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Representation theory of W-algebras, II

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Dedicated to Professor Akihiro Tsuchiya on the occasion of his retirement from Nagoya University

Abstract.

We study the (Ramond twisted) representations of the affine W-algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ in the case that f admits a good even grading. We establish the vanishing and the almost irreducibility of the corresponding BRST cohomology. This confirms some of the recent conjectures of Kac and Wakimoto [KW08]. In type A, our results give the characters of all irreducible ordinary (Ramond twisted) representations of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ for all nilpotent elements f and all non-critical k, and prove the existence of modular invariant representations conjectured in [KW08].

§1. Introduction

Let $\bar{\mathfrak{g}}$ be a complex simple Lie algebra, f a nilpotent element of $\bar{\mathfrak{g}}$, $\mathfrak{g}$ the non-twisted affine Kac–Moody Lie algebra associated with $\bar{\mathfrak{g}}$. Let $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ be the *affine W-algebra* associated with $(\bar{\mathfrak{g}}, f)$ at level $k \in \mathbb{C}$, defined by the method of the quantum BRST reduction [FF90, dBT94, KRW03].

The vertex algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ is in general $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded [KW04]. Therefore it is natural [KW08] to consider its *Ramond twisted representations*[1]. In fact it is in the Ramond twisted representations where the corresponding *finite W-algebra* $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$ [Lyn79, dBT93, Pre02] appears as its *Zhu algebra*, according to [DSK06].

In the previous paper [Ara07] we studied the representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ in the case that f is a principal nilpotent element. In the

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[1]If f is an even nilpotent element then $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ is $\mathbb{Z}_{\geq 0}$-graded and Ramond twisted representations are usual (untwisted) representations.

present paper we study the Ramond twisted representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ in the case that f admits a good even grading. All nilpotent elements in type A satisfy this condition.

There is a natural BRST (co)homology functor $H_0^{\mathrm{BRST}}(?)$ from a suitable category of representations of $\mathfrak{g}$ at level k to the category of Ramond twisted representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$. In our case $H_\bullet^{\mathrm{BRST}}(M)$ is essentially the same BRST cohomology studied in the recent work [KW08] of Kac and Wakimoto. In the case that f is a principal nilpotent element this functor is identical to the "−"-reduction functor studied in [FKW92, Ara04, Ara07].

The main result of this paper is the vanishing and the *almost irreduciblity* of the BRST cohomology (Theorem 5.5.4). Though our formulation is slightly different from that of [KW08], this result proves Conjecture B of [KW08], partially. Here, recall [DSK06] that a positive energy representation $M = \bigoplus_{d \in d_0 + \mathbb{Z}_{\geq 0}} M_d$, $M_{d_0} \neq 0$, of a vertex algebra V is called almost irreducible if M is generated by M_{d_0} and there is no graded submodule of M intersecting M_{d_0} trivially. In particular an almost irreducible module M is irreducible if and only if its "top part" M_{d_0} is irreducible over the Zhu algebra of V.

In our case the top part of the BRST cohomology functor is identical to the Lie algebra homology functor (the Whittaker functor [Mat90a, BK08]) from the highest weight category of $\bar{\mathfrak{g}}$ to the category of $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$-modules (see §5.4). Therefore our result reduces the study of the BRST cohomology functor to that of the Whittaker functor in the representations theory of finite W-algebras.

Although the representation theory of finite W-algebras has been rapidly developing (cf. [Pre07, Pre06, Los10, BGK08]), not much is known about the Whittaker functor associated with $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$ except for some special cases [Mat90a], unless $\bar{\mathfrak{g}} = \mathfrak{sl}_n$: In type A, Brundan and Kleshchev [BK08] determined the characters of all irreducible finite-dimensional representations of $\mathcal{W}^{\mathrm{fin}}(\mathfrak{sl}_n, f)$, by showing that the Whittaker functor sends an simple module to zero or a simple module, and any simple $\mathcal{W}^{\mathrm{fin}}(\mathfrak{sl}_n, f)$-module is obtained in this manner. It follows that in type A the almost irreduciblity of the BRST cohomology actually implies the irreduciblity, and furthermore, any irreducible ordinary[2] representation of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ is isomorphic to $H_0^{\mathrm{BRST}}(L(\lambda))$ for some irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{sl}}_n$ with highest weight λ (Theorem 5.7.1). Hence our result shows that the character of *every* irreducible ordinary Ramond twisted representation of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ at

[2]An irreducible positive energy representation of a vertex algebra is called ordinary if its all homogeneous subspaces are finite-dimensional.

any level $k \in \mathbb{C}$ is determined by that of the corresponding irreducible highest weight representation of $\mathfrak{g}$, which is known [KT00] (in terms of the Kazhdan–Lusztig polynomials) provided that k is non-critical. This generalizes the main results of [Ara05, Ara07].

