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Quantum groups and quantization of Weyl group symmetries of Painlevé systems

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Dedicated to Professor Akihiro Tsuchiya on his retirement

Abstract.

We shall construct the quantized q-analogues of the birational Weyl group actions arising from nilpotent Poisson algebras, which are conceptual generalizations, proposed by Noumi and Yamada, of the Bäcklund transformations for Painlevé equations. Consider a quotient Ore domain of the lower nilpotent part of a quantized universal enveloping algebra for any symmetrizable generalized Cartan matrix. Then non-integral powers of the image of the Chevalley generators generate the quantized q-analogue of the birational Weyl group action. Using the same method, we shall reconstruct the quantized Bäcklund transformations of q-Painlevé equations constructed by Hasegawa. We shall also prove that any subquotient integral domain of a quantized universal enveloping algebra of finite or affine type is an Ore domain.

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§0. Introduction

The main theme of this article is a representation theoretic method for quantizing discrete symmetries of Painlevé equations and isomonodromic deformations. Such discrete symmetries, often called Bäcklund transformations, play central roles in the theory of Painlevé equations and isomonodromic deformations. We could expect that this is the case in quantum settings.

In this article the term "quantization" means *canonical quantization* that replaces commutative Poisson algebras $\mathcal{A}^{\mathrm{cl}}$ with non-commutative algebras $\mathcal{A}$ and Poisson algebra automorphisms of $\mathcal{A}^{\mathrm{cl}}$ with algebra automorphisms of $\mathcal{A}$. If $\mathcal{A}^{\mathrm{cl}}$ is an integral domain and $Q(\mathcal{A}^{\mathrm{cl}})$ denotes the field of fractions of $\mathcal{A}^{\mathrm{cl}}$, then a birational action of a group G on $\operatorname{Spec}\mathcal{A}^{\mathrm{cl}}$ is identified with an algebra automorphism action of G on $Q(\mathcal{A}^{\mathrm{cl}})$. Therefore if a classical symmetry is represented by a birational action of G on a Poisson integral affine scheme, then its quantization should be an algebra automorphism action of G on a non-commutative skew field.

Note that q-difference analogue (q-analogue for short) or q-difference deformation (q-deformation) does not always mean quantization. In this article we shall deal with four types of classical and quantum systems, ordinary differential versions of classical systems and their quantizations, and q-analogues of classical systems and their quantizations.

Sections 1 and 2 are devoted to summarizing preparatory results on quantized universal enveloping algebras and localizations of noncommutative rings. We shall show that a quantized universal enveloping algebra of finite or affine type is an Ore domain (Theorem 2.14). The $q = 1$ cases were treated in [23] and [1]. Moreover we shall show that their subquotient integral domains are also Ore domains (Corollary 2.15). In Section 3, we shall explain how to justify non-integral powers in fields of fractions along the lines of the work [6] by Iohara and Malikov.

0.1. Quantized q-analogue of birational Weyl group actions

In Section 4, we shall construct a quantized q-analogue of the birational Weyl group action arising from a nilpotent Poisson algebra.

A series of works by Okamoto [19, 20, 21, 22] showed that the Painlevé equations $\mathrm{P_{II}}$, $\mathrm{P_{III}}$, $\mathrm{P_{IV}}$, $\mathrm{P_V}$, and $\mathrm{P_{VI}}$ have affine Weyl group symmetries of type $A_1^{(1)}$, $C_2^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, and $D_4^{(1)}$ respectively. Each of the affine Weyl groups birationally acts on dependent variables and parameters of the corresponding Painlevé equation. Its birational actions preserve the continuous flow generated by the Painlevé equation and are called Bäcklund transformations.

In [14], Noumi and Yamada generalized the birational affine Weyl group actions of type $A_2^{(1)}$, $A_3^{(1)}$, and $D_4^{(1)}$ (the cases of $\mathrm{P_{IV}}$, $\mathrm{P_V}$, and $\mathrm{P_{VI}}$) to the Weyl group associated to an arbitrary generalized Cartan matrix (GCM for short). If the size of GCM is equal to m, then the Weyl group birationally acts on a $2m$-dimensional space of m dependent variables and m parameters. For each $m \geqq 2$, they also constructed a system of differential equations with m dependent variables and m parameters on which the affine Weyl group of type $A_{m-1}^{(1)}$ acts as Bäcklund transformations [15]. It is called a higher order Painlevé equation of type $A_{m-1}^{(1)}$. Following this work, Nagoya constructed its quantization in [13].

In [17], Noumi and Yamada also proposed the further generalization of the birational Weyl group action.

Fix an arbitrary GCM. Let $\mathfrak{g}$ be the Kac–Moody algebra associated to the GCM, $\mathfrak{h}$ its Cartan subalgebra, and $\mathfrak{n}_\pm$ its upper and lower parts ([7]). The Kostant–Kirillov Poisson bracket $\{\ ,\ \}$ makes the symmetric algebra $S(\mathfrak{n}_-)$ Poisson and hence $\mathfrak{n}_-^* = \operatorname{Spec} S(\mathfrak{n}_-)$ is regarded as a Poisson scheme. Let J^{cl} be an arbitrary Poisson prime ideal of $S(\mathfrak{n}_-)$ and denote by $\mathcal{A}_0^{\mathrm{cl}} = S(\mathfrak{n}_-)/J^{\mathrm{cl}}$ the residue class ring modulo J^{cl}. Then $\mathcal{A}_0^{\mathrm{cl}}$ is a Poisson integral domain and hence $\operatorname{Spec} \mathcal{A}_0^{\mathrm{cl}}$ is a Poisson integral subscheme of $\mathfrak{n}_-^*$. Denote by $\mathcal{A}^{\mathrm{cl}} = \mathcal{A}_0^{\mathrm{cl}} \otimes S(\mathfrak{h})$ the tensor product algebra of $\mathcal{A}_0^{\mathrm{cl}}$ and $S(\mathfrak{h})$. The Poisson structure of $\mathcal{A}_0^{\mathrm{cl}}$ uniquely extends to that of $\mathcal{A}^{\mathrm{cl}}$ so that $S(\mathfrak{h})$ is Poisson-central in $\mathcal{A}^{\mathrm{cl}}$.

In [17], Noumi and Yamada constructed a birational Weyl group action on $\operatorname{Spec} \mathcal{A}^{\mathrm{cl}} = \operatorname{Spec} \mathcal{A}_0^{\mathrm{cl}} \times \mathfrak{h}$. They called $\mathcal{A}_0^{\mathrm{cl}}$ a nilpotent Poisson algebra. $\operatorname{Spec} \mathcal{A}_0^{\mathrm{cl}}$ and $\mathfrak{h}$ are identified with the space of dependent variables and that of parameters respectively.

Let us explain the quantized q-analogue of the above setting.

Let $A = [a_{ij}]_{i,j \in I}$ be a symmetrizable GCM symmetrized by a family $\{d_i\}_{i \in I}$ of positive rational numbers. Denote by $\{\alpha_i\}_{i \in I}$ the set of simple roots and by $\{\alpha_i^\vee\}_{i \in I}$ the set of simple coroots. Let d be the least common denominator of $\{d_i\}_{i \in I}$. Set the base field $\mathbb{F}$ by $\mathbb{F} = \mathbb{Q}(q^{1/d})$ and $q_i \in \mathbb{F}$ by $q_i = q^{d_i}$.

Let U_q be the quantized universal enveloping algebra of type A over $\mathbb{F}$, U_q^0 its Cartan subalgebra, and $U_q^\pm$ its upper and lower parts ([11]). Let J_q be an arbitrary completely prime ideal of U_q^- and denote by $\mathcal{A}_{q,0} = U_q^-/J_q$ the residue class ring modulo J_q. Assume that $\mathcal{A}_{q,0}$ is an Ore domain. For example, if A is of finite or affine type, then $\mathcal{A}_{q,0}$ is always an Ore domain. See Corollary 2.15. For the construction of examples for an arbitrary case, see Section 2.4.

Denote by $\mathcal{A}_q = \mathcal{A}_{q,0} \otimes U_q^0$ the tensor product algebra of $\mathcal{A}_{q,0}$ and U_q^0. Then $\mathcal{A}_q$ is also an Ore domain. Denote by $Q(\mathcal{A}_q)$ the skew field of fractions of $\mathcal{A}_q$. Let $\{f_i\}_{i\in I}$ be the images in $\mathcal{A}_q$ of the lower Chevalley generators $\{F_i\}_{i\in I}$ of U_q^-. In particular, f_i $(i \in I)$ satisfy the q-Serre relations. Assume that $f_i \neq 0$ for all $i \in I$.

Denote by $\tilde{s}_i$ the action on U_q^0 of the simple reflection $s_i \in W$ for $i \in I$. The action of $\tilde{s}_i$ naturally extends to the action on $Q(\mathcal{A}_q)$ so that $\tilde{s}_i$ trivially acts on $\mathcal{A}_{q,0}$.

In Section 4, we shall obtain the following results:

(1) For each $i \in I$, the conjugation action $\gamma(f_i^{\alpha_i^\vee})$ of the non-integral power $f_i^{\alpha_i^\vee}$ on $Q(\mathcal{A}_q)$ formally given by $\gamma(f_i^{\alpha_i^\vee})x = f_i^{\alpha_i^\vee} x f_i^{-\alpha_i^\vee}$ for $x \in Q(\mathcal{A}_q)$ is well-defined. (See Section 4.1.)

(2) For each $i \in I$, define the operator S_i acting on $Q(\mathcal{A}_q)$ by $S_i = \tilde{s}_i \circ \gamma(f_i^{-\alpha_i^\vee}) = \gamma(f_i^{\alpha_i^\vee}) \circ \tilde{s}_i$ Then S_i $(i \in I)$ satisfy the defining relations of the Weyl group. In particular the braid relations of S_i $(i \in I)$ are derived from the Verma relations of the Chevalley generators. (See the proof of Theorem 4.3.)

Thus we can construct a Weyl group action on $Q(\mathcal{A}_q)$, which is the quantized q-analogue of the birational Weyl group action arising from a nilpotent Poisson algebra. For details, see Section 4.

0.2. Quantized birational Weyl group actions of Hasegawa

In Section 5, we shall reconstruct the quantized (q-analogue of) birational Weyl group actions of Hasegawa [4].

In [8], Kajiwara, Noumi, and Yamada introduced a q-analogue of the fourth Painlevé equation $\mathrm{P_{IV}}$. It is called a q-Painlevé IV equation $q\mathrm{P_{IV}}$. They also constructed a birational action of the affine Weyl group of type $A_2^{(1)}$ preserving the discrete flow generated by $q\mathrm{P_{IV}}$.

Based on this work, in [4], Hasegawa constructed the quantized q-analogue of the birational Weyl group action associated to an arbitrary symmetrizable GCM.

Apparently Hasegawa's quantized q-analogues are different from those explained in the preceding subsection. But we can reconstruct the former by the same method as the latter.

We follow the notation and the assumptions on a GCM, roots, and coroots, etc. in the preceding subsection. Assume that if $i \neq j$ and $a_{ij} \neq 0$, then $\epsilon_{ij} = \pm 1$ and $\epsilon_{ji} = -\epsilon_{ij}$, and otherwise $\epsilon_{ij} = 0$.

Let $\mathcal{B}_q$ be the associative algebra generated by $\{k_i^{\pm 1}, f_i\}_{i\in I}$ with following defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$

$$k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \quad f_i f_j = q_i^{-\epsilon_{ij} a_{ij}} f_j f_i.$$

Note that the last formula is a sufficient condition for the q-Serre relation. Define the algebra U_q^0 by $U_q^0 = \mathbb{F}\left[\{a_i^{\pm 1}\}_{i\in I}\right]$, the Laurent polynomial ring over $\mathbb{F}$ generated by $\{a_i^{\pm 1}\}_{i\in I}$. We identify a_i with $q_i^{\alpha_i^\vee}$. Define the algebra $\widetilde{\mathcal{A}}_q$ by $\widetilde{\mathcal{A}}_q = \mathcal{B}_q \otimes \mathcal{B}_q \otimes U_q^0$, the tensor product algebra of $\mathcal{B}_q$, $\mathcal{B}_q$, and U_q^0. Then $\widetilde{\mathcal{A}}_q$ is an Ore domain and hence can be embedded in the skew field of fractions $Q(\widetilde{\mathcal{A}}_q)$.

Define $f_{i1}, f_{i2} \in \mathcal{B}_q \otimes \mathcal{B}_q$ by $f_{i1} = f_i \otimes 1$ and $f_{i2} = k_i^{-1} \otimes f_i$. Note that $f_{i1} + f_{i2}$ is the image of the coproduct of a lower Chevalley generator in the quantized universal enveloping algebra. Identify $f_{i1} \otimes 1, f_{i2} \otimes 1, 1 \otimes 1 \otimes a_i \in \widetilde{\mathcal{A}}_q$ with $f_{i1}, f_{i2} \in \mathcal{B}_q \otimes \mathcal{B}_q$, and $a_i \in U_q^0$ respectively. Define $F_i \in Q(\widetilde{\mathcal{A}}_q)$ by $F_i = a_i^{-1} f_{i1}^{-1} f_{i2}$ for $i \in I$. Do not confuse these F_i with the lower Chevalley generators. Let $\mathcal{A}_q$ be the subalgebra of $Q(\widetilde{\mathcal{A}}_q)$ generated by $\{F_i, a_i^{\pm 1}\}_{i\in I}$. Note that a_i is central in the field of fractions $Q(\mathcal{A}_q)$ and

$$F_i F_j = q_i^{-2\epsilon_{ij} a_{ij}} F_j F_i.$$

Therefore $\mathcal{A}_q$ is also an Ore domain. We have $Q(\mathcal{A}_q) \subset Q(\widetilde{\mathcal{A}}_q)$. Denote by $\tilde{s}_i$ the action on U_q^0 of the simple reflection $s_i \in W$ for $i \in I$. Then we have $\tilde{s}_i(a_j) = a_j a_i^{-a_{ij}}$. The action of $\tilde{s}_i$ extends to the action on $Q(\mathcal{A}_q)$ by $\tilde{s}_i(F_j) = a_i^{a_{ij}} F_j$.

In Section 5, we shall obtain the following results:

(1) For each $i \in I$, the conjugation action $\gamma\left((f_{i1} + f_{i2})^{\alpha_i^\vee}\right)$ of the non-integral power $(f_{i1} + f_{i2})^{\alpha_i^\vee}$ on $Q(\mathcal{A}_q)$ formally given by $\gamma\left((f_{i1} + f_{i2})^{\alpha_i^\vee}\right) x = (f_{i1} + f_{i2})^{\alpha_i^\vee} x (f_{i1} + f_{i2})^{-\alpha_i^\vee}$ for $x \in Q(\mathcal{A}_q)$ is well-defined (Section 5.1).

(2) For each $i \in I$, define the operator S_i acting on $Q(\mathcal{A}_q)$ by $S_i = \tilde{s}_i \circ \gamma\left((f_{i1} + f_{i2})^{-\alpha_i^\vee}\right) = \gamma\left((f_{i1} + f_{i2})^{\alpha_i^\vee}\right) \circ \tilde{s}_i$. Then S_i $(i \in I)$ satisfy the defining relations of the Weyl group. In particular the braid relations of S_i $(i \in I)$ are derived from the Verma relations of $\{f_{i1} + f_{i2}\}_{i\in I}$. (See Section 5.2.)

Thus we can construct a Weyl group action on $Q(\mathcal{A}_q)$, which coincides with Hasegawa's quantized q-analogue of the birational Weyl group action (Remark 5.3). For details, see Section 5.

§1. Quantized universal enveloping algebras

In this section, we shall summarize widely known results on quantized universal enveloping algebras. We shall mainly follow Lusztig's book [11].

1.1. Symmetrizable GCM and root datum

A matrix $A = [a_{ij}]_{i \in I}$ with integer entries defined to be a *generalized Cartan matrix* (*GCM* for short) if it satisfies, for any $i, j \in I$, (1) $a_{ii} = 2$, (2) $a_{ij} \leqq 0$ if $i \neq j$, and (3) $a_{ji} = 0$ if and only if $a_{ij} = 0$. Let A be a GCM. A is called *indecomposable* if, for any $i \neq j$ in I, there exists a sequence $i_0, i_1, \dots, i_s \in I$ such that $i = i_0$, $a_{i_k i_{k+1}} \neq 0$ ($k = 0, 1, \dots, s-1$), and $i_s = j$. If there exists a family $\{d_i\}_{i \in I}$ of positive rational numbers such that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in I$, then A is called *symmetrizable* and *symmetrized* by $\{d_i\}_{i \in I}$. If A is a GCM symmetrized by $\{d_i\}_{i \in I}$, then the transpose ${}^t A$ is a GCM symmetrized by $\{d_i^{-1}\}_{i \in I}$.

Let $A = [a_{ij}]_{i \in I}$ be a symmetrizable GCM symmetrized by $\{d_i\}_{i \in I}$. We say that A is *of finite type* (resp. *of affine type*) if its principal minors are positive (resp. its proper principal minors are positive and its determinant is equal to zero). All GCM's of finite and affine type are classified explicitly. For details, see Chapter 4 of [7].

