum_{j=1}^{n+2} a_{j,0}^2 - 2 \sum_{1 \le j < k \le n+2} a_{j,0} a_{k,0} + 2n = 0.$$

Proof. First remark that if $A(0, I) \le 0$ then $A(0, J) < 0$ for $J \subset I, J \neq I$. In fact it is sufficient to show this in case $J = \{1, \dots, r\}$ $(3 \le r \le n+1)$. This follows by lowering induction from the identity

$$A(0,1,\dots,r)A(1,\dots,r+1) - A^2\begin{pmatrix} 0,1,\dots,r \\ r+1,1,\dots,r \end{pmatrix}$$
$$= A(0,1,\dots,r+1)A(1,\dots,r)$$

because $A(1, \dots, r), A(1, \dots, r+1)$ are both negative. On the other hand

$$A(0, I) = 2^n \{(n-1) \sum_{j=1}^{n+2} a_{j,0}^2 - 2 \sum_{1 \le j < k \le n+2} a_{j,0} a_{k,0} + 2n\}.$$

Hence the Lemma follows. Q.E.D.

We now apply to it the formula (5.3) for $p = 2$.
For $I = \{i, j\}$ we have

$$A(0,i,j) = -(a_{i,0} + a_{j,0})^2, A\begin{pmatrix} 0,i,j \\ k,i,j \end{pmatrix} = 2(a_{i,0} + a_{j,0}).$$

Hence (5.3) and Lemma 5.2 give

$$(5.8) \quad (\lambda_\infty + n - 3 + \lambda_i + \lambda_j)(a_{i,0} + a_{j,0})\varphi_Q(i,j) \\ + \sum_{k \neq i,j} \lambda_k(a_{k,0} + a_{i,0})\varphi_Q(k,i) + \sum_{k \neq i,j} \lambda_k(a_{k,0} + a_{j,0})\varphi_Q(k,j) \sim w_{i,j}$$

where we put

$$w_{i,j} = 2\sum_{k \neq i,j} \lambda_k \varphi_Q(k,i,j) + (a_{i,0} + a_{j,0}) \sum_{k=1}^{n+2} \lambda_k a_{k,0} \varphi_Q(k) \\ - (\lambda_\infty + n - 1) \cdot (a_{i,0} + a_{j,0})\varphi_Q(\emptyset) \\ - (\lambda_\infty + n - 2)\{A(0,i)\varphi_Q(i) + A(0,j)\varphi_Q(j)\}.$$

To solve (5.4) with respect to $\varphi_Q(i,j)$ we put $v_{i,j} = (a_{i,0} + a_{j,0})\varphi_Q(i,j)$ and

$$V_i = \sum_{k \neq i} \lambda_k v_{k,i}, \ V_\infty = \sum_{i \neq j} \lambda_i \lambda_j v_{i,j}, \\ W_i = \sum_{k \neq i} \lambda_k w_{k,i}, \ W_\infty = \sum_{i \neq j} \lambda_i \lambda_j w_{i,j}.$$

Then (5.8) is equivalent to

$$(5.9) \qquad (\lambda_\infty + n - 3)v_{i,j} + V_i + V_j \sim w_{i,j}.$$

(5.9) can be uniquely solved for $v_{i,j}$:

$$(5.10) \qquad (\lambda_\infty + n - 3)v_{i,j} \sim w_{i,j} \\ + V_\infty \left(\frac{1}{2\lambda_\infty + n - 3 - 2\lambda_i} + \frac{1}{2\lambda_\infty + n - 3 - 2\lambda_j} \right) \\ - \left(\frac{W_i}{2\lambda_\infty + n - 3 - 2\lambda_i} + \frac{W_j}{2\lambda_\infty + n - 3 - 2\lambda_j} \right)$$

where V_i and V_∞ are uniquely determined by

$$(2\lambda_\infty + n - 3 - 2\lambda_i)V_i \sim W_i - V_\infty,$$
$$\left(1 + \sum_{k=1}^{n+2} \frac{\lambda_k}{2\lambda_\infty + n - 3 - 2\lambda_k}\right) V_\infty \sim \sum_{k=1}^{n+2} \frac{\lambda_k W_k}{2\lambda_\infty + n - 3 - 2\lambda_k}$$