The most important representations of a vertex algebra are those irreducible ordinary representations whose normalized characters are modular invariant. Kac and Wakimoto [KW08] have recently discovered the remarkable triples $(\bar{\mathfrak{g}}, f, k)$, for which the (nonzero) normalized Euler–Poincaré characters of the BRST cohomology $H^{\mathrm{BRST}}_{\bullet}(L(\lambda))$, with the coefficient in the irreducible principal admissible representations $L(\lambda)$ of $\mathfrak{g}$ at level k, are homomorphic functions on the complex upper half plane and span an $SL_2(\mathbb{Z})$-invariant space[3]. Our results show in type A that these Euler–Poincaré characters are indeed characters of irreducible Ramond twisted representations of $\mathcal{W}^k(\mathfrak{sl}_n, f)$, as conjectured in [KW08] (see Theorem 5.9.2)[4].

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Notation. Throughout this paper the ground field is the complex number $\mathbb{C}$ and tensor products and dimensions are always meant to be as vector spaces over $\mathbb{C}$.

§2. Preliminaries on vertex algebras and their twisted representations

In this section we collect the necessary information on vertex algebras and their (twisted) representations. The textbook [Kac98, FBZ04]

[3]In the case that f is a principal nilpotent the existence of modular invariant representation of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ was conjectured by Frenkel, Kac and Wakimoto [FKW92] and proved in [Ara07].

[4]It seems that the "top parts" of modular invariant representations are in general "generic" representations of $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$, see Theorem 5.8.1.

and the papers [Li96, BK04, DSK06] are our basic references in this section.

2.1. Fields

Let V be a vector space. For a formal series $a(z) \in (\operatorname{End} V)[[z, z^{-1}]]$, we set $a_{(n)} = \operatorname{Res}_z z^n a(z)$, where Res_z denotes the coefficient of z^{-1}.

An element $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in (\operatorname{End} V)[[z, z^{-1}]]$ is called a *field* on V if $a_{(n)}v = 0$ for all $v \in V$ and $n \gg 0$.

The normally ordered product

$$:a(z)b(z) := a(z)_- b(z) + b(z)a(z)_+ \tag{1}$$

of two fields $a(z)$ and $b(z)$ is also a field, where $a(z)_- = \sum_{n<0} a_{(n)} z^{-n-1}$ and $a(z)_+ = \sum_{n\geq 0} a_{(n)} z^{-n-1}$.

Two fields $a(z)$ and $b(z)$ are called *mutually local* if

$$(z-w)^r [a(z), b(w)] = 0 \quad \text{for } r \gg 0 \tag{2}$$

in $(\operatorname{End} V)[[z, z^{-1}, w, w^{-1}]]$.

Set

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]. \tag{3}$$

The locality (2) gives

$$[a(z), b(w)] = \sum_{n\geq 0} (a(w)_{(n)} b(w)) \partial_w^{[n]} \delta(z-w), \tag{4}$$

where $\partial_w^{[n]} = \partial_w^n / n!$, $\partial_w = \frac{\partial}{\partial w}$, and

$$a(w)_{(n)} b(w) = \operatorname{Res}_z (z-w)^n [a(z), b(w)].$$

2.2. Vertex algebras

A *vertex algebra* is a vector space V equipped with the following data:

- A vector $\mathbf{1} \in V$ (vacuum vector),
- $T \in \operatorname{End} V$ (translation operator),
- A collection $\{a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_{(n)} z^{-n-1}; \alpha \in A\}$ of fields on V, where A is an index set (generating fields),

These data are subject to the following:

(i) $T\mathbf{1} = 0$,
(ii) $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all $\alpha \in A$,

(iii) $a^{\alpha}(z)\mathbf{1} \in V[[z]]$ for all $\alpha \in A$,
(iv) the vectors $a^{\alpha_1}_{(m_1)} \dots a^{\alpha_r}_{(m_r)}\mathbf{1}$ with $r \geq 0$, $\alpha_i \in A$ and $m_i \in \mathbb{Z}$ span V,
(v) for any $\alpha, \beta \in A$ the fields $a^{\alpha}(z)$ and $a^{\beta}(z)$ mutually local.