Let X and Y be finitely generated free $\mathbb{Z}$-modules and $\langle\ ,\ \rangle : Y \times X \to \mathbb{Z}$ a perfect bilinear pairing. (X can be identified with $\mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Z})$.) Let $\{\alpha_i^\vee\}_{i \in I}$ and $\{\alpha_i\}_{i \in I}$ be families of elements in Y and X respectively. A *root datum* of type A is defined to consist of $(Y, X, \langle\ ,\ \rangle, \{\alpha_i^\vee\}_{i \in I}, \{\alpha_i\}_{i \in I})$ satisfying $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for any $i, j \in I$. Then $\alpha_i^\vee$ and α_i are called a *coroot* and a *root* respectively. The *dual root datum* of type ${}^t A$ is defined to be $(X, Y, \langle\ ,\ \rangle, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$.

If $\{\alpha_i^\vee\}_{i \in I}$ (resp. $\{\alpha_i\}_{i \in I}$) is linearly independent in Y (resp. X), then the root datum called *Y-regular* (resp. *X-regular*) and the submodule of Y (resp. X) generated by $\{\alpha_i^\vee\}_{i \in I}$ (resp. $\{\alpha_i\}_{i \in I}$) is called a *coroot lattice* (resp. a *root lattice*) and denoted by Q (resp. $Q^\vee$). We set $X^+ = \{\, \lambda \in X \mid \langle \alpha_i^\vee, \lambda \rangle \geqq 0 \text{ for all } i \in I \,\}$ and call its elements *dominant*. We set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geqq 0} \alpha_i$.

1.2. Braid group and Weyl group

Let $A = [a_{ij}]_{i,j \in I}$ be a symmetrizable GCM.

The braid group $B(A)$ of type A is the group generated by $\{s_i\}_{i \in I}$ with the following defining relations: for any $i \neq j$ in I,

$$
\begin{aligned}
s_i s_j &= s_j s_i && \text{if } a_{ij} a_{ji} = 0, \\
s_i s_j s_i &= s_j s_i s_j && \text{if } a_{ij} a_{ji} = 1,
\end{aligned}
$$

$$
\begin{aligned}
s_i s_j s_i s_j &= s_j s_i s_j s_i && \text{if } a_{ij}a_{ji} = 2, \\
s_i s_j s_i s_j s_i s_j &= s_j s_i s_j s_i s_j s_i && \text{if } a_{ij}a_{ji} = 3.
\end{aligned}
$$

These relations are called braid relations.

The Weyl group $W(A)$ of type A is the group generated by $\{s_i\}_{i\in I}$ satisfying the braid relations together with $s_i^2 = 1$ for all $i \in I$. When A is indecomposable, $W(A)$ is finite if and only if A is of finite type.

Denote by B the braid group of type A and by W the Weyl group of type A.

For $w \in W$, the length $\ell(w)$ of w is the smallest integer $p \geqq 0$ such that there exists $i_1, \dots, i_p \in I$ with $w = s_{i_1} \cdots s_{i_p}$. Then $s_{i_1} \cdots s_{i_p}$ is called a reduced expression of w.

If $s_{i_1} \cdots s_{i_p}$ and $s_{i'_1} \cdots s_{i'_p}$ are reduced expressions of $w \in W$, then the equality $s_{i_1} \cdots s_{i_p} = s_{i'_1} \cdots s_{i'_p}$ holds in the braid group B. Therefore the mapping from W to B sending $w \in W$ to the element of B represented by a reduced expression of w is well-defined.

Let $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ be a root datum of type A. Then the Weyl group $W(A)$ acts on Y and X by $s_i(y) = y - \langle y, \alpha_i\rangle\alpha_i^\vee$ for $y \in Y$ and $s_i(x) = x - \langle\alpha_i^\vee, x\rangle\alpha_i$ for $x \in X$. Moreover we have $\langle w(y), x\rangle = \langle y, w^{-1}(x)\rangle$ for $w \in W(A)$, $y \in Y$, and $x \in X$.

If $s_{i_1} \cdots s_{i_p}$ is a reduced expression in W, then $s_{i_p} s_{i_{p-1}} \cdots s_{i_2}(\alpha_{i_1}^\vee) \in \sum_{i\in I} \mathbb{Z}_{\geqq 0}\alpha_i^\vee$ and $s_{i_p} s_{i_{p-1}} \cdots s_{i_2}(\alpha_{i_1}) \in \sum_{i\in I} \mathbb{Z}_{\geqq 0}\alpha_i$.

Example 1.1. Assume that $i, j \in I$ and $i \neq j$.

(1) If $(a_{ij}, a_{ji}) = (0, 0)$, then both sides of $s_i s_j = s_j s_i$ are reduced expressions and

$$
\begin{cases} 1(\alpha_j^\vee) = \alpha_j^\vee, \\ s_j(\alpha_i^\vee) = \alpha_i^\vee, \end{cases}
\qquad
\begin{cases} 1(\alpha_i^\vee) = \alpha_i^\vee, \\ s_i(\alpha_j^\vee) = \alpha_j^\vee. \end{cases}
$$

(2) If $(a_{ij}, a_{ji}) = (-1, -1)$, then both sides of $s_i s_j s_i = s_j s_i s_j$ are reduce expressions and

$$
\begin{cases} 1(\alpha_i^\vee) = \alpha_i^\vee, \\ s_i(\alpha_j^\vee) = \alpha_i^\vee + \alpha_j^\vee, \\ s_i s_j(\alpha_i^\vee) = \alpha_j^\vee, \end{cases}
\qquad
\begin{cases} 1(\alpha_j^\vee) = \alpha_j^\vee, \\ s_j(\alpha_i^\vee) = \alpha_i^\vee + \alpha_j^\vee, \\ s_j s_i(\alpha_j^\vee) = \alpha_i^\vee. \end{cases}
$$

(3) If $(a_{ij}, a_{ji}) = (-1, -2)$, then both sides of $s_i s_j s_i s_j = s_j s_i s_j s_i$ are reduce expressions and

$$\begin{cases} 1(\alpha_j^\vee) = \alpha_j^\vee, \\ s_j(\alpha_i^\vee) = \alpha_i^\vee + \alpha_j^\vee, \\ s_j s_i(\alpha_j^\vee) = 2\alpha_i^\vee + \alpha_j^\vee, \\ s_j s_i s_j(\alpha_i^\vee) = \alpha_i^\vee, \end{cases} \qquad \begin{cases} 1(\alpha_i^\vee) = \alpha_i^\vee, \\ s_i(\alpha_j^\vee) = 2\alpha_i^\vee + \alpha_j^\vee, \\ s_i s_j(\alpha_i^\vee) = \alpha_i^\vee + \alpha_j^\vee, \\ s_i s_j s_i(\alpha_j^\vee) = \alpha_j^\vee. \end{cases}$$

(4) If $(a_{ij}, a_{ji}) = (-1, -3)$, then both sides of $s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$ are reduce expressions and

$$\begin{cases} 1(\alpha_j^\vee) = \alpha_j^\vee, \\ s_j(\alpha_i^\vee) = \alpha_i^\vee + \alpha_j^\vee, \\ s_j s_i(\alpha_j^\vee) = 3\alpha_i^\vee + 2\alpha_j^\vee, \\ s_j s_i s_j(\alpha_i^\vee) = 2\alpha_i^\vee + \alpha_j^\vee, \\ s_j s_i s_j s_i(\alpha_j^\vee) = 3\alpha_i^\vee + \alpha_j^\vee, \\ s_j s_i s_j s_i s_j(\alpha_i^\vee) = \alpha_i^\vee, \end{cases} \qquad \begin{cases} 1(\alpha_i^\vee) = \alpha_i^\vee, \\ s_i(\alpha_j^\vee) = 3\alpha_i^\vee + \alpha_j^\vee, \\ s_i s_j(\alpha_i^\vee) = 2\alpha_i^\vee + \alpha_j^\vee, \\ s_i s_j s_i(\alpha_j^\vee) = 3\alpha_i^\vee + 2\alpha_j^\vee, \\ s_i s_j s_i s_j(\alpha_i^\vee) = \alpha_i^\vee + \alpha_j^\vee, \\ s_i s_j s_i s_j s_i(\alpha_j^\vee) = \alpha_j^\vee. \end{cases}$$

These formulae shall be applied to the Verma relations.

1.3. Kac–Moody algebra

For details of Kac–Moody algebras, see Kac's book [7].

Let $A = [a_{ij}]_{i,j \in I}$ be a symmetrizable GCM and $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ a root datum of type A. We set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Y$ and identify $\mathfrak{h}^*$ with $\mathbb{C} \otimes_{\mathbb{Z}} X$ by $\langle\,,\rangle$.

The Kac–Moody (Lie) algebra $\mathfrak{g}$ associated to the root datum is defined to be the Lie algebra over $\mathbb{C}$ generated by E_i, F_i $(i \in I)$ and $H \in \mathfrak{h}$ with following defining relations:

$$\begin{aligned} &\mathfrak{h} \text{ is an Abelian Lie subalgebra of } \mathfrak{g}; \\ &[H, E_i] = \langle H, \alpha_i \rangle E_i, \quad [H, F_i] = -\langle H, \alpha_i \rangle F_i \quad \text{for } i \in I,\ H \in \mathfrak{h}; \\ &[E_i, F_j] = \delta_{ij}\alpha_i^\vee \quad \text{for } i, j \in I; \\ &\mathrm{ad}(E_i)^{1-a_{ij}} E_j = 0, \quad \mathrm{ad}(F_i)^{1-a_{ij}} F_j = 0 \quad \text{if } i \neq j. \end{aligned}$$

Here we set $\mathrm{ad}(X)Y = [X, Y]$, for example, $\mathrm{ad}(X)^3 Y = [X, [X, [X, Y]]]$. The last two relations are called *Serre relations.*

Denote by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the Lie subalgebra of $\mathfrak{g}$ generated by $\{E_i\}_{i\in I}$ (resp. $\{F_i\}_{i\in I}$). We call $\mathfrak{n}_\pm$ the *upper and lower parts* of $\mathfrak{g}$ and E_i (resp. F_i) the Chevalley generators of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$). The Abelian subalgebra $\mathfrak{h}$ is called the *Cartan subalgebra* of $\mathfrak{g}$. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. Define the *upper and lower Borel subalgebras* $\mathfrak{b}_\pm$ by $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$.

We can define the $\mathbb{Z}$-gradation of $\mathfrak{g}$ by $\deg E_i = 1$, $\deg F_i = -1$ $(i \in I)$ and $\deg H = 0$ $(H \in \mathfrak{h})$ and call it the *principal gradation* of $\mathfrak{g}$. Denote by $\mathfrak{g}_k$ the degree-k part of $\mathfrak{g}$ for $k \in \mathbb{Z}$. Then we have $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$, $\dim \mathfrak{g}_k < \infty$, $\mathfrak{n}_\pm = \bigoplus_{k>0} \mathfrak{g}_{\pm k}$, and $\mathfrak{h} = \mathfrak{g}_0$. The induced $\mathbb{Z}$-gradation $U(\mathfrak{g}) = \bigoplus_{k\in\mathbb{Z}} U(\mathfrak{g})_k$ is also called the principal gradation. Define the principal gradations of $U(\mathfrak{n}_\pm)$ by $U(\mathfrak{n}_\pm)_{\pm k} = U(\mathfrak{g})_{\pm k} \cap U(\mathfrak{n}_\pm)$ for $k \in \mathbb{Z}_{\geqq 0}$.

Assume that the root datum is Y-regular and X-regular. Denote by $\mathfrak{g}'$ the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ of the Kac–Moody algebra $\mathfrak{g}$. Then $\mathfrak{g}'$ is a finite dimensional Lie algebra (resp. a central extension of a (possibly twisted) loop algebra of a finite dimensional simple Lie algebra) if and only if A is of finite type (resp. of affine type). Then $\mathfrak{g}$ is called *a Kac–Moody algebra of finite type* (resp. *a Kac–Moody algebra of affine type* or *an affine Lie algebra* for short). These lead to the following result.

Lemma 1.2. *Assume that the GCM A is of finite or affine type. Then $\{\dim \mathfrak{g}_k\}_{k\in\mathbb{Z}}$ is bounded, namely there exists a positive integer N such that $\dim \mathfrak{g}_k \leqq N$ for all $k \in \mathbb{Z}$. Define the positive integers $C_k^{(N)}$ $(k \in \mathbb{Z}_{\geqq 0})$ by $\left(\prod_{i=1}^\infty (1-t^i)\right)^{-N} = \sum_{k=0}^\infty C_k^{(N)} t^k$. Then we have $\dim U(\mathfrak{n}_\pm)_{\pm k} \leqq C_k^{(N)}$ for $k \in \mathbb{Z}_{\geqq 0}$.*

1.4. q-Binomial theorem

We define q-numbers, q-factorials, q-binomial coefficients, and q-shifted factorials as follows:

$$
\begin{aligned}
&[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \quad \text{for } n \in \mathbb{Z}, \\
&[n]_q! = \prod_{k=1}^n [k]_q \quad \text{for } n \in \mathbb{Z}_{\geqq 0}, \\
&\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[k]_q!} \quad \text{for } k \in \mathbb{Z}_{\geqq 0}, \\
&(x)_{q,k} = (1+x)(1+q^2x)\cdots(1+q^{2(k-1)}x) \quad \text{for } k \in \mathbb{Z}_{\geqq 0}, \\
&(x)_{q,\infty} = (1+x)(1+q^2x)(1+qx^4)\cdots = \prod_{\mu=0}^\infty (1+q^{2\mu}x).
\end{aligned}
$$

Note that our q-shifted factorials are different from usual ones defined by $(x;q)_k = \prod_{\mu=0}^{k-1}(1-q^\mu x)$. Then we can prove the following lemma by induction on n.

Lemma 1.3 (q-binomial theorem). *Assume that x, y are elements of an $\mathbb{Q}(q)$-algebra satisfying $yx = q^2xy$. Then, for $n = 0, 1, 2, \ldots$,*

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$
$$= \sum_{k=0}^{\infty} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} = \sum_{k=0}^{\infty} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k.$$

Moreover, if x is invertible, then, for $n = 0, 1, 2, \ldots$,

$$(x+y)^n = x^n (x^{-1}y)_{q,n} = x^n \frac{(x^{-1}y)_{q,\infty}}{(q^{2n}x^{-1}y)_{q,\infty}} = \frac{(q^{-2n}x^{-1}y)_{q,\infty}}{(x^{-1}y)_{q,\infty}} x^n,$$

where the infinite products cancel out except finite factors.

1.5. Quantized universal enveloping algebra

Let $A = [a_{ij}]_{i,j\in I}$ be a symmetrizable GCM symmetrized by $\{d_i\}_{i\in I}$ and $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ a root datum of type A. Let d be the least common denominator of $\{d_i\}_{i\in I}$. We set the base field $\mathbb{F}$ by $\mathbb{F} = \mathbb{Q}(q^{1/d})$ and $q_i \in \mathbb{F}$ by $q_i = q^{d_i}$. Then we have $d_i\alpha_i^\vee \in d^{-1}Y$ and we extend naturally the perfect bilinear pairing $\langle\,,\rangle : Y \times X \to \mathbb{Z}$ to $\langle\,,\rangle : d^{-1}Y \times X \to d^{-1}\mathbb{Z}$.

Then the *quantized universal enveloping algebra* $U_q = U_q(\mathfrak{g})$ associated to the root datum is defined to be the associative algebra over the base field $\mathbb{F}$ generated by E_i, F_i, q^λ for $i \in I$ and $\lambda \in d^{-1}Y$ with following defining relations:

$$q^0 = 1, \quad q^{\lambda+\mu} = q^\lambda q^\mu \quad \text{for } \lambda, \mu \in d^{-1}Y;$$
$$q^\lambda E_i q^{-\lambda} = q^{\langle\lambda,\alpha_i\rangle} E_i, \quad q^\lambda F_i q^{-\lambda} = q^{-\langle\lambda,\alpha_i\rangle} F_i \quad \text{for } i \in I,\ \lambda \in d^{-1}Y;$$
$$E_iF_j - F_jE_i = \delta_{ij}[\alpha_i^\vee]_{q_i} = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \text{for } i, j \in I;$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0 \quad \text{if } i \neq j;$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0 \quad \text{if } i \neq j,$$

where we set $K_i = q_i^{\alpha_i^\vee} = q^{d_i\alpha_i^\vee}$. The last two relations are called *q-Serre relations*. (For this definition, see Corollary 33.1.5 in [11].) In particular, we have $K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j$ and $K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j$ for $i,j \in I$. By induction on $m, n \in \mathbb{Z}_{\geqq 0}$ we can prove the following formula:

$$
\begin{aligned}
E_i^{(m)}F_i^{(n)} &= \sum_{k=0}^{\min\{m,n\}} F_i^{(n-k)} \begin{bmatrix} \alpha_i^\vee - (m-k) - (n-k) \\ k \end{bmatrix}_{q_i} E_i^{(m-k)}, \\
F_i^{(n)}E_i^{(m)} &= \sum_{k=0}^{\min\{m,n\}} E_i^{(m-k)} \begin{bmatrix} -\alpha_i^\vee - (m-k) - (n-k) \\ k \end{bmatrix}_{q_i} F_i^{(n-k)},
\end{aligned}
\tag{1.1}
$$

where we set $E_i^{(m)} = E_i^m/[m]_{q_i}!$ and $F_i^{(n)} = F_i^n/[n]_{q_i}!$. (See Corollary 3.1.9 in [11].)