provided none of $2\lambda_\infty + n - 3 - 2\lambda_k$ or the symmetric polynomial

$$G(\lambda) = \prod_{k=1}^{n+2}(2\lambda_\infty + n - 3 - 2\lambda_k) + \sum_{k=1}^{n+2} \lambda_k \prod_{j\neq k}(2\lambda_\infty + n - 3 - 2\lambda_j)$$

vanishes. In this way we can conclude

Proposition 5.10. *For $m = n+2, n \geq 2$, suppose that in addition to $(\mathcal{H}1)$, $a_{i,j} = -1$ for all i,j $(i \neq j)$, and $A(I) < 0$ for all admissible I with $3 \leq |I| \leq n+2$. Suppose further that neither of $2\lambda_\infty + n - 3 - 2\lambda_k$ or $\lambda_\infty + n - 3$ or $G(\lambda)$ vanish. Then all the Lefschetz cycles $\mathfrak{l}^*(I)$ $(|I| = 2)$ vanish. $H^n(Y, \nabla_0)$ is of dimension $\kappa_n - \binom{n+2}{2}$ and has a basis of representatives $\varphi_Q(I)$ with $0 \leq |I| \leq n+1, |I| \neq 2$ satisfying the fundamental relations* (2.3),(2.4). *$\varphi_Q(i,j)$ can be described as a linear combination of these representatives as in* (5.10).

§6. Two problems

As is seen from Propositon 2.2, $H^n(Y, \nabla_0)$ is spanned by the representatives $\varphi_Q(I)$. The result due to Orlik–Terao (see [9]) suggests that this fact still holds in general in the following sense:

Conjecture 6.1. *Without any of the hypotheses $(\mathcal{H}1)$ or $(\mathcal{H}2)$, $H^n(Y, \nabla_0)$ is spanned by the representatives $\varphi_Q(I)$, $I \subset \{1, 2, \ldots, m\}$ including $\varphi_Q(\emptyset)$.*

The complex hypersphere Q has the Kähler metric

$$ds^2 = \sum_{\nu=1}^{n+1} |d\xi_\nu|^2 = \sum_{\mu,\nu=1}^{n} g_{\mu,\bar{\nu}} d\zeta^\mu d\overline{\zeta^\nu}$$

with respect to local coordinates $\zeta = (\zeta^\nu)_{1 \leq \nu \leq n}$. We put

$$\lambda_j = N l_j + \lambda'_j \quad (N \in \mathbf{Z}_{>0})$$

for fixed $l = (l_j)_j \in (\mathbf{Z}_{>0})^m, \lambda' = (\lambda'_j)_j \in \mathbf{R}^m$. For a large N the asymptotic behavior of the integral (2.2) can be explicitly evaluated if the cycle $\mathbf{c}$ is a stable cycle defined by the gradient vector field on Y:

$$d\zeta^{\mu} = \sum_{\nu=1}^{n} \frac{\partial}{\partial \overline{\zeta_{\nu}}} (\sum_{j=1}^{m} l_j \log |f_j|) g^{\mu,\overline{\nu}} dt$$

where $g^{\mu,\overline{\nu}}$ denotes the inverse matrix of the metric tensor $g_{\mu,\overline{\nu}}$. The critical points are determined by the equations on Y (see [1], [10], [14])

$$\sum_{j=1}^{m} l_j \, d \log f_j = 0.$$

Every cycle mentioned in Theorem 3.11 seems to have one-to-one relation with a stable cycle corresponding to these critical points.

Question 6.2. *How can be found the critical points of the gradient vector field in Y? When are they real or imaginary? Are the stable cycles corresponding to them homological to the ones mentioned in* Theorem 3.11*?*

Under the conditions $(\mathcal{H}1), (\mathcal{H}2a)$ every critical point is real and lies one-to-one in a connected component of $\Re Y$. On the other hand under the conditions $(\mathcal{H}1), (\mathcal{H}2b)$ all critical points are not real. It can be proved by a perturbative method that all of the critical points lie in $\Re Y$, at least in the case where $m \leq n+1$ and all admissible $A(I)$, $(2 \leq |I| \leq n+1)$ are negative and near 0.

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