Let V be a vertex algebra. There exists a unique linear map

$$V \to (\operatorname{End} V)[[z, z^{-1}]], \quad a \mapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \tag{5}$$

such that

(i) $Y(a,z)$ is a field on V for any $a \in V$,
(ii) $Y(a,z)$ and $Y(b,z)$ are mutually local for any $a, b \in V$,
(iii) $[T, Y(a,z)] = \partial_z Y(a,z)$ for any $a \in V$,
(iv) $Y(a,z)\mathbf{1} \in V[[z]]$ and $\lim_{z \to 0} Y(a,z)\mathbf{1} = a$ for any $a \in V$,
(v) $Y(a^{\alpha}_{(-1)}\mathbf{1}, z) = a^{\alpha}(z)$ for any generating filed $a^{\alpha}(z)$.

The map $Y(?, z)$ is called the *state-field correspondence.*

A *Hamiltonian* of a vertex algebra V is a diagonalizable operator $H \in \operatorname{End} V$ such that

$$[H, Y(a,z)] = Y(Ha, z) + z\partial_z Y(a,z) \quad \text{for all } a \in V.$$

A vertex algebra with a Hamiltonian H is called graded. If a is a eigenvector of H its eigenvalue is called the *conformal weight* of a and denoted by Δ_a. Let[5]

$$V_{\Delta} = \{a \in V; Ha = \Delta a\},$$

so that $V = \bigoplus_{\Delta \in \mathbb{C}} V_{\Delta}$.

2.3. Twisted representations of vertex algebras

Let $N \in \mathbb{N}$. An *N-twisted field* $a(z)$ on a vector space M is a formal power series in $z^{1/N}$, $z^{-1/N}$ of the form

$$a(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \operatorname{End}(M) \tag{6}$$

such that $a_{(n)}m = 0$ for all $m \in M$ and $n \gg 0$.

Two N-twisted fields $a(z)$ and $b(z)$ on M are called mutually local if they satisfy (2) in $(\operatorname{End} M)[[z^{1/N}, z^{-1/N}, w^{1/N}, w^{-1/N}]]$.

[5] This differs from the notation in [Ara07].

Let V be a vertex algebra, σ an automorphism of V of order N. A σ*-twisted representation* of V is a vector space M equipped with a linear map from V to the space of N-twisted fields on M,

$$V \to (\operatorname{End} M)[[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]], \quad a \mapsto Y^M(a,z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a^M_{(n)} z^{-n-1},$$

such that

$$Y^M(\sigma a, z) = Y^M(a, e^{2\pi i} z), \tag{7}$$

$$Y^M(\mathbf{1}, z) = \mathrm{id}_M, \tag{8}$$

and

$$\sum_{i=0}^{\infty} \binom{m}{i} (a_{(r+i)}b)^M_{(m+n-i)} = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(a^M_{(m+r-i)} b^M_{(n+i)} - (-1)^r b^M_{(n+r-i)} a^M_{(m+i)} \right) \tag{9}$$

for $a \in V_{\bar{j}}$, $b \in V$, $m \in \frac{j}{N} + \mathbb{Z}$, $n \in \frac{1}{N}\mathbb{Z}$, $r \in \mathbb{Z}$, where

$$V_{\bar{j}} = \{\sigma(a) = (e^{\frac{2\pi\sqrt{-1}}{N}})^{-j} a\}. \tag{10}$$

The relation (9) is called the *twisted Borcherds identity.*

By setting $r = 0$ in (9), one obtains

$$[a^M_{(m)}, b^M_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)^M_{(m+n-i)}, \tag{11}$$

or equivalently,

$$[Y^M(a,z), Y^M(b,w)] = \sum_{i=0}^{\infty} Y^M(a_{(i)}b, w) \partial_w^{[i]} \delta_j(z-w) \tag{12}$$

for $a \in V_{\bar{j}}$, where

$$\delta_j(z-w) = z^{-j/N} w^{j/N} \delta(z-w) = \sum_{n \in j/N + \mathbb{Z}} w^n z^{-n-1}.$$

In particular $Y^M(a,z)$ and $Y^M(b,z)$ are mutually local.

The relation (11) gives [Li96]

(13)

$$\begin{aligned}&Y^M(a_{(n)}b,w)\\&=\operatorname{Res}_z\sum_{k=0}^{\infty}\binom{-j/N}{k}z^{j/N-k}w^{-j/N}(z-w)^{n+k}[Y^M(a,z),Y^M(b,w)]\end{aligned}$$

for all $n\geq 0$. The sum in (13) is finite because of the locality. (In reality (13) holds for all $n\in\mathbb{Z}$ in an appropriate sense, see [Li96]).