Denote by $U_q^+ = U_q(\mathfrak{n}_+)$ (resp. $U_q^- = U_q(\mathfrak{n}_-)$, $U_q^0 = U_q(\mathfrak{h})$) the subalgebra of $U_q = U_q(\mathfrak{g})$ generated by $\{E_i\}_{i\in I}$ (resp. $\{F_i\}_{i\in I}$, $\{q^\lambda\}_{\lambda\in d^{-1}Y}$). We call $U_q^\pm$ the *upper and lower parts* of U_q and E_i, F_i the upper and lower *Chevalley generators* respectively. The commutative subalgebra U_q^0 is called the *Cartan subalgebra* of U_q. We have the triangular decomposition $U_q \cong U_q^- \otimes U_q^0 \otimes U_q^+$ of U_q. Note that $U_q^\pm$ are determined by the symmetrized GCM only and their structure does not depend on the choice of a root datum. Define the *upper and lower Borel subalgebras* $U_q(\mathfrak{b}_\pm)$ to be the subalgebra of $U_q = U_q(\mathfrak{g})$ generated by $U_q^\pm = U_q(\mathfrak{n}_\pm)$ and $U_q^0 = U_q(\mathfrak{h})$. Then we have $U_q(\mathfrak{b}_\pm) \cong U_q^0 \otimes U_q^\pm \cong U_q^\pm \otimes U_q^0$. We say that U_q is *of finite type* (resp. *of affine type* or *affine* for short) if A is of finite type (resp. of affine type).

We can define the $\mathbb{Z}$-gradation of U_q by $\deg E_i = 1$, $\deg F_i = -1$ ($i \in I$) and $\deg q^\lambda = 0$ ($\lambda \in d^{-1}Y$) and call it the *principal gradation* of U_q. Denote by $(U_q)_k$ the degree-k part of U_q for $k \in \mathbb{Z}$. Then we have $U_q = \bigoplus_{k\in\mathbb{Z}} U_k$. Define the principal gradations of $U_q^\pm$ by $(U_q^\pm)_{\pm k} = (U_q)_{\pm k} \cap U_q^\pm$ for $k \in \mathbb{Z}_{\geqq 0}$. Then we have $U_q^\pm = \bigoplus_{k\geqq 0}(U_q^\pm)_{\pm k}$.

We can regard U_q (resp. $U_q^\pm$, U_q^0) as a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$, (resp. $U(\mathfrak{n}_\pm)$, $U(\mathfrak{h})$) of a Kac–Moody algebra $\mathfrak{g}$ (resp. its upper and lower parts $\mathfrak{n}_\pm$, its Cartan subalgebra $\mathfrak{h}$).

Define the local ring $\mathbb{A}_1$ by $\mathbb{A}_1 = \{\, f(q^{1/d}) \in \mathbb{F} = \mathbb{Q}(q^{1/d}) \mid f$ is regular at $q^{1/d} = 1\,\} = \mathbb{Q}(q^{1/d}) \cap \mathbb{Q}[[q^{1/d} - 1]]$. We regard $\mathbb{C}$ as an algebra over $\mathbb{A}_1$ by acting $q^{1/d}$ on $\mathbb{C}$ as 1 and denote the algebra $\mathbb{C}$ over $\mathbb{A}_1$ by $\mathbb{C}_1$. Assume that $\{y_1, \dots, y_M\}$ is a $\mathbb{Z}$-free basis of $d^{-1}Y$. Set $(x)_q = (1-q^x)/(1-q)$. Then we have the following results. For the proof, see Sections 3.3 and 3.4 of [5], for example.

Lemma 1.4. *Let $U_{\mathbb{A}_1}$ be the subalgebra of U_q over $\mathbb{A}_1$ generated by $\{E_i, F_i\}_{i\in I}$ and $\{(y_\mu)_q, q^{-y_\mu}\}_{\mu=1}^M$. Let $U_{\mathbb{A}_1}^+$ (resp. $U_{\mathbb{A}_1}^-$) be the subalgebra* *of U_q^+ over $\mathbb{A}_1$ (resp. U_q^-) generated by $\{E_i\}_{i\in I}$ (resp. $\{F_i\}_{i\in I}$) and $U_{\mathbb{A}_1}^0$ the subalgebra of U_q^0 over $\mathbb{A}_1$ generated by $\{q^{y_\mu}, (y_\mu)_q\}_{\mu=1}^M$. Set $(U_{\mathbb{A}_1}^\pm)_k = (U_q^\pm)_k \cap U_{\mathbb{A}_1}$ for $k \in \mathbb{Z}$.*

(1) *The multiplication gives an isomorphism $U_{\mathbb{A}_1}^- \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^0 \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^+ \xrightarrow{\sim} U_{\mathbb{A}_1}$ of $\mathbb{A}_1$-modules.*
(2) *$(U_{\mathbb{A}_1}^\pm)_k$ are free $\mathbb{A}_1$-modules and $U_{\mathbb{A}_1}^\pm = \bigoplus_{k=0}^\infty (U_{\mathbb{A}_1}^\pm)_{\pm k}$.*
(3) *$U_{\mathbb{A}_1}^0 = \mathbb{A}_1[(y_1)_q, \dots, (y_M)_q, q^{-y_1}, \dots, q^{-y_M}]$ properly contains $\mathbb{A}_1[q^{\pm y_1}, \dots, q^{\pm y_M}]$.*
(4) *$\mathbb{F} \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1} = U_q$, $\mathbb{F} \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^\pm = U_q^\pm$, $\mathbb{F} \otimes_{\mathbb{A}_1} (U_{\mathbb{A}_1}^\pm)_k = (U_q^\pm)_k$, and $\mathbb{F} \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^0 = U_q^0$.*
(5) *$\mathbb{C}_1 \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1} = U(\mathfrak{g})$, $\mathbb{C}_1 \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^\pm = U(\mathfrak{n}_\pm)$, $\mathbb{C}_1 \otimes_{\mathbb{A}_1} (U_{\mathbb{A}_1}^\pm)_k = U(\mathfrak{n}_\pm)_k$, and $\mathbb{C}_1 \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^0 = U(\mathfrak{h})$.*
(6) *$\dim_{\mathbb{F}} (U_q^\pm)_{\pm k} = \mathrm{rank}_{\mathbb{A}_1} (U_{\mathbb{A}_1}^\pm)_{\pm k} = \dim_{\mathbb{C}} U(\mathfrak{n}_\pm)_{\pm k}$ for $k \in \mathbb{Z}_{\geqq 0}$.*

In particular, we obtain the following results.

Lemma 1.5. *Let A be a symmetrizable GCM. A quantized universal enveloping algebra U_q of type A is always an integral domain. If A is of finite or affine type, then $\dim_{\mathbb{F}}(U_q^\pm)_k \leqq C_k^{(N)}$ for $k \in \mathbb{Z}_{\geqq 0}$, where N and $C_k^{(N)}$ are given in Lemma 1.2.*

Proof. Assume that $a, b \in U_q$ are non-zero. Then there exist $\lambda, \mu \in d^{-1}Y$ such that $q^\lambda a, bq^\mu \in U_q^- \otimes \mathbb{F}[q^{y_1}, \dots, q^{y_M}] \otimes U_q^+$. Let $\{u_s^\pm\}_{s=0}^\infty$ be $\mathbb{A}_1$-free bases of $U_{\mathbb{A}_1}^\pm$. Set $u_\mu^0 = (y_1)_q^{\mu_1} \cdots (y_M)_q^{\mu_M}$ for $\mu = (\mu_1, \dots, \mu_M) \in (\mathbb{Z}_{\geqq 0})^n$. Then $\{u_\mu^0\}_{\mu \in (\mathbb{Z}_{\geqq 0})^n}$ is an $\mathbb{A}_1$-free basis of $\mathbb{A}_1[(y_1)_q, \dots, (y_M)_q]$. Note that $\mathbb{F}[q^{y_1}, \dots, q^{y_M}] = \mathbb{F}[(y_1)_q, \dots, (y_M)_q]$. Because of Lemma 1.4 (1) and (4), we can uniquely write $q^\lambda a$ and bq^μ in the following forms: $q^\lambda a = \sum_{r,\mu,s} c_{r,\mu,s} u_r^- u_\mu^0 u_s^+$, $bq^\mu = \sum_{r,\mu,s} d_{r,\mu,s} u_r^- u_\mu^0 u_s^+$ ($c_{r,\mu,s}, d_{r,\mu,s} \in \mathbb{F} = \mathbb{Q}(q^{1/d})$), where only finitely many $c_{r,\mu,s}$ and $d_{r,\mu,s}$ are non-zero. Since any $c \in \mathbb{F}^\times$ is uniquely expressed as $c = (q-1)^{-k}\tilde{c}$ with $k \in \mathbb{Z}$ and $\tilde{c} \in \mathbb{A}_1^\times$ (i.e. $\tilde{c} \in \mathbb{A}_1$ and $\tilde{c}(1) \neq 0$), we set $\mathrm{ord}(c) = k$ and $\mathrm{ord}(0) = -\infty$. Setting $l = \max\{\mathrm{ord}(c_{s,\mu,s})\}$ and $m = \max\{\mathrm{ord}(d_{s,\mu,s})\}$, we have $(q-1)^l q^\lambda a, (q-1)^m bq^\mu \in U_{\mathbb{A}_1}^- \otimes_{\mathbb{A}_1} \mathbb{A}_1[(y_1)_q, \dots, (y_M)_q] \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}^+$. Moreover their images in $\mathbb{C}_1 \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1} = U(\mathfrak{g})$ are non-zero and hence their product in $U(\mathfrak{g})$ is non-zero. Therefore $(q-1)^{m+l} q^\lambda ab q^\mu \neq 0$, namely $ab \neq 0$. This means that U_q is an integral domain. The second statement immediately follows from Lemma 1.4 (6). Q.E.D.

If the root datum is Y-regular and X-regular, then the highest weight integral representations of $\mathfrak{g}$ are deformed to those of U_q. For details, see Chapter 33 of [11] and Section 3.4 of [5].

We can define the coproduct $\Delta : U_q \to U_q \otimes U_q$ (an algebra homomorphism), the counit $\varepsilon : U_q \to \mathbb{F}$ (an algebra homomorphism), and the antipode $S : U_q \to U_q$ (an anti-algebra automorphism) by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad \text{for } i \in I,$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \quad \text{for } i \in I,$$
$$\Delta(q^\lambda) = q^\lambda \otimes q^\lambda \quad \text{for } \lambda \in d^{-1}Y,$$
$$\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(q^\lambda) = 1 \quad \text{for } i \in I,\ \lambda \in d^{-1}Y,$$
$$S(E_i) = -E_i K_i^{-1},\ S(F_i) = -K_i F_i,\ S(q^\lambda) = q^{-\lambda} \text{ for } i \in I,\ \lambda \in d^{-1}Y.$$

These give a Hopf algebra structure on U_q.

Remark 1.6. The above definition of a Hopf algebra structure on U_q is different from that in Lusztig's book [11]. Denote by Δ^L, ε^L, and S^L the coproduct, the counit, and the antipode of [11] respectively. We can uniquely define the involutive algebra automorphism ω of U_q by $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, $\omega(q^\lambda) = q^{-\lambda}$ for $i \in I$, $\lambda \in d^{-1}Y$. Then the Hopf algebra structure of [11] is related to ours by $\Delta^L = (\omega \otimes \omega) \circ \Delta \circ \omega$, $\varepsilon^L = \varepsilon \circ \omega$, and $S^L = \omega \circ S \circ \omega$.

Example 1.7 (affine gl_m case). Assume $m \in \mathbb{Z}_{\geqq 2}$ and set $I = \{0, 1, \ldots, m-1\}$. Let Y be the free $\mathbb{Z}$-module generated by $\{\varepsilon_i\}_{i=1}^m$, c, and d. Let X be the dual lattice of Y. Define the non-degenerate symmetric bilinear form $(\,,) : Y \times Y \to \mathbb{Z}$ by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(c, d) = 1$, and $(\varepsilon_i, c) = (\varepsilon_i, d) = (c, c) = (d, d) = 0$. We can identify X with Y by $(\,,)$. We set $\alpha_i^\vee = \alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $i = 1, \ldots, m-1$, $\alpha_0^\vee = \alpha_0 = c - \varepsilon_1 + \varepsilon_m$. Define the matrix $A_{m-1}^{(1)}$ by $A_{m-1}^{(1)} = [a_{ij}] = [(\alpha_i^\vee, \alpha_j)]_{i,j \in I}$. If $m = 2$, then $a_{00} = a_{11} = 2$ and $a_{01} = a_{10} = -2$. If $m \geqq 3$, then $a_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{j+1,i} - \delta_{i0}\delta_{j,m-1} - \delta_{j0}\delta_{i,m-1}$. Thus $A_{m-1}^{(1)}$ is a symmetric GCM and $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i \in I})$ is a Y-regular and X-regular root datum of type $A_{m-1}^{(1)}$. The Kac–Moody algebra associated to the root datum can be identified with the affine Lie algebra $\widehat{\mathrm{gl}}_m = \mathrm{gl}_m(\mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}td/dt$ by

$$E_0 = tE_{m1},\ F_0 = t^{-1}E_{1m},$$
$$E_i = E_{i,i+1},\ F_i = E_{i+1,i} \quad \text{for } i = 1, \ldots, m-1,$$
$$\varepsilon_i = E_{ii} \quad \text{for } i = 1, \ldots, m, \quad d = td/dt,$$

where E_{ij} $(i, j = 1, \ldots, m)$ are unit matrices. We set all $d_i = 1$. The quantized universal enveloping algebra associated to the root datum is called *the quantized universal enveloping algebra of* gl_m and denoted by $U_q(\widehat{\mathrm{gl}}_m)$.

Example 1.8 (affine sl_m and psl_m cases). Assume $m \in \mathbb{Z}_{\geqq 2}$ and set $I = \{0, 1, \dots, m-1\}$. Let $A^{(1)}_{m-1} = [a_{ij}]_{i,j\in I}$ be the symmetric GCM given above. We set all $d_i = 1$. Let Y be the free $\mathbb{Z}$-module generated by $\{\alpha_i^\vee\}_{i=1}^{m-1}$, c, and d. Set $\alpha_0^\vee = c - \alpha_1^\vee - \cdots - \alpha_{m-1}^\vee$. Let X be the dual lattice of Y. Define the non-degenerate symmetric bilinear form $(\,,\,) : Y \times Y \to \mathbb{Z}$ by $(\alpha_i^\vee, \alpha_j^\vee) = a_{ij}$, $(c, d) = 1$, and $(\alpha_i^\vee, c) = (\alpha_i^\vee, d) = (c, c) = (d, d) = 0$ for $i = 1, \dots, m-1$. Identifying X with a sublattice of $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Y$ by $(\,,\,)$, we have $Y \subsetneqq X$. Define $\alpha_j \in X$ for $j \in I$ by $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, $\langle c, \alpha_j \rangle = 0$, $\langle d, \alpha_j \rangle = \delta_{j0}$ for $i, j \in I$. Then $(Y, X, \langle\,,\,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ is a Y-regular and X-regular root datum of type $A^{(1)}_{m-1}$. The Kac–Moody algebra associated to the root datum can be identified with the affine Lie algebra $\widehat{\mathrm{sl}}_m = \mathrm{sl}_m(\mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}td/dt$ by

$$
\begin{aligned}
&E_0 = tE_{m1}, \ F_0 = t^{-1}E_{1m}, \\
&E_i = E_{i,i+1}, \ F_i = E_{i+1,i} \quad \text{for } i = 1, \dots, m-1, \\
&\alpha_i^\vee = E_{ii} - E_{i+1,i+1} \quad \text{for } i = 1, \dots, m-1, \\
&\alpha_0^\vee = c - E_{11} + E_{mm}, \quad d = td/dt.
\end{aligned}
$$

The quantized universal enveloping algebra associated to the root datum is called *the quantized universal enveloping algebra of* $\widehat{\mathrm{sl}}_m$ and denoted by $U_q(\widehat{\mathrm{sl}}_m)$. Associating to the dual root datum $(X, Y, \langle\,,\,\rangle, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$, we define the quantized universal enveloping algebra $U_q(\widehat{\mathrm{psl}}_m)$.

1.6. Adjoint action

For an arbitrary Hopf algebra H, the adjoint action $\mathrm{ad} : H \to \mathrm{End}\, H$ is defined by $\mathrm{ad}(x)y = \sum_{(x)} x_{(1)} y S(x_{(2)})$ for $x, y \in H$, where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ (the Swedler notation).

In a quantized universal enveloping algebra U_q, we have

$$
\begin{aligned}
&\mathrm{ad}(E_i)x = E_i x K_i^{-1} - x E_i K_i^{-1} \quad \text{for } i \in I,\ x \in U_q, \\
&\mathrm{ad}(F_i)x = F_i x - K_i^{-1} x K_i F_i \quad \text{for } i \in I,\ x \in U_q, \\
&\mathrm{ad}(q^\lambda)x = q^\lambda x q^{-\lambda} \quad \text{for } \lambda \in d^{-1}Y,\ x \in U_q.
\end{aligned}
$$

Setting $x = F_i \otimes 1$ and $y = K_i^{-1} \otimes F$, we have $yx = q_i^2 xy$. Using the q-binomial theorem, we obtain $\Delta(F_i)^n = \sum_{k=0}^{n} q_i^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} F_i^{n-k} K_i^{-k} \otimes K_i^k$ and hence $(1 \otimes S)(\Delta(F_i^n)) = \sum_{k=0}^{n} (-1)^k q_i^{k(n-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} F_i^{n-k} K_i^{-k} \otimes K_i^k$

F_i^k. We conclude that

$$\operatorname{ad}(F_i)^n x = \operatorname{ad}(F_i^n)x = \sum_{k=0}^{n}(-1)^k q_i^{k(n-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} F_i^{n-k} K_i^{-k} x K_i^k F_i^k.$$

In particular, we have

$$\operatorname{ad}(F_i)^n F_j = \sum_{k=0}^{n}(-1)^k q_i^{k(n-1+a_{ij})} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} F_i^{n-k} F_j F_i^k. \tag{1.2}$$

Hence the q-Serre relations for F_i $(i \in I)$ are rewritten as $\operatorname{ad}(F_i)^{1-a_{ij}} F_j = 0$ for $i \neq j$. Similar results hold for E_i $(i \in I)$. The following lemma is equivalent to Formula (14) of [6].