Set $b=\mathbf{1}$, $r=-2$, $n=0$ in (9). It follows that

$$Y^M(Ta,z)=\partial_z Y^M(a,z). \tag{14}$$

Suppose that V is graded by a Hamiltonian H. A σ-twisted representation M is called *graded* if there exists an diagonalizable operator H^M on M such that

$$[H^M,a^M_{(n)}]=(Ta)^M_{(n+1)}+(Ha)_{(n)} \tag{15}$$

for all $a\in V$ and $n\in\frac{1}{N}\mathbb{Z}$. If a is homogeneous, (15) is equivalent to

$$[H^M,a^M_{(n)}]=-(n-\Delta_a+1)a^M_{(n)}. \tag{16}$$

We set

$$M_d=\{m\in M;H^Mm=dm\} \tag{17}$$

for $d\in\mathbb{C}$.

A *positive energy σ-twisted representation*[6] of V is a graded σ-twisted representation M of V such that there exists a finite set $d_1,\dots,$ $d_r\in\mathbb{C}$ such that $M_d=0$ unless $d\in\bigcup_i d_i+\mathbb{Z}_{\geq 0}$. Let $V\text{-}\mathfrak{Mod}_\sigma$ be the category of positive energy σ-twisted representations of V, whose morphisms are graded homomorphisms of σ-twisted representations.

An *ordinary* σ-twisted representation of V is a positive energy σ-twisted representation of V such that $\dim M_d<\infty$ for all d. Let $V\text{-}\mathfrak{mod}_\sigma$ be the full subcategory of $V\text{-}\mathfrak{Mod}_\sigma$ consisting of ordinary σ-twisted representations.

When $\sigma=\mathrm{id}_V$, σ-twisted representations are just usual (non-twisted) representations. We set $V\text{-}\mathfrak{Mod}=V\text{-}\mathfrak{Mod}_{\mathrm{id}_V}$ and $V\text{-}\mathfrak{mod}=V\text{-}\mathfrak{mod}_{\mathrm{id}_V}$.

[6] A positive energy representations is also called an admissible representation in the literature.

2.4. H-twisted Zhu algebras

Let V be a vertex algebra graded by a Hamiltonian H. Assume that $V_\Delta \neq 0$ unless $\Delta \in \frac{1}{N}\mathbb{Z}$. Then $\sigma_H := e^{2\pi i H} : V \to V$ is an automorphism of order at most N.

If M is a graded σ_H-twisted representations of V then the number $n - \Delta_a + 1$ in (16) is always an integer. Set $a_n^M = a^M_{(n+\Delta_a-1)}$, so that

$$Y^M(a,z) = \sum_{n\in\mathbb{Z}} a_n^M z^{-n-\Delta_a}, \quad [H^M, a_n^M] = -n a_n^M. \tag{18}$$

Define the H-twisted Zhu algebra [Zhu96, DSK06] $\mathrm{Zh}_H V$ by

$$\mathrm{Zh}_H V = V/V \circ V, \tag{19}$$

where $V \circ V$ is the span of the vectors

$$a \circ b := \sum_{r\geq 0} \binom{\Delta_a}{r} a_{(r-2)} b$$

with homogeneous vectors $a, b \in V$. The $\mathrm{Zh}_H V$ is an associative algebra with the multiplication

$$a * b = \sum_{r\geq 0} \binom{\Delta_a}{r} a_{(r-1)} b.$$

Let M be an object of V-$\mathfrak{Mod}_{\sigma_H}$. Denote by V_{top} the sum of homogeneous subspace V_d such that $V_{d'} = 0$ for all $d' \in d - \mathbb{N}$. Then V_{top} is naturally a module over $\mathrm{Zh}_H V$ by the following action:

$$(a + V \circ V)m = a^M_{(\Delta_a - 1)} m = a_0^M m. \tag{20}$$

Theorem 2.4.1 ([Zhu96, DSK06]). *The map $M \mapsto M_{\mathrm{top}}$ gives a bijective correspondence between simple objects of V-$\mathfrak{Mod}_{\sigma_H}$ and irreducible $\mathrm{Zh}_H V$-modules.*

The M is said to be *almost highest weight* if (1) $M_{\mathrm{top}} = M_d$ for some d and (2) M is generated by M_{top} over V. The M is said to be *almost co-highest weight* if (1) $M_{\mathrm{top}} = M_d$ for some d and (2) M contains no graded submodule intersecting M_{top} trivially. The M is called *almost irreducible* [DSK06] if M is both almost highest weight and almost co-highest weight. Clearly, an almost irreducible module is simple if and only if M_{top} is irreducible over $\mathrm{Zh}_H V$.