Lemma 1.9 ([6]). *Assume that $x \in U_q$ and $K_i^{-1} x K_i = q_i^a x$. Then*

$$F_i^n x = \sum_{k=0}^{\infty} q_i^{(k+a)(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \operatorname{ad}(F_i)^k(x) F_i^{n-k} \quad \text{for } n \in \mathbb{Z}_{\geqq 0},$$

where the left-hand side is a finite sum with respect to $k = 0, 1, \ldots, n$. In particular, if $i \neq j$ in I, then

$$F_i^n F_j = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \operatorname{ad}(F_i)^k(F_j) F_i^{n-k} \quad \text{for } n \in \mathbb{Z}_{\geqq 0}.$$

The first formula of this lemma is proved by induction on n. The case for $n = 1$ leads to the cases for any $n \in \mathbb{Z}_{>0}$. The second immediately follows from the q-Serre relations.

The following example can be found as Formula (24) of [6].

Example 1.10 ([6]). If $a_{ij} = -1$, then

$$\begin{aligned} F_i^n F_j &= q_i^{-n} F_j F_i^n + [n]_{q_i} \operatorname{ad}(F_i)(F_j) F_i^n - 1 \\ &= [1-n]_{q_i} F_j F_i^n + [n]_{q_i} F_i F_j F_i^{n-1}. \end{aligned}$$

1.7. Verma relations

Let $A = [a_{ij}]_{i,j \in I}$ be a symmetrizable GCM symmetrized by $\{d_i\}_{i \in I}$ and $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i \in I}, \{\alpha_i\}_{i \in I})$ a Y-regular and X-regular root datum of type A.

Denote by W the Weyl group of type A and by s_i $(i \in I)$ its generators. Denote by U_q^- the lower part of the quantized universal enveloping algebra associated to the root datum and by F_i $(i \in I)$ its Chevalley generators. Let $\lambda \in X^+$.

Assume that $s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression in W. We set $k_p\in\mathbb{Z}$ for $p=1,2,\dots,n$ by

$$k_p=\langle s_{i_n}s_{i_{n-1}}\cdots s_{i_{p+1}}(\alpha_{i_p}^\vee),\lambda\rangle.$$

For examples, $k_n=\langle\alpha_{i_n}^\vee,\lambda\rangle$, $k_{n-1}=\langle s_{i_n}(\alpha_{i_{n-1}}^\vee),\lambda\rangle$, $k_{n-2}=\langle s_{i_n}s_{i_{n-1}}(\alpha_{i_{n-2}}^\vee),\lambda\rangle$, and so on. Since $s_{i_n}\cdots s_{i_{p+1}}s_{i_p}$ is also a reduced expression, $s_{i_n}\cdots s_{i_{p+1}}(\alpha_{i_p}^\vee)\in\sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i^\vee$. Therefore $k_p\in\mathbb{Z}_{\geq 0}$ for $p=1,2,\dots,n$.

Assume that $s_{j_1}s_{j_2}\cdots s_{j_n}$ is another reduced expression with $s_{i_1}s_{i_2}\cdots s_{i_n}=s_{j_1}s_{j_2}\cdots s_{j_n}$. We similarly set $l_p\in\mathbb{Z}$ for $p=1,2,\dots,n$ by

$$l_p=\langle s_{j_n}s_{j_{n-1}}\cdots s_{j_{p+1}}(\alpha_{j_p}^\vee),\lambda\rangle.$$

Then we have the following identity in U_q^-:

$$F_{i_1}^{k_1}F_{i_2}^{k_2}\cdots F_{i_n}^{k_n}=F_{j_1}^{l_1}F_{j_2}^{l_2}\cdots F_{j_n}^{l_n}. \tag{1.3}$$

Furthermore the sequence $((i_1,k_1),(i_2,k_2),\dots,(i_n,k_n))$ is equal to the sequence $((j_1,l_1),(j_2,l_2),\dots,(j_n,l_n))$ up to permutation of order. In order to prove these results, it is sufficient to show them for each pair of reduced expressions in Example 1.1. For the proof, see Section 39.3 of [11] and Lemma 2 of [2]. These results are called *Verma relations.*

Example 1.11 (Verma relations). Assume that $i,j\in I$ and $i\neq j$. Let k and l be arbitrary non-negative integers. Then Example 1.1 leads to the following formulae of the Chevalley generators of U_q^-:

(1) $F_i^kF_j^l=F_j^lF_i^k$ if $(a_{ij},a_{ji})=(0,0)$;
(2) $F_i^lF_j^{k+l}F_i^k=F_j^kF_i^{k+l}F_j^l$ if $(a_{ij},a_{ji})=(-1,-1)$;
(3) $F_i^kF_j^{2k+l}F_i^{k+l}F_j^l=F_j^lF_i^{k+l}F_j^{2k+l}F_i^k$ if $(a_{ij},a_{ji})=(-1,-2)$;
(4) $F_i^kF_j^{3k+l}F_i^{2k+l}F_j^{3k+2l}F_i^{k+l}F_j^l=F_j^lF_i^{k+l}F_j^{3k+2l}F_i^{2k+l}F_j^{3k+l}F_i^k$ if $(a_{ij},a_{ji})=(-1,-3)$.

These formulae shall be used in the construction of the quantized birational Weyl group actions.

Remark 1.12. For $\beta_1,\dots,\beta_n\in\sum_{i\in I}\mathbb{Z}_{\geqq 0}\alpha_i^\vee$, denote by $F_{i_1}^{\beta_1}\cdots F_{i_n}^{\beta_n}$ the mapping from X^+ to U_q^- sending λ to $F_{i_1}^{\langle\beta_1,\lambda\rangle}\cdots F_{i_n}^{\langle\beta_n,\lambda\rangle}$. We introduce the formal symbols $\tilde{s}_i$ $(i\in I)$ satisfying the braid relations and $\tilde{s}_i^{-1}F_j^{\beta_j}\tilde{s}_i=F_j^{s_i(\beta_j)}$. Then the Verma identity (1.3) can be formally rewritten in the following form:

$$\tilde{s}_{i_1}F_{i_1}^{\alpha_{i_1}^\vee}\tilde{s}_{i_2}F_{i_2}^{\alpha_{i_2}^\vee}\cdots\tilde{s}_{i_n}F_{i_n}^{\alpha_{i_n}^\vee}=\tilde{s}_{j_1}F_{j_1}^{\alpha_{j_1}^\vee}\tilde{s}_{j_2}F_{j_2}^{\alpha_{j_2}^\vee}\cdots\tilde{s}_{j_n}F_{j_n}^{\alpha_{j_n}^\vee}.$$

This means that $\tilde{s}_i F_i^{\alpha_i^\vee}$ $(i \in I)$ formally satisfy the braid relations. Moreover, if we have $\tilde{s}_i^2 = 1$ and $F_i^{-\alpha_i^\vee} F_i^{\alpha_i^\vee} = 1$, then we obtain, at least formally,

$$\tilde{s}_i F_i^{\alpha_i^\vee} \tilde{s}_i F_i^{\alpha_i^\vee} = \tilde{s}_i^2 F_i^{-\alpha_i^\vee} F_i^{\alpha_i^\vee} = 1.$$

This means that $\tilde{s}_i F_i^{\alpha_i^\vee}$ $(i \in I)$ formally satisfy the defining relations of the Weyl group. If we can justify the above heuristic consideration, then we can construct the braid or Weyl group representations.

§2. Localizations of non-commutative rings

In this section, we shall summarize results on localizations of non-commutative rings necessary to quantize birational actions. Most of the proofs omitted below can be found, for examples, in Chapter 10 of [3] and Chapter 2 of [12].

2.1. Localization at an Ore subset

Let A be a (possibly non-commutative) ring. A is called an *integral domain* (or a *domain* for short) if $A \neq 0$ and the products of non-zero elements of A are always non-zero. A proper two-sided ideal I of A is called *completely prime* if A/I is an integral domain. We say that A is *left Noetherian* if there is no infinite properly ascending chain of left ideals of A. A *right Noetherian* ring is similarly defined.

A subset S of A is called *multiplicative* if S contains 1 and is closed with respect to multiplication. Let A be an integral domain and S its multiplicative subset. We say that S satisfies *the left* (resp. *right*) *Ore condition* if $Sa \cap As \neq \emptyset$ (resp. $aS \cap sA \neq \emptyset$) for any $a \in A$ and $s \in S$. A multiplicative subset satisfying the left (resp. right) Ore condition is called a *left* (resp. *right*) *Ore subset* for short. A left and right Ore subset is simply called an *Ore subset.*

Assume that S is a left Ore subset of A. Then we can define the ring $S^{-1}A$ as follows. As a set, $S^{-1}A$ is defined to be the quotient set $(S \times A)/\sim$, where the equivalence relation $\sim$ is defined by $(s,a) \sim (s',a')$ $\Leftrightarrow$ there exists $u, u' \in A$ such that $us = u's' \in S$ and $ua = u'a'$. Denote by $s\backslash a$ the element of $S^{-1}A$ represented by $(s,a) \in S \times A$. We can define the ring structure of $S^{-1}A$ by

$$\begin{aligned} (s\backslash a)(s'\backslash a') &= (s''s)\backslash(a''a'), \qquad & s''a = a''s',\ a'' \in A,\ s'' \in S; \\ s\backslash a + s'\backslash a' &= (us)\backslash(ua + u'a'), & us = u's',\ u' \in A,\ u \in S. \end{aligned}$$

Identifying $a \in A$ with $1\backslash a \in S^{-1}A$, we can embed A into $S^{-1}A$. Then the ring $S^{-1}A$ contains A as a subring and satisfies that any element of S is invertible in $S^{-1}A$ and $S^{-1}A = \{ s^{-1}a = s\backslash a \mid s \in S, a \in A \}$. Furthermore $S^{-1}A$ has the following universality: for any ring B and any ring homomorphism $f : A \to B$ with the property that $f(s)$ is invertible in B for any $s \in S$, there exists a unique ring homomorphism $\phi : S^{-1}A \to B$ with $\phi|_A = f$. In particular $S^{-1}A$ is uniquely, up to isomorphism, determined by A and S. We call $S^{-1}A$ the *left localization* of A at S. If S is an Ore subset, then the left localization at S can be identified with the right one, namely $S^{-1}A = \{ as^{-1} \mid s \in S, a \in A \}$.

Lemma 2.1. *Let A be an integral domain generated by $\{a_j\}_{j\in J}$ over a field $\mathbb{F}$ and S its multiplicative subset generated by $\{s_i\}_{i\in I}$. Then we have the following results:*

(1) *If $Sa \cap As_i \neq \emptyset$ for any $a \in A$ and $i \in I$, then S is a left Ore subset of A.*

(2) *Assume that for any $i \in I$, $j \in J$, and $n \in \mathbb{Z}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ with $s_i^N a_j \in As_i^n$. Then for any $i \in I$, $a \in A$, and $n \in \mathbb{Z}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ with $s_i^N a \in As_i^n$. Therefore S is a left Ore subset of A.*

Proof. (1) Take $i_1, \dots, i_n \in I$. By induction on n, let us show that $Sa \cap Af_{i_n} \cdots f_{i_1} \neq \emptyset$ for any $a \in A$. The case of $n = 1$ is just the the assumption. Assume that it holds for $n - 1$. Then there exist $t \in S$ and $b \in A$ with $ta = bf_{i_{n-1}} \cdots f_{i_1}$. By the case of $n = 1$, there exist $c \in A$ and $u \in S$ with $ub = cf_{i_n}$. Then $ut \in S$ and $uta = cf_{i_n} \cdots f_{i_1}$.

(2) Fix any $i \in I$. Let $\widetilde{A}$ be the subset of A consisting of the elements $a \in A$ such that for any $n \in \mathbb{Z}_{>0}$ there exists $N \in \mathbb{Z}_{>0}$ with $s_i^N a \in As_i^n$. It is sufficient for the proof of the first statement to show that $\widetilde{A}$ is a subalgebra of A. Take any $a, b \in \widetilde{A}$. For any $n \in \mathbb{Z}_{>0}$, there exists $M, N \in \mathbb{Z}_{>0}$ such that $s_i^M a \in As_i^n$ and $s_i^N b \in As_i^n$. Then $s_i^{M+N}(a + b) \in As_i^n$ and hence $a + b \in \widetilde{A}$. There exists $L \in \mathbb{Z}_{>0}$ such that $s_i^L a \in As_i^N$. Then $s_i^L ab \in As_i^N b \subset As_i^n$ and hence $ab \in \widetilde{A}$. We have shown that $\widetilde{A}$ is a subalgebra of A. The second statement follows from (1). Q.E.D.

Example 2.2 (the inverse of F_i)**.** Consider a quantized universal enveloping algebra U_q and its lower part U_q^-. Let J be any subset of I and S_J the multiplicative subset generated by $\{F_j\}_{j\in J}$. Using Formula (1.1), $F_i q_i^\lambda = q_i^{\lambda+\langle\lambda,\alpha_i\rangle} F_i$ ($\lambda \in d^{-1}Y$), and the q-Serre relations of $\{F_i\}_{i\in I}$, we can find that both (U_q^-, S_J) and (U_q, S_J) satisfy the assumption of Lemma 2.1 (2). Therefore S_J is a left Ore subset of U_q^- and U_q. The anti-algebra involution given by $E_i \mapsto E_i$, $F_i \mapsto F_i$ ($i \in I$) and $q^\lambda \mapsto q^{-\lambda}$ ($\lambda \in d^{-1}Y$) proves that S_J is also a right Ore subset of U_q^- and U_q. By the universality of $S_J^{-1}U_q^-$, we can regard $S_J^{-1}U_q^-$ as a subalgebra of $S_J^{-1}U_q$. For $J = \{i_1, \dots, i_r\}$, we denote $S_J^{-1}U_q$ by $U_q[F_{i_1}^{-1}, \dots, F_{i_r}^{-1}]$ and $S_J^{-1}U_q^-$ by $U_q^-[F_{i_1}^{-1}, \dots, F_{i_r}^{-1}]$.

Using the inverse of F_i, we can state the following generalization of Lemma 1.9 for negative integral powers of F_i.

Lemma 2.3. *Assume that $x \in U_q$, $K_i^{-1}xK_i = q_i^a x$, and $\mathrm{ad}(F_i)^k x = 0$ for sufficiently large k. Then we have the following formula in $U_q[F_i^{-1}]$:*

$$F_i^n x = \sum_{k=0}^{\infty} q_i^{(k+a)(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \mathrm{ad}(F_i)^k(x) F_i^{n-k} \quad \textit{for } n \in \mathbb{Z},$$

where the left-hand side is a finite sum. In particular, if $i \neq j$ in I, then

$$F_i^n F_j = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \mathrm{ad}(F_i)^k(F_j) F_i^{n-k} \quad \textit{for } n \in \mathbb{Z}.$$

Proof. The second formula immediately follows from the first formula and the q-Serre relations. By Lemma 1.9, we can assume that n is negative. By induction on $N \in \mathbb{Z}_{>0}$, we can obtain the following formula:

$$\begin{aligned} F_i^{-1}x = &\sum_{k=0}^{N-1} (-1)^k q_i^{-(k+1)(k+a)} \mathrm{ad}(F_i)^k(x) F_i^{-(k+1)} \\ &+ q_i^{-N(N-1+a)} F_i^{-1} \mathrm{ad}(F_i)^N(x) F_i^{-N}. \end{aligned}$$

Since $\mathrm{ad}(F_i)^k(x) = 0$ for sufficiently large k and $\left[\begin{smallmatrix} -1 \\ k \end{smallmatrix}\right]_{q_i} = (-1)^k$, the first formula for $n = -1$ has been proved. This leads to the first formulae for all negative n by induction on $-n$. Q.E.D.

2.2. Ore domains

An integral domain A is called a *left* (resp. *right*) *Ore domain* if $Aa \cap Ab \neq 0$ (resp. $aA \cap bA \neq 0$) for any non-zero $a, b \in A$. In other words, an integral domain A is a left (resp. right) Ore domain if and only if $A \setminus \{0\}$ is a left (resp. right) Ore subset. A left and right Ore domain is simply called an *Ore domain.*

Assume that A is an Ore domain. Let K be the localization of A at $A \setminus \{0\}$. Then K is a skew field and $K = \{ s^{-1}a \mid a, s \in A,\ s \neq 0 \} = \{ as^{-1} \mid a, s \in A,\ s \neq 0 \}$. We call K the *(skew) field of fractions* of A and denote K by $Q(A)$.

Lemma 2.4 (2.1.15 of [12]). *A left (resp. right) Noetherian domain is a left (resp. right) Ore domain. In particular a left and right Noetherian domain is an Ore domain.*

Example 2.5. The following are left and right Noetherian domains (Chapter 1 of [12]):

(1) the skew polynomial ring $R[x;\sigma,\delta]$ associated to a left and right Noetherian domain R, an algebra automorphism σ of R, and a σ-derivation δ of R ($\delta(ab)=\delta(a)b+\sigma(a)\delta(b)$ for $a,b\in R$), defined to be the ring generated by $a\in R$ and x with defining relations: (a) R is a subring of $R[x;\sigma,\delta]$, (b) $xa=\sigma(a)x+\delta(a)$ for $a\in R$;

(2) the skew Laurent polynomial ring $R[x,x^{-1};\sigma]$ associated to a left and right Noetherian domain R and an algebra automorphism σ of R, defined to be the ring generated by $a\in R$ and $x^{\pm1}$ with defining relations: (a) R is a subring of $R[x,x^{-1};\sigma]$, (b) $xa=\sigma(a)x$ for $a\in R$, (c) $xx^{-1}=x^{-1}x=1$;

(3) the Weyl algebras over a filed $\mathbb{F}$ of characteristic 0 generated by $x_1,\dots,x_n,\partial_1,\dots,\partial_n$ with defining relations: $x_ix_j=x_jx_i$, $\partial_i\partial_j=\partial_j\partial_i$, and $\partial_ix_j-x_j\partial_i=\delta_{ij}$;

(4) the universal enveloping algebra $U(\mathfrak{g})$ of any finite dimensional Lie algebra $\mathfrak{g}$ over a field.

Let $\mathbb{F}$ be a field and $q_{ij}\in\mathbb{F}^\times$ for $i,j=1,\dots,n$. Assume that $q_{ii}=1$ and $q_{ji}=q_{ij}^{-1}$. From the first and second examples above, we obtain, by induction on n, the following examples of left and right Noetherian domains respectively:

(5) the q-polynomial ring over $\mathbb{F}$ defined to be the algebra over $\mathbb{F}$ generated by $x_1,\dots,x_n$ with defining relations $x_jx_i=q_{ij}x_ix_j$ for any i,j;

(6) the q-Laurent polynomial ring over $\mathbb{F}$ defined to be the algebra over $\mathbb{F}$ generated by $x_1^{\pm1},\dots,x_n^{\pm1}$ with defining relations $x_i^{-1}x_i=x_ix_i^{-1}=1$ for any i and $x_jx_i=q_{ij}x_ix_j$ for any i,j.

All of these examples are Ore domains.

In the next subsection, we shall deal with Ore domains which are not always left and right Noetherian.

2.3. Tempered domains

Let A be an associative algebra over a field $\mathbb{F}$ and $\{F_kA\}_{k=0}^\infty$ a family of $\mathbb{F}$-vector subspaces of A. Set $A_k=0$ for $k\in\mathbb{Z}_{<0}$. We say that $\{F_kA\}_{k=0}^\infty$ is a *filtration* of A if $F_0A\subset F_1A\subset F_2A\subset\cdots$, $\bigcup_{k=0}^\infty F_kA=A$, $1\in F_0A$, and $F_kAF_lA\subset F_{k+l}A$ for any k,l. Let $\{F_kA\}_{k=0}^\infty$ be a filtration of A. Set $\mathrm{gr}_k A = F_k A / F_{k-1} A$ and $\mathrm{gr}\, A = \bigoplus_{k=0}^{\infty} \mathrm{gr}_k A$. Then $\mathrm{gr}\, A$ has a natural graded algebra structure. If $\mathrm{gr}\, A$ is an integral domain (resp. left Noetherian, right Noetherian), then A is so.

Definition 2.6 (tempered domain). An associative algebra A over a field $\mathbb{F}$ has a *slowly increasing filtration* if there exists a filtration $\{F_k A\}_{k=0}^{\infty}$ of A such that $\limsup_k (\dim_{\mathbb{F}} F_k A)^{1/k} \leqq 1$. This is equivalent to the condition that $\dim_{\mathbb{F}} \mathrm{gr}_k A < \infty$ for all k and $\limsup_k (\dim_{\mathbb{F}} \mathrm{gr}_k A)^{1/k} \leqq 1$. See Remark 2.8 (1) below. An associative algebra with slowly increasing filtration is called a *tempered algebra* for short. In addition, if A is an integral domain, then A is called a *tempered domain.*

From the definition we can immediately obtain the following result.

Lemma 2.7. *Assume that A is a tempered domain over a field. Then subalgebras of A and quotient integral domains of A are also tempered domains.*

Remark 2.8. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of complex numbers and ρ the convergence radius of the power series $\sum_{k=0}^{\infty} a_k z^k$. The Cauchy–Hadamard theorem says $\limsup_k |a_k|^{1/k} = \rho^{-1}$. The absolute convergence of $\sum_{k=0}^{\infty} a_k$ is equivalent to the condition that, for sufficiently large k_0, the infinite product $\prod_{k=k_0}^{\infty}(1+a_k)$ is absolutely convergent to a non-zero complex number. Therefore the following conditions are mutually equivalent:

(a) $\limsup_k |a_k|^{1/k} \leqq 1$.
(b) There exists a holomorphic function in $|z| < 1$ such that its Maclaurin expansion is equal to $\sum_{k=0}^{\infty} a_k z^k$.
(c) For sufficiently large k_0, the infinite product $\prod_{k=k_0}^{\infty}(1 + a_k z^k)$ is absolutely and uniformly convergent in wide sense to a non-vanishing holomorphic function in $|z| < 1$.

These observations lead to the following results:

(1) $\limsup_k |a_k|^{1/k} \leqq 1$ implies $\limsup_k |a_0 + a_1 + \cdots + a_k|^{1/k} \leqq 1$.
(2) Fix $N \in \mathbb{Z}$. Then $\limsup_k |a_k|^{1/k} \leqq 1$ is equivalent to $\limsup_k |a_k|^{1/(k+N)} \leqq 1$.
(3) Assume that $|a_k| \leqq 1$ for all k. For any positive integer N, define the sequence $\{b_k^{(N)}\}_{k=0}^{\infty}$ of complex numbers by the Maclaurin expansion $\left(\prod_{k=0}^{\infty}(1 - a_k z^k)\right)^{-N} = \sum_{k=0}^{\infty} b_k^{(N)} z^k$ in $|z| < 1$. Then $\limsup_k |b_k|^{1/k} \leqq 1$.
(4) Set $c_k = \sum_{i=0}^{k} a_i b_{k-i}$. Then $\limsup_k |a_k|^{1/k} \leqq 1$ and $\limsup_k |b_k|^{1/k} \leqq 1$ implies $\limsup_k |c_k|^{1/k} \leqq 1$.

Rocha–Caridi and Wallach found the following criterion of Ore domains (Lemma 1.2 of [23]) to prove that the universal enveloping algebras of affine (or Euclidean) Lie algebras are Ore domains.

Lemma 2.9 ([23]). *A tempered domain over a field is always an Ore domain.*

Proof. Let A be an integral domain over a field $\mathbb{F}$ and $\{F_k A\}_{k=0}^{\infty}$ an arbitrary increasing filtration of A. Assume that A is not an Ore domain. Then there exist $r \in \mathbb{Z}_{\geqq 0}$ and non-zero $a, b \in F_r A$ such that $Aa \cap Ab = 0$ or $aA \cap bA = 0$. Assume that $Aa \cap Ab = 0$. Set $a_k = \dim_{\mathbb{F}} F_k A$ and take $k_0 \in \mathbb{Z}_{\geqq 0}$ so that $a_{k_0} \geqq 1$. Since $(F_k A)a + (F_k A)b \subset F_{k+r} A$ and $(F_k A)a \cap (F_k A)b = 0$, we have $a_{k+r} = \dim_{\mathbb{F}} F_{k+r} A \geqq \dim((F_k A)a \oplus (F_k A)b) = 2a_k$ and hence $a_{k_0+pr} \geqq 2^p a_{k_0} \geqq 2^p$, i.e. $a_{k_0+pr}^{1/(pr)} \geqq 2^{1/r} > 1$ for all $p \in \mathbb{Z}_{\geqq 0}$. This leads to $\limsup_k a_k^{1/(k-k_0)} > 1$ and hence Remark 2.8 (2) shows $\limsup_k a_k^{1/k} > 1$. Therefore A is not tempered. When $aA \cap bA = 0$, similarly A is not. We complete the proof of the lemma. Q.E.D.

Example 2.10. *Let A be a q-Laurent polynomial ring over a field. (See Example 2.5.) Then A is a tempered domain. Therefore its subalgebras and quotient integral domains are also tempered domains. Furthermore all of these are Ore domains.*

Remark 2.8 (4) immediately lead to the following result.

Lemma 2.11. *For any tempered algebras A and B over a field $\mathbb{F}$, the tensor product algebra $A \otimes B$ over $\mathbb{F}$ is also tempered. Therefore, if A is a tempered domain and B is a q-polynomial or q-Laurent polynomial ring, then $A \otimes B$ is also a tempered domain.*

Theorem 2.12. *The following algebras are tempered domains:*

(1) *the universal enveloping algebras $U(\mathfrak{n}_\pm)$ of the upper and lower parts $\mathfrak{n}_\pm$ of a Kac–Moody algebra of finite or affine type (Theorem 1.10 of* [23]*),*

(2) *the upper and lower parts $U_q^\pm$ of a quantized universal enveloping algebra of finite or affine type.*

Therefore these are Ore domains.

Proof. Denote $U(\mathfrak{n}_\pm)$ or $U_q^\pm$ by A. Let us define a filtration of A by using principal gradations. Set $F_k A = \bigoplus_{i=0}^{k} U(\mathfrak{n}_\pm)_{\pm i}$ if $A = U(\mathfrak{n}_\pm)$ and $F_k A = \bigoplus_{i=0}^{k} (U_q^\pm)_{\pm i}$ if $A = U_q^\pm$. Then $\{F_k A\}_{k=0}^{\infty}$ is a filtration of A. Because of $U(\mathfrak{n}_-)$ and U_q^- are of finite or affine type, from Lemma 1.2 and Lemma 1.5 together with Remark 2.8 (3), we obtain that $\{F_k A\}_{k=0}^{\infty}$

is slowly increasing. This means that A is a tempered domain. Therefore Lemma 2.9 completes the proof. Q.E.D.

In [1], Berman and Cox showed that the universal enveloping algebras of Kac–Moody Lie algebras of affine type, as well as those of toroidal Lie algebras, are tempered domains. Using an associative algebra version of Lemma 1.4 (a) in [1], we shall show that quantized universal enveloping algebras of affine type are also tempered domains.

Lemma 2.13. *Let A be a domain over a field and $A^{\pm}$ and A^0 its subalgebras. Assume that $A^{\pm}$ and A^0 have a slowly increasing filtration denoted by $\{F_i A^{\pm}\}_{i=0}^{\infty}$ and $\{F_i A^0\}_{i=0}^{\infty}$ respectively. Assume that these satisfy the following conditions:*

(a) *The multiplication gives an isomorphism $A^- \otimes A^0 \otimes A^+ \xrightarrow{\sim} A$ of vector spaces.*

(b) *$F_j A^0 F_l A^- = F_l A^- F_j A^0$ and $F_k A^+ F_m A^0 = F_m A^0 F_k A^+$ for any j, k, l, m.*

(c) *$F_k A^+ F_l A^- \subset \sum_{p=0}^{\max\{k,l\}} F_{l-p} A^- F_p A^0 F_{k^p} A^+$ for any k, l.*

Then A is also a tempered domain and hence an Ore domain.

Proof. Using the above conditions, we can define a filtration $\{F_l A\}_{l=0}^{\infty}$ of A by $F_l A = \sum_{i+j+k=l} F_i A^- F_j A^0 F_k A^+$. From Remark 2.8 (4) we obtain that $\{F_l A\}_{l=0}^{\infty}$ is slowly increasing and hence A is a tempered domain. Therefore Lemma 2.9 completes the proof. Q.E.D.

Theorem 2.14. *The following algebras are tempered domains:*

(1) *the universal enveloping algebra $U(\mathfrak{g})$ of a Kac–Moody algebra of finite or affine type (part of Proposition 1.7 of* [1]*),*

(2) *a quantized universal enveloping algebra U_q of finite or affine type.*

Therefore these are Ore domains.

Proof. Since $U(\mathfrak{g})$ and U_q are integral domains, it is sufficient for the proof to construct slowly increasing filtrations of $U(\mathfrak{g})$ and U_q.

First we assume that $A = U(\mathfrak{g})$, a Kac–Moody algebra of finite or affine type. Set $A^{\pm} = U(\mathfrak{n}_{\pm})$ and $A^0 = U(\mathfrak{h})$. Using the principal gradations of $U(\mathfrak{n}_{\pm})$, we can define the increasing filtrations $\{F_k A^{\pm}\}_{k=0}^{\infty}$ of $A^{\pm}$ by $F_k A^{\pm} = \bigoplus_{i=0}^{k} U(\mathfrak{n}_{\pm})_{\pm k}$. Then we can find from the proof of Theorem 2.12 that $\{F_k A^{\pm}\}_{k=0}^{\infty}$ are slowly increasing. Let $\{y_1, \dots, y_M\}$ a basis of $\mathfrak{h}$. Define the degree by $\deg y_i = 1$ for $i = 1, \dots, M$. Let $F_k A^0$ be the subspace of $A^0 = U(\mathfrak{h}) = \mathbb{C}[y_1, \dots, y_M]$ spanned by the elements of degree $\leqq k$. Then $\{F_k A^0\}_{k=0}^{\infty}$ is a slowly increasing filtration of A^0. These satisfies the conditions (a), (b), and (c) of Lemma 2.13. Therefore $A = U(\mathfrak{g})$ is a tempered domain.

Second we assume that $A = U_q$, the quantized universal enveloping algebra associated to a root datum $(Y, Z, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ of finite or affine type. Using the principal gradations of $U_q^\pm$, we can define the filtrations $\{F_kA^\pm\}_{k=0}^\infty$ of $A^\pm$ by $F_kA^\pm = \bigoplus_{i=0}^k (U_q^\pm)_{\pm k}$. Then we can find from the proof of Theorem 2.12 that $\{F_kA^\pm\}_{k=0}^\infty$ are slowly increasing. Let $\{y_1, \dots, y_M\}$ be a $\mathbb{Z}$-free basis of $d^{-1}Y$. Define the degree by $\deg q^{\pm y_i} = 1$ for $i = 1, \dots, M$. Let F_kA^0 be the subspace of $A^0 = U_q^0 = \mathbb{F}[q^{\pm y_1} \dots, q^{\pm y_M}]$ consisting of the elements of degree $\leqq k$. Then $\{F_kA^0\}_{k=0}^\infty$ is a slowly increasing filtration of A^0. These satisfies the conditions (a), (b), and (c) of Lemma 2.13. Therefore $A = U_q$ is a tempered domain. Q.E.D.

From Lemma 2.7 and the above theorem, we immediately obtain the following.

Corollary 2.15. *Let A be the universal enveloping algebra $U(\mathfrak{g})$ of a Kac–Moody algebra of finite or affine type or a quantized universal enveloping algebra U_q of finite or affine type. Assume that B is a subalgebra of A and I is a completely prime ideal of B. Then B/I is a tempered domain and hence an Ore domain.*

2.4. Truncated q-Serre relations

In this subsection, we shall explain a method for constructing quotient tempered domains of $U(\mathfrak{n}_-)$ and U_q^- for any symmetrizable GCM.

First let us consider the case of $q = 1$.

Let $A = \{a_{ij}\}_{i,j\in I}$ be a symmetrizable GCM symmetrized by $\{d_i\}_{i\in I}$, $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ a root datum of type A, and $\mathfrak{g}$ the Kac–Moody algebra associated to the root datum. Denote by $\mathfrak{n}_-$ (resp. $\mathfrak{b}_-$) the lower part (resp. the lower Borel subalgebra) of $\mathfrak{g}$. Let $\{y_1, \dots, y_M\}$ be a basis of $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Y$. Assume that if $i \neq j$ and $a_{ij} \neq 0$, then $\epsilon_{ij} = \pm 1$ and $\epsilon_{ji} = -\epsilon_{ij}$, otherwise $\epsilon_{ij} = 0$.

Define the algebra $\mathcal{B}$ to be the associative algebra over $\mathbb{C}$ generated by f_i ($i \in I$) and $h \in \mathfrak{h}$ with defining relations:

$$\begin{aligned} &U(\mathfrak{h}) = S(\mathfrak{h}) \text{ is a subalgebra of } \mathcal{B}; \\ &[h, f_i] = -\langle h, \alpha_i\rangle f_i \quad \text{for } i \in I,\ h \in \mathfrak{h}; \\ &[f_i, f_j] = -\epsilon_{ij} d_i a_{ij} \quad \text{for } i, j \in I. \end{aligned}$$

The last relations are sufficient conditions of Serre relations for $\{f_i\}_{i\in I}$ and called *truncated Serre relations*. Sending F_i to f_i for each $i \in I$, we can regard $\mathcal{B}$ as a quotient algebra of $U(\mathfrak{b}_-)$.