§3. Affine W-algebras

3.1. The setting

Let $\bar{\mathfrak{g}}$ be a complex simple Lie algebra, f a nilpotent element of $\bar{\mathfrak{g}}$. The corresponding affine W-algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ at the level $k \in \mathbb{C}$ is defined by the method of the quantum BRST reduction. This method was discovered by Feigin and Frenkel [FF90] who used it to define the W-algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ associated with the principal nilpotent elements f. The most general definition of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ was given by Kac, Roan and Wakimoto [KRW03], and the definition of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ given in [KRW03, KW04] involves another data, namely a *good grading* of $\bar{\mathfrak{g}}$ for f. However, thanks to the results [BG07] of Brundan and Goodwin, the definition of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ does *not* depend on the choice of a good grading for f.

Throughout this paper *we assume that f admits a good even grading* unless otherwise stated, that is, there exists a $\mathbb{Z}$-grading

$$\bar{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \bar{\mathfrak{g}}_j \tag{21}$$

of $\bar{\mathfrak{g}}$ such that $f \in \bar{\mathfrak{g}}_{-1}$ and $\operatorname{ad} f : \bar{\mathfrak{g}}_{\leq 0} \to \bar{\mathfrak{g}}_{<0}$ is surjective, where $\bar{\mathfrak{g}}_{\leq 0} = \bigoplus_{j \leq 0} \bar{\mathfrak{g}}_j$ $\bar{\mathfrak{g}}_{<0} = \bigoplus_{j<0} \bar{\mathfrak{g}}_j$. The last condition is equivalent to that $\operatorname{ad} f : \bar{\mathfrak{g}}_{>0} \to \bar{\mathfrak{g}}_{\geq 0}$ is injective, where $\bar{\mathfrak{g}}_{\geq 0} = \bigoplus_{j \geq 0} \bar{\mathfrak{g}}_j$ and $\bar{\mathfrak{g}}_{>0} = \bigoplus_{j>0} \bar{\mathfrak{g}}_j$. By definition there exists an exact sequence

$$0 \to \bar{\mathfrak{g}}^f \hookrightarrow \bar{\mathfrak{g}}_{\leq 0} \overset{\operatorname{ad} f}{\to} \bar{\mathfrak{g}}_{<0} \to 0, \tag{22}$$

where $\bar{\mathfrak{g}}^f$ is the centralizer of f in $\bar{\mathfrak{g}}$.

One can find a $\mathfrak{sl}_2$-triple (e, h, f) in $\bar{\mathfrak{g}}$ such that $e \in \bar{\mathfrak{g}}_1$, $h \in \bar{\mathfrak{g}}_0$, see Lemma 1.1 of [EK05]. Below *we write h_0 for h*. Also, there exists a semisimple element $x_0 \in \bar{\mathfrak{g}}_0$ that defines the $\mathbb{Z}$-grading, i.e.,

$$\bar{\mathfrak{g}}_j = \{a \in \bar{\mathfrak{g}}; [x_0, a] = ja\}. \tag{23}$$

Let $(\ |\ )$ be the normalized invariant bilinear form on $\bar{\mathfrak{g}}$, that is, $(\ |\ ) = 1/2h^\vee \times$ the Killing form on $\bar{\mathfrak{g}}$, where $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$. Set

$$\bar{\chi} = \bar{\chi}_f = (f|?) \in \bar{\mathfrak{g}}^*, \tag{24}$$

and let $\mathbb{O}_{\bar{\chi}} \subset \bar{\mathfrak{g}}^*$ be the coadjoint orbit of $\bar{\chi}$,

$$d_{\bar{\chi}} = \frac{1}{2} \dim \mathbb{O}_{\bar{\chi}}. \tag{25}$$

By (22) one has

$$\dim \bar{\mathfrak{g}}_{<0} = \frac{1}{2}(\dim \bar{\mathfrak{g}} - \dim \bar{\mathfrak{g}}^f) = d_{\bar{X}}. \tag{26}$$