Define the degree by $\deg f_i = \deg h = 1$ for $i \in I$ and $h \in \mathfrak{h}$. Let $F_k\mathcal{B}$ be the subspace of $\mathcal{B}$ spanned by the elements of degree $\leqq$

k. Then $\{F_k\mathcal{B}\}_{k=0}^{\infty}$ is a slowly increasing filtration of $\mathcal{B}$ and $\operatorname{gr}\mathcal{B} = \bigoplus_{k=0}^{\infty} F_k\mathcal{B}/F_{k-1}\mathcal{B}$ is isomorphic to the commutative polynomial ring generated by f_i ($i \in I$) and y_μ ($\mu = 1, \dots, M$). Therefore $\mathcal{B}$ is a tempered domain. By Lemma 2.11, $\mathcal{B}^{\otimes N}$ is also a tempered domain for any positive integer N.

We can define the algebra homomorphism $\phi_N : U(\mathfrak{n}_-) \to \mathcal{B}^{\otimes N}$ by $\phi_N(F_i) = \sum_{\nu=1}^{N} f_{i\nu}$, where $f_{i\nu} = 1^{\otimes(\nu-1)} \otimes F_i \otimes 1^{\otimes(N-\nu)}$. Denote the image of ϕ_N by $\mathcal{N}_N$. Then $\mathcal{N}_N$ is also a tempered domain and hence an Ore domain. Denote $\mathcal{N}_1$ by $\mathcal{N}$ for short.

Second let us consider a q-analogue of the above construction.

Let d be the least common denominator of $\{d_i\}_{i\in I}$. Set the base field $\mathbb{F}$ by $\mathbb{F} = \mathbb{Q}(q^{1/d})$ and $q_i \in \mathbb{F}$ by $q_i = q^{d_i}$. Let U_q be the quantized universal enveloping algebra associated to the root datum. Denote by U_q^- (resp. $U_q(\mathfrak{b}_-)$) the lower part (resp. the lower Borel subalgebra) of U_q. Let $\{y_1, \dots, y_M\}$ be a $\mathbb{Z}$-free basis of $d^{-1}Y$.

Define the algebra $\mathcal{B}_q$ to be the associative algebra over $\mathbb{F}$ generated by f_i ($i \in I$) and q^λ ($\lambda \in d^{-1}Y$) with defining relations:

$$
\begin{aligned}
&U_q^0 \text{ is a subalgebra of } \mathcal{B}_q;\\
&q^\lambda f_i q^{-\lambda} = q^{-\langle\lambda,\alpha_i\rangle} f_i \quad \text{for } i \in I,\ \lambda \in d^{-1}Y;\\
&f_i f_j = q_i^{-\epsilon_{ij} a_{ij}} f_j f_i \quad \text{for } i, j \in I.
\end{aligned}
$$

The last relations are sufficient conditions of q-Serre relations for $\{f_i\}_{i\in I}$ and called *truncated q-Serre relations.* Sending F_i to f_i for each $i \in I$, we can regard $\mathcal{B}_q$ as a quotient algebra of $U_q(\mathfrak{b}_-)$. Denote the image of $K_i = q_i^{\alpha_i^\vee}$ in $\mathcal{B}_q$ by k_i.

Then $\mathcal{B}_q$ is a subalgebra of a q-Laurent polynomial ring over $\mathbb{F}$ generated by f_i ($i \in I$) and q^{y_μ} ($\mu = 1, \dots, M$) and hence a tempered domain. By Lemma 2.11, $\mathcal{B}_q^{\otimes N}$ is also a tempered domain for any positive integer N.

We can define the algebra homomorphism $\Delta_N : U_q \to U_q^{\otimes N}$ by $\Delta_1 = \mathrm{id}_{U_q} = 1$, $\Delta_\nu = (\Delta_{\nu-1} \otimes 1) \circ \Delta$ for $\nu = 2, \dots, N$. Then we have $\Delta_N(F_i) = \sum_{\nu=1} F_{i\nu}$, where $F_{i\nu} = (K_i^{-1})^{\otimes(\nu-1)} \otimes F_i \otimes 1^{\otimes(N-\nu)}$. Therefore we can define the algebra homomorphism $\phi_{q,N} : U_q^- \to \mathcal{B}_q^{\otimes N}$ by $\phi_{q,N}(F_i) = \sum_{\nu=1}^{N} f_{i\nu}$, where $f_{i\nu} = (k_i^{-1})^{\otimes(\nu-1)} \otimes f_i \otimes 1^{\otimes(N-\nu)}$. Denote the image of $\phi_{q,N}$ by $\mathcal{N}_{q,N}$. Then $\mathcal{N}_{q,N}$ is also a tempered domain and hence an Ore domain. Denote $\mathcal{N}_{q,1}$ by $\mathcal{N}_q$ for short.

§3. Non-integral powers

3.1. Evaluation mapping between fields of fractions

Let $\mathbb{F}$ be any base field. Let A be a tempered domain over $\mathbb{F}$. Set $\mathcal{A} = A \otimes \mathbb{F}[x_1, \ldots, x_M] = A[x_1, \ldots, x_M]$. Then $\mathcal{A}$ is also a tempered domain over $\mathbb{F}$. Denote by K the field of fractions of A and by $\mathcal{K}$ that of $\mathcal{A}$. Using the universality of K, we can regard K as a subfield of $\mathcal{K}$. $\mathcal{A}' = A[x_1^{\pm 1}, \ldots, x_M^{\pm 1}]$ is also a tempered domain. We can identifies the field of fractions of $\mathcal{A}'$ with $\mathcal{K}$.

For $c = (c_1, \ldots, c_M) \in \mathbb{F}^M$, we define the evaluation algebra homomorphism $\mathrm{ev}_c : \mathcal{A} \to A$ at c by $\mathrm{ev}_c(f) = f(c) = f(c_1, \ldots, c_M) \in A$ for $f \in \mathcal{A} = A[x_1, \ldots, x_M]$.

Any element f of $\mathcal{K}$ can be represented as $f = g^{-1}h$ for some $g, h \in \mathcal{A}$ with $g \neq 0$. Fix $c \in \mathbb{F}^M$. Assume that $g, h, g', h' \in \mathcal{A}$, $g(c), g'(c) \neq 0$, and $g^{-1}h = g'^{-1}h'$. Since $\mathcal{A}$ is an Ore domain, there exist non-zero $u, u' \in \mathcal{A}$ such that $ug = u'g'$. Then $uh = ugg^{-1}h = ug'g'^{-1}h' = u'h'$. We have $u(c)g(c) = u'(c)g'(u)$ and $u(c)h(c) = u'(c)h'(c)$. Therefore $g(c)^{-1}h(c) = (u(c)g(c))^{-1}u(c)h(c) = (u'(c)g'(c))^{-1}u'(c)h'(c) = g'(c)^{-1}h'(c)$ in K. This means that if $f \in \mathcal{K}$ can be represented as $f = g^{-1}h$ for some $g, h \in \mathcal{A}$ with $g(c) \neq 0$, then $\mathrm{ev}_c(f) = f(c) = g(c)^{-1}h(c) \in K$ is well-defined and does not depend on the choice of g and h.

Let C be a subset of $\mathbb{F}^M$ with the following *Zariski dense property*:

(D) For every $a \in \mathbb{F}[x_1, \ldots, x_M]$, if $a(c) = 0$ for all $c \in C$, then $a = 0$ in $\mathbb{F}[x_1, \ldots, x_M]$.

For example, for any infinite subset C_1 of $\mathbb{F}$, the direct product $C_1^M \subset \mathbb{F}^M$ has the property (D). For every $f \in \mathcal{A} = A \otimes \mathbb{F}[x_1, \ldots, x_M]$, if $\mathrm{ev}_c(f) = f(c) = 0$ for all $c \in C$, then $f = 0$ in $\mathcal{A}$. Immediately we obtain the following result.

Lemma 3.1. *Let $g, h \in \mathcal{A}$, $g \neq 0$, and $f = g^{-1}h$. Assume that there exit a subset C of $\mathbb{F}^M$ with the property (D) such that $g(c) \neq 0$ for all $c \in C$. If $\mathrm{ev}_c(f) = f(c) = 0$ for all $c \in C$, then $f = 0$ in $\mathcal{K}$.*

Let C be a subset of $\mathbb{F}^M$ with the property (D). Then, for any non-zero $b \in \mathbb{F}[x_1, \ldots, x_M]$, the subset $C_{b\neq 0} = \{\, c \in C \mid b(c) \neq 0 \,\}$ of C also has the property (D). In fact, for every $a \in \mathbb{F}[x_1, \ldots, x_M]$, if $a(c) = 0$ for all $c \in C_{b\neq 0}$, then $a(c)b(c) = 0$ for all $c \in C$. Therefore $ab = 0$ and hence $a = 0$ in $\mathbb{F}[x_1, \ldots, x_M]$. It follows that, for any non-zero $g_1, \ldots, g_N \in \mathcal{A}$, the subset $C_{g_1,\ldots,g_N\neq 0} = \{\, c \in C \mid g_i(c) \neq 0 \text{ for all } i = 1, \ldots, N \,\}$ of C also has the property (D). From this we can obtain the following result.

Lemma 3.2. *Let C be a subset of $\mathbb{F}^M$ with the property (D). Take any $f, f' \in \mathcal{K}$. By the definition of the field of fractions $\mathcal{K}$, there exist*

$g, h, g', h', g'', h'', u, u' \in \mathcal{A}$ such that $g, g', g'', u \neq 0$, $f = g^{-1}h$, $f' = g'^{-1}h'$, $ff' = (g''g)^{-1}h''h'$, and $f + f' = (ug)^{-1}(uh + u'h')$. Then the subset $C' = C_{g,g',g'',u\neq 0}$ of C satisfies the property (D) and that $\mathrm{ev}_c(f)\,\mathrm{ev}_c(f') = \mathrm{ev}_c(ff')$ and $\mathrm{ev}_c(f) + \mathrm{ev}_c(f') = \mathrm{ev}_c(f + f')$ for all $c \in C'$.

We shall use these results to justify the conjugation actions of non-integral powers in Section 3.2.

3.2. Non-integral powers in fields of fractions

In this subsection, we shall justify the conjugation actions of non-integral powers along the lines of the work [6] by Iohara and Malikov.

Let $\mathbb{F}$ be any base field. Let A be a tempered domain over $\mathbb{F}$ with generators f_i ($i \in I$) and defining relations $R_\lambda(\{f_i\}_{i\in I}) = 0$ ($\lambda \in \Lambda$), where R_λ ($\lambda \in \Lambda$) are elements of the tensor algebra $T(V)$ of $V = \bigoplus_{i\in I} \mathbb{F} f_i$. That is, A is the quotient algebra of $T(V)$ modulo the two-sided ideal generated by $\{R_\lambda\}_{\lambda\in\Lambda}$. The polynomial ring $A[x] = A \otimes \mathbb{F}[x]$ of one variable over $\mathcal{A}$ is also a tempered domain over $\mathbb{F}$. We can regard the field of fractions $Q(A)$ as a subfield of $Q(A[x])$.

Assume that a non-zero element g in A, a countable family $\{c_n\}_{n=0}^{\infty}$ of mutually distinct elements in $\mathbb{F}$, and an infinite subset Γ of $\mathbb{Z}_{\geqq 0}$ satisfy the following condition:

($*$) For any $i \in I$, there exists $\phi_i \in Q(A[x])$ such that, for all $n \in \Gamma$, $\mathrm{ev}_{c_n}(\phi_i) = \phi_i(c_n) \in Q(A)$ is well-defined and $g^n f_i g^{-n} = \phi_i(c_n)$.

For any $\lambda \in \Lambda$ and $k \in \Gamma$, we have $R_\lambda(\{\phi_i(c_n)\}_{i\in I}) = R_\lambda(\{g^n f_i g^{-n}\}_{i\in I}) = g^k R_\lambda(\{f_i\}_{i\in I}) g^{-k} = 0$ in $Q(A)$. By Lemma 3.1, $R_\lambda(\{\phi_i(x_1)\}_{i\in I}) = 0$ in $Q(A[x])$. Therefore we can define the algebra homomorphism $\gamma_{g,x} : A[x] \to Q(A[x])$ by $\gamma_{g,x}(f_i) = \phi_i(x)$ for $i \in I$ and $\gamma_{g,x}(x) = x$. Using the universality of the field of fractions $Q(A[x])$, we can extend $\gamma_{g,x}$ to the algebra automorphism of $Q(A[x])$. We call $\gamma_{g,x}$ the *conjugation action of non-integral power* of g on $Q(A[x])$.

Assume that $\mathbb{F}[x]$ is identified with a subalgebra of the polynomial algebra $\mathbb{F}[x_1^{\pm 1}, \dots, x_M^{\pm 1}]$ of M-variables over $\mathbb{F}$. Then we can also define the algebra automorphism $\gamma_{g,x}$ of $Q(A[x_1, \dots, x_M])$ by $\gamma_{g,x}(f_i) = \phi_i(x)$ for $i \in I$ and $\gamma_{g,x}(x_\mu) = x_\mu$ for $\mu = 1, \dots, M$.

If $c_n = n$, then $\gamma_{g,x}$ is denoted by $\gamma(g^x)$. If $c_n = q^n$ and x is identified with q^λ, then $\gamma_{g,x}$ is denoted by $\gamma(g^\lambda)$.

§4. Quantized q-analogues of birational Weyl group actions

In this section, we shall construct quantized q-analogues of the birational Weyl group actions arising from nilpotent Poisson algebras proposed by Noumi and Yamada [17].

Let $A = [a_{ij}]$ be a symmetrizable GCM symmetrized by $\{d_i\}_{i\in I}$. Let $\mathcal{A}_{q,0}$ be a quotient tempered domain of the lower part U_q^- of a universal enveloping algebra U_q of type A. Then $\mathcal{A}_{q,0}$ is generated by the images $\{f_i\}_{i\in I}$ of the lower Chevalley generators $\{F_i\}_{i\in I}$. If A is of finite or affine type, then any quotient integral domain of U_a^- is a tempered domain. See Corollary 2.15. In order to construct the examples for arbitrary cases, see Section 2.4.

4.1. Non-integral power of f_i

Fix $i \in I$ and assume that $f_i \neq 0$. Lemma 2.3 leads to the following formulae:

$$f_i^n f_j f_i^{-n} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \operatorname{ad}(f_i)^k(f_j) f_i^{-k} \quad \text{for } n \in \mathbb{Z} \text{ if } i \neq j,$$

where $\operatorname{ad}(f_i)^k(f_j)$ denotes the image of $\operatorname{ad}(F_i)^k(F_j) \in U_q^-$ in $\mathcal{A}_{q,0}$:

$$\operatorname{ad}(f_i)^k(f_j) = \sum_{\nu=0}^{k} (-1)^\nu q_i^{\nu(k-1+a_{ij})} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q_i} f_i^{k-\nu} f_j f_i^\nu.$$

For $j \in I$, define $\phi_{ij}(x) \in Q(A[x])$ by

$$\phi_{ij}(x) = \begin{cases} \displaystyle\sum_{k=0}^{-a_{ij}} q_i^{-(k+a_{ij})k} x^{k+a_{ij}} a_{ij;k}(x) \operatorname{ad}(f_i)^k(f_j) f_i^{-k} & \text{if } i \neq j, \\ f_i & \text{if } i = j, \end{cases}$$

where $a_{ij;k}(x) \in \mathbb{F}[x, x^{-1}]$ for $k \in \mathbb{Z}_{\geqq 0}$ are given by

$$a_{ij;k}(x) = \frac{[x;0]_{q_i}[x;-1]_{q_i} \cdots [x;-k+1]_{q_i}}{[k]_{q_i}!}, \quad [x;\nu]_{q_i} = \frac{x q_i^\nu - x^{-1} q_i^{-\nu}}{q_i - q_i^{-1}}.$$

Then there exist $g_{ij} \in \mathcal{A}_{q,0}$ and $h_{ij}(x) \in \mathcal{A}_{q,0}[x, x^{-1}]$ such that $\phi_{ij}(x) = g_{ij}^{-1} h_{ij}(x)$. Therefore $\phi_{ij}(q^n)$ is well-defined and $f_i^n f_j f_i^{-n} = \phi_{ij}(q_i^n)$ for all $j \in I$ and $n \in \mathbb{Z}$.

Let $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$ be a root datum of type A and $\{y_1, \dots, y_M\}$ a $\mathbb{Z}$-free basis of $d^{-1}Y$. Let $\mathcal{A}_q$ be the tensor product algebra $\mathcal{A}_{q,0} \otimes U_q^0 = \mathcal{A}_{q,0}[q^{\pm y_1}, \dots, q^{\pm y_M}]$. Note that q^λ ($\lambda \in d^{-1}Y$)

commute f_j $(j \in I)$ in $\mathcal{A}_q$. Take any $\lambda \in Y$. Identifying x with $q_i^\lambda = q^{d_i\lambda}$, we regard $\mathbb{F}[x]$ as a subalgebra of U_q^0. Using the result of Section 3.2, we can define the algebra automorphism $\gamma(f_i^\lambda)$ of $Q(\mathcal{A}_q)$ by $\gamma(f_i^\lambda)(f_j) = \phi_{ij}(q_i^\lambda)$ for $j \in I$ and $\gamma(f_i^\lambda)(q^\mu) = q^\mu$ for $\mu \in d^{-1}Y$. More explicitly we have

$$\gamma(f_i^\lambda)(f_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\lambda-k)} \begin{bmatrix} \lambda \\ k \end{bmatrix}_{q_i} \mathrm{ad}(f_i)^k(f_j)f_i^{-k} & \text{if } i \neq j, \\ f_i & \text{if } i = j. \end{cases} \tag{4.1}$$

Note that the right-hand side is a Laurent polynomial in q_i^λ.

For the $q = 1$ cases, we have the same construction as the above. Let $\mathcal{A}_0$ be a quotient tempered domain of the universal enveloping algebra of the lower part $\mathfrak{n}_-$ of a Kac–Moody algebra $\mathfrak{g}$ of type A. Then $\mathcal{A}_0$ is generated by the images $\{f_i\}_{i\in I}$ of the lower Chevalley generators $\{F_i\}_{i\in I}$. Let $\mathcal{A}$ be the tensor product algebra $\mathcal{A}_0 \otimes U(\mathfrak{h}) = \mathcal{A}_0[y_1, \dots, y_\mu]$. Then we can define the algebra automorphism $\gamma(f_i^\lambda)$ of $Q(\mathcal{A})$ by $\gamma(f_i^\lambda)(h) = h$ for $h \in \mathfrak{h}$ and the $q \to 1$ limit of (4.1):

$$\gamma(f_i^\lambda)(f_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} \binom{\lambda}{k} \mathrm{ad}(f_i)^k(f_j)f_i^{-k} & \text{if } i \neq j, \\ f_i & \text{if } i = j, \end{cases} \tag{4.2}$$

where $\mathrm{ad}(X)(Y) = [X, Y]$ and $\binom{\lambda}{k} = \lambda(\lambda - 1)\cdots(\lambda - k + 1)/k!$. The left-hand side of (4.2) is a polynomial in λ.

Remark 4.1. Formula (4.2) (resp. (4.1)) can be regarded as a quantization (resp. quantized q-analogue) of Formula (1.9) in [17] proposed by Noumi and Yamada.

Simply Formula (4.2) is the $q \to 1$ limit of Formula (4.1). Note that the $q \to 1$ limit is not a classical limit because (4.2) is a formula in a non-commutative algebra.

Let us explain how to obtain Formula (1.9) in [17] as the classical limit of Formula (4.2). We replace f_i by $\hbar^{-1}\varphi_i$ and λ by $\hbar^{-1}\lambda_i$ and define $\mathrm{ad}_\hbar$ by $\mathrm{ad}_\hbar(X)(Y) = \hbar^{-1}[X, Y]$, where $\hbar$ denotes the Planck constant. Assume that $i \neq j$. Then Formula (4.2) is equivalent to

(4.3)

$$\gamma(\varphi_i^{\hbar^{-1}\lambda_i})(\varphi_j) = \sum_{k=0}^{-a_{ij}} \frac{\lambda_i(\lambda_i - \hbar)\cdots(\lambda_i - (k-1)\hbar)}{k!} \mathrm{ad}_\hbar(\varphi_i)^k(\varphi_j)\varphi_i^{-k}.$$

The classical limit of $\hbar^{-1}[X,Y]$ should be the Poisson bracket $\{X,Y\}$. Thus, as the classical limit of (4.3), we can obtain Formula (1.9) in [17]:

$$s_i(\varphi_j) = \sum_{k=0}^{-a_{ij}} \frac{1}{k!}\left(\frac{\lambda_i}{\varphi_i}\right)^k \operatorname{ad}_{\{\}}(\varphi_i)^k(\varphi_j), \quad \operatorname{ad}_{\{\}}(X)(Y) = \{X,Y\}.$$

4.2. Quantization of birational Weyl group actions

In the previous subsection, we have constructed the conjugation action $\gamma(f_i^\lambda)$ of a non-integral powers f_i^λ on the field of fractions $Q(\mathcal{A}_q)$, where $i \in I$, $\lambda \in Y$, and $\mathcal{A}_q$ is the tensor product algebra of a quotient tempered domain $\mathcal{A}_{q,0}$ of U_q^- and the Cartan subalgebra U_q^0 of U_q. We denote by f_i the image of F_i in $\mathcal{A}_{q,0}$. We identify $\mathcal{A}_q$ with the Laurent polynomial ring $\mathcal{A}_{q,0}[q^{\pm y_1},\dots,q^{\pm y_M}]$, where $\{y_1,\dots,y_M\}$ is a $\mathbb{Z}$-free basis of $d^{-1}Y$. Note that q^λ ($\lambda \in d^{-1}Y$) commute f_i in $\mathcal{A}_q$.

The Weyl group $W = \langle s_i | i \in I\rangle$ acts on Y. (See Section 1.2.) This action naturally extends to those on $d^{-1}Y$ and $U_q^0 = \bigoplus_{\lambda\in d^{-1}Y} \mathbb{F}q^\lambda$. In this subsection, we denote by $\tilde{w}$ the action of $w \in W$ on U_q^0 regarded as a subalgebra of $\mathcal{A}_q$:

$$\tilde{s}_i(q^\lambda) = q^{s_i(\lambda)}, \quad s_i(\lambda) = \lambda - \langle\lambda,\alpha_i\rangle\alpha_i^\vee \quad \text{for } i \in I,\ \lambda \in d^{-1}Y.$$

The action $\tilde{w}$ of $w \in W$ on U_q^0 is extended to the action on $\mathcal{A}_q$ by $\tilde{w}(f_i) = f_i$ for $i \in I$. The induced action of $\tilde{w}$ on $Q(\mathcal{A}_q)$ is also denoted by $\tilde{w}$.

Lemma 4.2. *For any $i,j \in I$ and $\lambda \in Y$, $\gamma(f_j^\lambda)\circ\tilde{s}_i = \tilde{s}_i\circ\gamma(f_j^{s_i(\lambda)})$ on $Q(\mathcal{A}_q)$.*

Proof. Take any $k \in I$ and $\mu \in d^{-1}Y$. Then we have

$$\begin{aligned}
&\gamma(f_j^\lambda)\circ\tilde{s}_i(f_k) = \gamma(f_j^\lambda)(f_k) = \varphi_{jk}(q_j^\lambda),\\
&\tilde{s}_i\circ\gamma(f_j^{s_i(\lambda)})(f_k) = \tilde{s}_i(\varphi_{jk}(q_j^{s_i(\lambda)})) = \varphi_{jk}(q_j^{s_i^2(\lambda)}) = \varphi_{jk}(q_j^\lambda),\\
&\gamma(f_j^\lambda)\circ\tilde{s}_i(q^\mu) = \gamma(f_j^\lambda)(q^{s_i(\mu)}) = q^{s_i(\mu)},\\
&\tilde{s}_i\circ\gamma(f_j^{s_i(\lambda)})(q^\mu) = \tilde{s}_i(q^\mu) = q^{s_i(\mu)}.
\end{aligned}$$

This proves the above lemma. Q.E.D.

Theorem 4.3 (quantized birational Weyl group action). *Assume that $f_i \neq 0$ for all $i \in I$. For each $i \in I$ we define the algebra automorphism S_i of $Q(\mathcal{A}_q)$ by $S_i = \tilde{s}_i\circ\gamma(f_i^{-\alpha_i^\vee}) = \gamma(f_i^{\alpha_i^\vee})\circ\tilde{s}_i$. Then the action of the Weyl group W on $Q(\mathcal{A}_q)$ is defined by $s_i(x) = S_i(x)$ for $i \in I$ and*

$x \in Q(\mathcal{A}_q)$. Explicitly, the following formulae define a representation of the Weyl group in algebra automorphisms of $Q(\mathcal{A}_q)$:

$$s_i(f_j) = \begin{cases} \displaystyle\sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} \mathrm{ad}(f_i)^k(f_j) f_i^{-k} & \text{if } i \neq j, \\ f_i & \text{if } i = j, \end{cases}$$

$$s_i(q^\lambda) = q^{s_i(\lambda)} = q^{\lambda - \langle \lambda, \alpha_i \rangle \alpha_i^\vee} \quad \lambda \in d^{-1}Y.$$

Proof. It is sufficient to show the braid relations of $\{S_i\}_{i \in I}$ and $S_i^2 = 1$ for $i \in I$.

First let us prove $S_i^2 = 1$. It is sufficient to show that $S_i^2(f_j) = f_j$. Since $S_i^2(f_i) = f_i$ is trivial, we can assume $i \neq j$. Using Lemma 4.2 we have $S_i^2 = \gamma(f_i^{\alpha_i^\vee}) \circ \gamma(f_i^{-\alpha_i^\vee})$. $S_i^2(f_j)$ is a Laurent polynomial $\Phi(q_i^{\alpha_i^\vee})$ of $q_i^{\alpha_i^\vee}$ with coefficients in $Q(\mathcal{A}_q)$. Then we have $\Phi(q_i^n) = f_i^n(f_i^{-n} f_j f^n) f^{-n} = f_j$ for all $n \in \mathbb{Z}$. Therefore we obtain $S_i^2(f_j) = \Phi(q_i^{\alpha_i^\vee}) = f_j$.

Second let us prove the braid relations for $\{S_i\}_{i \in I}$. Assume that $i \neq j$ and $(a_{ij}, a_{ji}) = (0,0), (-1,-1), (-1,-2)$, or $(-1,-3)$. We define the sequences $(i_1, \ldots, i_n)$, $(j_1, \ldots, j_n)$ as follows. If $(a_{ij}, a_{ji}) = (0,0)$, then $n = 2$, $(i_1, i_2) = (i, j)$, and $(j_1, j_2) = (j, i)$. If $(a_{ij}, a_{ji}) = (-1,-1)$, then $n = 3$, $(i_1, i_2, i_3) = (i, j, i)$, and $(j_1, j_2, j_3) = (j, i, j)$. If $(a_{ij}, a_{ji}) = (-1,-2)$, then $n = 4$, $(i_1, \ldots, i_4) = (i, j, i, j)$, and $(j_1, \ldots, j_4) = (j, i, j, i)$. If $(a_{ij}, a_{ji}) = (-1,-3)$, then $n = 6$, $(i_1, \ldots, i_6) = (i, j, i, j, i, j)$, and $(j_1, \ldots, j_6) = (j, i, j, i, j, i)$. Then the braid relation to be shown is written as $S_{i_1} \ldots S_{i_n} = S_{j_1} \ldots S_{j_n}$.

For $p = 1, \ldots, n$, we set $\lambda_p = s_{i_n} s_{i_{n-1}} \cdots s_{i_{p+1}}(\alpha_{i_p}^\vee)$ and $\mu_p = s_{j_n} s_{j_{n-1}} \cdots s_{j_{p+1}}(\alpha_{j_p}^\vee)$. Then $\lambda_p = a_p \alpha_i^\vee + b_p \alpha_j^\vee$ and $\mu_p = c_p \alpha_i^\vee + d_p \alpha_j^\vee$ for some $a_p, b_p.c_p, d_p \in \mathbb{Z}_{\geqq 0}$. (See Example 1.1.) For $k, l \in \mathbb{Z}_{\geqq 0}$, we set $u_{k,l} = f_{i_1}^{a_1 k + b_1 l} \cdots f_{i_n}^{a_n k + b_n l}$ and $v_{k,l} = f_{j_1}^{c_1 k + d_1 l} \cdots f_{j_n}^{c_n k + d_n l}$. Then $u(k,l)$ (resp. $v(k,l)$) is the image in $\mathcal{A}_{q,0}$ of the left-hand side (resp. right-hand side) of the corresponding formula in Example 1.11. For example, if $(a_{ij}, a_{ji}) = (-1,-1)$, then $u(k,l) = f_i^l f_j^{k+l} f_i^k$ The Verma relations mean that $u(k,l) = v(k,l)$ for all $k, l \in \mathbb{Z}_{\geqq 0}$.

By Remark 1.12 and Lemma 4.2, the condition $S_{i_1} \ldots S_{i_n} = S_{j_1} \ldots S_{j_n}$ is equivalent to $\gamma(f_{i_1}^{-\lambda_1}) \cdots \gamma(f_{i_n}^{-\lambda_n}) = \gamma(f_{j_1}^{-\mu_1}) \cdots \gamma(f_{j_n}^{-\mu_n})$. Denote the left-hand side by ϕ and right-hand side by ψ. Fix any $t \in I$. Then $\phi(f_t)$ and $\psi(f_t)$ belong to $Q(\mathcal{A}_{q,0}[q_i^{-\alpha_i^\vee}, q_j^{-\alpha_j^\vee}])$. We denote $\phi(f_k)$ by $\Phi(q_i^{-\alpha_i^\vee}, q_j^{-\alpha_j^\vee})$ and $\psi(f_k)$ by $\Psi(q_i^{-\alpha_i^\vee}, q_j^{-\alpha_j^\vee})$.

From Lemma 3.2 and the definition of $\gamma(f_i^\lambda)$, it follows that there exists a subset Γ of $\mathbb{Z}^2_{\geqq 0}$ with the following properties:

(1) If $f(x,y) \in \mathcal{A}_{q,0}[x,y]$ and $f(q_i^k, q_j^l) = 0$ for all $(k,l) \in \Gamma$, then $f(x,y) = 0$ in $\mathcal{A}_{q,0}[x,y]$.
(2) $\Phi(q_i^k, q_j^l)$ and $\Psi(q_i^k, q_j^l)$ are well-defined for all $(k,l) \in \Gamma$.
(3) $\Phi(q_i^k, q_j^l) = u(k,l) f_t u(k,l)^{-1}$ and $\Phi(q_i^k, q_j^l) = v(k,l) f_t v(k,l)^{-1}$ for all $(k,l) \in \Gamma$.

Using the Verma relations, we obtain that $\Phi(q_i^k, q_j^l) = \Psi(q_i^k, q_j^l)$ for all $(k,l) \in \Gamma$. From Lemma 3.1 it follows that $\phi(f_t) = \Phi(q_i^{-\alpha_i^\vee}, q_j^{-\alpha_j^\vee}) = \Psi(q_i^{-\alpha_i^\vee}, q_j^{-\alpha_j^\vee}) = \psi(f_t)$. We have just completed the proof. Q.E.D.

Remark 4.4. In Theorem 4.3, we construct the representation of the Weyl group W in algebra automorphisms of $Q(\mathcal{A}_q)$. This can be regarded as a quantized q-analogue of the birational Weyl group action arising from a nilpotent Poisson algebra proposed by Noumi and Yamada in [17]. See also Remark 4.1.

§5. Quantized birational Weyl group actions of Hasegawa

In this section, we shall reconstruct the quantized birational Weyl group actions of Hasegawa [4].

Let $A = \{a_{ij}\}_{i,j \in I}$ be a symmetrizable GCM symmetrized by $\{d_i\}_{i \in I}$, d the least common denominator of $\{d_i\}_{i \in I}$, $(Y, X, \langle\,,\rangle, \{\alpha_i^\vee\}_{i \in I}, \{\alpha_i\}_{i \in I})$ a root datum of type A. Let $\{y_1, \dots, y_M\}$ be a $\mathbb{Z}$-free basis of $d^{-1}Y$. Assume that if $i \neq j$ and $a_{ij} \neq 0$, then $\epsilon_{ij} = \pm 1$ and $\epsilon_{ji} = -\epsilon_{ij}$, otherwise $\epsilon_{ij} = 0$. Set the base field $\mathbb{F}$ by $\mathbb{F} = \mathbb{Q}(q^{1/d})$ and $q_i \in \mathbb{F}$ by $q_i = q^{d_i}$.

Consider the tempered domain $\mathcal{B}_q$ defined in Section 2.4. For $i \in I$, define $f_{i1}, f_{i2} \in \mathcal{B}_q \otimes \mathcal{B}_q$ by $f_{i1} = f_i \otimes 1$ and $f_{i2} = k_i^{-1} \otimes f_i$. Note that $f_{i1} + f_{i2}$ $(i \in I)$ are the images of the lower Chevalley generators F_i in $\mathcal{B}_q \otimes \mathcal{B}_q$. Therefore $f_{i1} + f_{i2}$ $(i \in I)$ satisfy the q-Serre relations (Section 1.5) and hence the Verma relations (Section 1.7).

Let $\widetilde{\mathcal{A}}_{q,0}$ be the subalgebra of $\mathcal{B}_q \otimes \mathcal{B}_q$ generated by f_{i1}, f_{i2} $(i \in I)$. Then $\widetilde{\mathcal{A}}_{q,0}$ is identified with the algebra over $\mathbb{F}$ generated by f_{i1}, f_{i2} $(i \in I)$ with defining relations:

$$
\begin{aligned}
f_{i\nu} f_{j\nu} &= q_i^{-\epsilon_{ij} a_{ij}} f_{j\nu} f_{i\nu} && \text{for } i,j \in I,\ \nu = 1,2, \\
f_{i2} f_{j1} &= q_i^{a_{ij}} f_{j1} f_{i2} && \text{for } i,j \in I.
\end{aligned}
$$

Note that $q_i^{a_{ij}} = q_j^{a_{ji}}$ because $d_i a_{ij} = d_j a_{ji}$.

Let $\widetilde{\mathcal{A}}_q$ be the tensor product algebra $\widetilde{\mathcal{A}}_{q,0} \otimes U_q^0$. Then $\widetilde{\mathcal{A}}_q$ can be identified with the Laurent polynomial ring $\widetilde{\mathcal{A}}_{q,0}[q^{\pm y_1}, \dots, q^{\pm y_M}]$ with coefficients in $\widetilde{\mathcal{A}}_{q,0}$. For $i \in I$ and $\lambda \in d^{-1}Y$, we identify $f_{i1} \otimes 1, f_{i2} \otimes 1, 1 \otimes 1 \otimes q^\lambda \in \widetilde{\mathcal{A}}_q$ with $f_{i1}, f_{i2}, q^\lambda \in \widetilde{\mathcal{A}}_{q,0}[q^{\pm y_1}, \dots, q^{\pm y_M}]$ respectively. Note that q^λ commutes f_{i1} and f_{i2} in $\widetilde{\mathcal{A}}_q$ for $\lambda \in d^{-1}Y$ and $i \in I$. Since $\widetilde{\mathcal{A}}_q$ is also a tempered domain and hence an Ore domain, there exists the field of fractions $Q(\widetilde{\mathcal{A}}_q)$ of $\widetilde{\mathcal{A}}_q$.

For $i \in I$, define $g_i \in Q(\widetilde{\mathcal{A}}_q)$ by $g_i = f_{i1}^{-1} f_{i2}$. Let $\mathcal{A}_{q,0}$ (resp. $\mathcal{A}_q$) be the subalgebra of $Q(\widetilde{\mathcal{A}}_q)$ generated by g_i $(i \in I)$ (resp. generated by g_i $(i \in I)$ and q^λ $(\lambda \in d^{-1}Y)$). Then $\mathcal{A}_{q,0}$ can be identified with the algebra over $\mathbb{F}$ generated by g_i $(i \in I)$ with defining relations:

$$g_i g_j = q_i^{-2\epsilon_{ij} a_{ij}} g_j g_i \quad \text{for } i, j \in I. \tag{5.1}$$

Furthermore $\mathcal{A}_q$ can be identified with the Laurent polynomial ring $\mathcal{A}_{q,0}[q^{\pm y_1}, \dots, q^{\pm y_M}]$. Note that q^λ commutes g_i in $\mathcal{A}_q$ for $\lambda \in d^{-1}Y$ and $i \in I$. Since $\mathcal{A}_{q,0}$ and $\mathcal{A}_q$ are tempered domains, there exist the fields of fractions $Q(\mathcal{A}_{q,0})$ and $Q(\mathcal{A}_q)$. We have also $Q(\mathcal{A}_{q,0}) \subset Q(\mathcal{A}_q) \subset Q(\widetilde{\mathcal{A}}_q)$.

In [4], Hasegawa constructed a representation of the Weyl group $W = W(A)$ in algebra automorphisms of $Q(\mathcal{A}_q)$. Our aim is to reconstruct it by the same method as in Section 4.

5.1. Non-integral power of $f_{i1} + f_{i2}$

Applying the q-binomial theorem (Lemma 1.3) to $f_{i2} f_{i1} = q_i^2 f_{i1} f_{i2}$, we obtain

$$(f_{i1} + f_{i2})^n = \frac{(q_i^{-2n} g_i)_{q_i,\infty}}{(g_i)_{q_i,\infty}} f_{i1}^n \in Q(\widetilde{\mathcal{A}}_q) \quad \text{for } n \in \mathbb{Z},$$

where $(x)_{i,\infty} = \prod_{\nu=0}^{\infty} (1 + q_i^{2\nu} x)$. The infinite products in the right-hand side cancel each other out except finite factors. Using (5.1), and $f_{i1} g_j = q_i^{(\epsilon_{ij}-1)a_{ij}} g_j f_{i1}$, we obtain

$$\begin{aligned}
&(f_{i1} + f_{i2})^n g_j (f_{i1} + f_{i2})^{-n} \\
&= q_i^{(\epsilon_{ij}-1)a_{ij} n} g_j \frac{(q_i^{-2\epsilon_{ij} a_{ij}} q_i^{-2n} g_i)_{q_i,\infty}}{(q_i^{-2\epsilon_{ij} a_{ij}} g_i)_{q_i,\infty}} \frac{(g_i)_{q_i,\infty}}{(q_i^{-2n} g_i)_{q_i,\infty}} \\
&= q_i^{(\epsilon_{ij}-1)a_{ij} n} \frac{(q_i^{-2n} g_i)_{q_i,\infty}}{(g_i)_{q_i,\infty}} \frac{(q_i^{2\epsilon_{ij} a_{ij}} g_i)_{q_i,\infty}}{(q_i^{2\epsilon_{ij} a_{ij}} q_i^{-2n} g_i)_{q_i,\infty}} g_j
\end{aligned}$$

for $n \in \mathbb{Z}$. More explicitly we have $(f_{i1}+f_{i2})^n g_j (f_{i1}+f_{i2})^{-n} = \phi_{ij}(q_i^n)$ for $n \in \mathbb{Z}$, where $\phi_{ij}(x) \in Q(\mathcal{A}_{q,0}[x])$ $(i,j \in I)$ are defined by

$$\phi_{ij}(x) = \begin{cases} g_j \left(\displaystyle\prod_{\nu=0}^{-a_{ij}-1} \frac{1+q_i^{2\nu} g_i}{1+q_i^{2\nu} x^{-2} g_i} \right) & \text{if } \epsilon_{ij} = +1, \\ x^{2(-a_{ij})} \left(\displaystyle\prod_{\nu=0}^{-a_{ij}-1} \frac{1+q_i^{2\nu} x^{-2} g_i}{1+q_i^{2\nu} g_i} \right) g_j & \text{if } \epsilon_{ij} = -1, \\ x^{-2} g_i & \text{if } i = j, \\ g_j & \text{if } a_{ij} = 0. \end{cases}$$

Take any $\lambda \in Y$. Identifying x with $q_i^\lambda = q^{d_i \lambda}$, we regard $\mathbb{F}[x]$ as a subalgebra of U_q^0. Using the result of Section 3.2, we can define the algebra automorphism $\gamma((f_{i1}+f_{i2})^\lambda)$ of $Q(\mathcal{A}_q)$ by $\gamma((f_{i1}+f_{i2})^\lambda)(g_j) = \phi_{ij}(q_i^\lambda)$ for $j \in I$ and $\gamma((f_{i1}+f_{i2})^\lambda)(q^\mu) = q^\mu$ for $\mu \in d^{-1}Y$.

Remark 5.1. We have shown that the conjugation action of a non-integral power $(f_{i1}+f_{i2})^\lambda$ on $Q(\mathcal{A}_q)$ is well-defined. Recall that the subalgebra of $\mathcal{B}_q \otimes \mathcal{B}_q$ generated by $\{f_{i1}+f_{i2}\}_{i \in I}$ is denoted by $\mathcal{N}_{q,2}$ in Section 2.4. Although the conjugation action of $(f_{i1}+f_{i2})^\lambda$ on $Q(\mathcal{N}_{q,2} \otimes U_q^0)$ is well-defined by Theorem 4.3, it does not reconstruct Hasegawa's action.

5.2. Reconstruction of Hasegawa's Weyl group actions

The Weyl group $W = \langle s_i | i \in I \rangle$ acts on U_q^0. This is naturally extended to the action on $Q(\mathcal{A}_q)$ so that each $w \in W$ trivially acts on $\{g_i\}_{i \in I}$. In this subsection, we denote by $\tilde{w}$ the action of $w \in W$ on $Q(\mathcal{A}_q)$: $\tilde{w}(g_i) = g_i$ for $i \in I$ and $\tilde{w}(q^\lambda) = q^{w(\lambda)}$ for $\lambda \in d^{-1}Y$.

Theorem 5.2. *For $i \in I$ we define the algebra automorphism S_i of $Q(\mathcal{A}_q)$ by $S_i = \tilde{s}_i \circ \gamma((f_{i1}+f_{i2})^{-\alpha_i^\vee}) = \gamma((f_{i1}+f_{i2})^{\alpha_i^\vee}) \circ \tilde{s}_i$. Then the action of the Weyl group W on $Q(\mathcal{A}_q)$ is defined by $s_i(x) = S_i(x)$ for $i \in I$ and $x \in Q(\mathcal{A}_q)$. Explicitly, the following formulae define a representation of the Weyl group in algebra automorphisms of $Q(\mathcal{A}_q)$:*

$$s_i(g_j) = \begin{cases} g_j \left(\displaystyle\prod_{\nu=0}^{-a_{ij}-1} \frac{1+q_i^{2\nu} g_i}{1+q_i^{2\nu} q_i^{-2\alpha_i^\vee} g_i} \right) & \textit{if } \epsilon_{ij} = +1, \\ q_i^{2(-a_{ij})\alpha_i^\vee} \left(\displaystyle\prod_{\nu=0}^{-a_{ij}-1} \frac{1+q_i^{2\nu} q_i^{-2\alpha_i^\vee} g_i}{1+q_i^{2\nu} g_i} \right) g_j & \textit{if } \epsilon_{ij} = -1, \\ q_i^{-2\alpha_i^\vee} g_i & \textit{if } i = j, \\ g_j & \textit{if } a_{ij} = 0, \end{cases}$$

$$s_i(q^\lambda) = q^{s_i(\lambda)} = q^{\lambda - \langle \lambda, \alpha_i \rangle \alpha_i^\vee} \quad \lambda \in d^{-1}Y.$$

Proof. Since $f_{i1} + f_{i2}$ $(i \in I)$ satisfy the Verma relations, we can prove the theorem by the same argument as in the proof of Theorem 4.3. Appropriately replacing $f_i^{\alpha_i^\vee}$ and f_i in the proof of Theorem 4.3 by $(f_{i1} + f_{i2})^{\alpha_i^\vee}$ and g_i respectively, we obtain the proof of the above theorem. Q.E.D.

Remark 5.3 (Hasegawa's Weyl group action). In Theorem 5.2, we construct the representation of the Weyl group W in algebra automorphisms of $Q(\mathcal{A}_q)$. This can be regarded as a reconstruction of the Weyl group action constructed by Hasegawa in [4]. Set $a_i = q_i^{\alpha_i^\vee}$ and $F_i = a_i^{-1} g_i$ for $i \in I$. Do not confuse these F_i with the lower Chevalley generators. Then $\mathcal{A}_q$ can be identified with the algebra generated by F_i $(i \in I)$ and q^λ $(\lambda \in d^{-1}Y)$ with defining relations:

$$F_i F_j = q_i^{-2\epsilon_{ij} a_{ij}} F_j F_i \quad \text{for } i, j \in I,$$
$$q^0 = 1, \ q^\lambda q^\mu = q^{\lambda+\mu}, \ q^\lambda F_i = F_i q^\lambda \quad \text{for } \lambda, \mu \in d^{-1}Y.$$

The explicit formulae of the Weyl group action in Theorem 5.2 can be rewritten as

$$s_i(F_j) = \begin{cases} F_j \left(\prod_{\nu=0}^{-a_{ij}-1} \dfrac{1 + q_i^{2\nu} a_i F_i}{a_i + q_i^{2\nu} F_i} \right) & \text{if } \epsilon_{ij} = +1, \\ \left(\prod_{\nu=0}^{-a_{ij}-1} \dfrac{a_i + q_i^{2\nu} F_i}{1 + q_i^{2\nu} a_i F_i} \right) F_j & \text{if } \epsilon_{ij} = -1, \\ F_j & \text{otherwise}, \end{cases}$$
$$s_i(q^\lambda) = q^{s_i(\lambda)}, \quad \text{in particular } s_i(a_j) = a_j a_i^{-a_{ij}}.$$

These formulae essentially coincide with those of Hasegawa. Compare these with Hasegawa's formulae, Equation (8) in [4] for the $A_l^{(1)}$ case and the example for the B_2 case below Theorem 4 in [4].

Let us explain the classical limit of Hasegawa's action. Set $q = e^\eta$. In the above setting, replace q by $q^\hbar$ and $\lambda \in d^{-1}Y$ by $\hbar^{-1}\lambda$. Then q^λ is replaced by itself and

$$\hbar^{-1}[F_i, F_j] = \hbar^{-1}(F_i F_j - F_j F_i) \equiv -2\eta \epsilon_{ij} d_i a_{ij} F_i F_j \mod \hbar.$$

Hence the classical limit $\mathcal{A}_q^{\mathrm{cl}}$ of the algebra $\mathcal{A}_q$ is the commutative Poisson algebra generated by F_i $(i \in I)$ and q^λ $(\lambda \in d^{-1}Y)$ with Poisson brackets defined by

$$\{F_i, F_j\} = -2\eta \epsilon_{ij} d_i a_{ij} F_i F_j \quad \text{for } i, j \in I,$$

$$\{q^{\lambda}, q^{\mu}\} = \{q^{\lambda}, F_i\} = 0 \qquad \text{for } \lambda, \mu \in d^{-1}Y,\ i \in I.$$

The classical limit of the above Weyl group action can be simply written as

$$s_i(F_j) = F_j \left(\frac{1 + a_i F_i}{a_i + F_i} \right)^{\varepsilon_{ij} a_{ij}}, \quad s_i(q^{\lambda}) = q^{s_i(\lambda)},$$

where $a_i = q_i^{\alpha_i^{\vee}} = q^{d_i a_{ij}}$. This action preserves the Poisson brackets of $Q(\mathcal{A}_q^{\mathrm{cl}})$. The classical case of type $A_2^{(1)}$ was found by Kajiwara, Noumi, and Yamada in [8]. See Equation (6) of [8]. In [4], Hasegawa quantized its generalization to an arbitrary symmetrizable GCM case.

References

[1] S. Berman and B. Cox, Enveloping algebras and representations of toroidal Lie algebras, Pacific J. Math., **165** (1994), 239–267.

[2] P. Etingof and A. Varchenko, Dynamical Weyl groups and applications, Adv. Math., **167** (2002), 74–127; preprint, arXiv:math.QA/0011001v1.

[3] K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings. Second ed., London Math. Soc. Stud. Texts, **61**, Cambridge Univ. Press, Cambridge, 2004, xxiv+344 pp.

[4] K. Hasegawa, Quantizing the Bäcklund transformations of Painlevé equations and the quantum discrete Painlevé VI equation, this volume; preprint, arXiv:math.QA/0703037v1.

[5] J. Hong and S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Grad. Stud. Math., **42**, Amer. Math. Soc., Providence, RI, 2002, xviii+307 pp.

[6] K. Iohara and F. Malikov, Rings of skew polynomials and Gel'fand–Kirillov conjecture for quantum groups, Comm. Math. Phys., **164** (1994), 217–237; preprint, arXiv:hep-th/9306138v1.

[7] V. G. Kac, Infinite-dimensional Lie Algebras. Third ed., Cambridge Univ. Press, Cambridge, 1990, xxii+400 pp.

[8] K. Kajiwara, M. Noumi and Y. Yamada, A study on the fourth q-Painlevé equation, J. Phys. A, **34** (2001), 8563–8581; preprint, arXiv:nlin.SI/0012063v1.

[9] K. Kajiwara, M. Noumi and Y. Yamada, Discrete dynamical systems with $W(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ symmetry, Lett. Math. Phys., **60** (2002), 211–219; preprint, arXiv:nlin.SI/0106029v1.

[10] K. Kajiwara, M. Noumi and Y. Yamada, q-Painlevé systems arising from q-KP hierarchy, Lett. Math. Phys., **62** (2002), 259–268; preprint, arXiv:nlin.SI/0112045v1.

[11] G. Lusztig, Introduction to Quantum Groups, Progr. Math., **110**, Birkhäuser Boston, Inc., Boston, MA, 1993, xii+341 pp.

[12] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, With the Cooperation of L. W. Small, Revised ed., Grad. Stud. Math., **30**, Amer. Math. Soc., Providence, RI, 2001, xx+636 pp.

[13] H. Nagoya, Quantum Painlevé systems of type $A_l^{(1)}$, Internat. J. Math., **15** (2004), 1007–1031; preprint, arXiv:math.QA/0402281v1.

[14] M. Noumi and Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, Comm. Math. Phys., **199** (1998), 281–295; preprint, arXiv:math.QA/9804132v1.

[15] M. Noumi and Y. Yamada, Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial. Ekvac., **41** (1998), 483–503; preprint, arXiv:math.QA/9808003v1.

[16] M. Noumi and Y. Yamada, Symmetries in the fourth Painleve equation and Okamoto polynomials, Nagoya Math. J., **153** (1999), 53–86; preprint, arXiv:q-alg/9708018v1.

[17] M. Noumi and Y. Yamada, Birational Weyl group action arising from a nilpotent Poisson algebra, In: Physics and Combinatorics 1999 (Nagoya), World Sci. Publ., River Edge, NJ, 2001, pp. 287–319; preprint, arXiv:math.QA/0012028v1.

[18] M. Noumi and Y. Yamada, Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions, In: Representation Theory of Algebraic Groups and Quantum Groups, Adv. Stud. Pure Math., **40**, Math. Soc. Japan, Tokyo, 2004, pp. 371–442; preprint, arXiv:math.QA/0203030v2.

[19] K. Okamoto, Studies on the Painlevé equations I: Sixth Painlevéquation P_{VI}, Ann. Mat. Pura Appl. (4), **146** (1987), 337–381.

[20] K. Okamoto, Studies on the Painlevé equations II: Fifth Painlevé equation P_{V}, Japan. J. Math. (N.S.), **13** (1987), 47–76.

[21] K. Okamoto, Studies on the Painlevé equations III: Second and fourth Painlevé equations, P_{II} and P_{IV}, Math. Ann., **275** (1986), 221–255.

[22] K. Okamoto, Studies on the Painlevé equations IV: Third Painlevé equation P_{III}, Funkcial. Ekvac., **30** (1987), no. 2-3, 305–332.

[23] A. Rocha-Caridi and N. R. Wallach, Characters of irreducible representations of the Virasoro algebra, Math. Z., **185** (1984), 1–21.

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