3.2. Root data

Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}_0$ containing x_0 and h_0 (see above). Then $\bar{\mathfrak{h}}$ is a Cartan subalgebra of $\bar{\mathfrak{g}}$. Let $\bar{\Delta}$ be the set of roots of $\bar{\mathfrak{g}}$. One has

$$\bar{\Delta} = \sqcup_{j\in\mathbb{Z}} \bar{\Delta}_j,$$

where $\bar{\Delta}_j = \{\alpha \in \bar{\Delta}; \langle \alpha, x_0\rangle = j\}$. The $\bar{\Delta}_0$ is the set of roots of the reductive subalgebra $\bar{\mathfrak{g}}_0$. Let $\bar{\Delta}_{0,+}$ be a set of positive roots of $\bar{\mathfrak{g}}_0$, $\bar{\Delta}_{0,-} = -\bar{\Delta}_{0,+}$. Then $\bar{\Delta}_+ = \bar{\Delta}_{0,+} \sqcup \bar{\Delta}_{>0}$ is a set of positive roots of $\bar{\mathfrak{g}}$, where $\bar{\Delta}_{>0} = \sqcup_{j>0}\bar{\Delta}_j$. Likewise, $\bar{\Delta}_- = \bar{\Delta}_{0,-} \sqcup \bar{\Delta}_{<0}$ is a set of negative roots of $\bar{\mathfrak{g}}$, where $\bar{\Delta}_{<0} = \sqcup_{j<0}\bar{\Delta}_j$. Let $\bar{\mathfrak{g}}_0 = \bar{\mathfrak{n}}_{0,-} \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_{0,+}$ and $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ be the corresponding triangular decompositions of $\bar{\mathfrak{g}}_0$ and $\bar{\mathfrak{g}}$, respectively.

Let $\bar{\rho}$ be the half sum of positive roots of $\bar{\mathfrak{g}}$.

Let $\bar{Q}$, $\bar{Q}^\vee$, $\bar{P}$ and $\bar{P}^\vee$ be the root lattice, the coroot lattice, the weight lattice and the coweight lattice of $\bar{\mathfrak{g}}$, respectively. Denote by $\bar{W}$ the Weyl group of $\bar{\mathfrak{g}}$.

Set $\bar{I} = \{1, \dots, \operatorname{rank} \bar{\mathfrak{g}}\}$. Let $\{J_a; a \in \bar{I} \sqcup \bar{\Delta}_\pm\}$ be a Chevalley basis of $\bar{\mathfrak{g}}$ such that J_α with $\alpha \in \bar{\Delta}$ is a root vector of root α and $\{J_i; i \in \bar{I}\}$ is a basis of $\bar{\mathfrak{h}}$. Denote by $c_{a,b}^d$ the corresponding structure constant. Let $\bar{\mathfrak{g}} \ni x \mapsto x^t \in \bar{\mathfrak{g}}$ be the anti-Lie algebra involution defined by $J_\alpha^t = J_{-\alpha}$ $(\alpha \in \bar{\Delta})$ and $J_i^t = J_i$ $(i \in \bar{I})$.

3.3. Kac–Moody Lie algebras

Let $\mathfrak{g}$ be the Kac–Moody affinization of $\bar{\mathfrak{g}}$:

$$\mathfrak{g} = \bar{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D, \tag{27}$$

where $\bar{\mathfrak{g}}[t, t^{-1}] = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$. The commutation relations are given by

$$[X(m), Y(n)] = [X, Y](m+n) + m\delta_{m+n,0}(X|Y)K, \tag{28}$$

$$[D, X(m)] = mX(m), \quad [K, \mathfrak{g}] = 0 \tag{29}$$

for $X, Y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, where $X(m) = X \otimes t^n$. The subalgebra $\bar{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g}$ is naturally identified with $\bar{\mathfrak{g}}$.

The form $(\ |\ )$ is extended from $\bar{\mathfrak{g}}$ to the invariant symmetric bilinear of $\mathfrak{g}$ as follows:

$$(X(m)|Y(n)) = (X|Y)\delta_{m+n,0}, \quad (D|K) = 1,$$
$$(X(m)|D) = (X(m)|K) = (D|D) = (K|K) = 0.$$

We fix the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as usual:

(30) $$\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

(31) $$\mathfrak{n}_- = \bar{\mathfrak{n}}_-[t^{-1}] \oplus \bar{\mathfrak{h}}[t^{-1}]t^{-1} \oplus \bar{\mathfrak{n}}_+[t^{-1}]t^{-1},$$

(32) $$\mathfrak{n}_+ = \bar{\mathfrak{n}}_-[t]t \oplus \bar{\mathfrak{h}}[t]t \oplus \bar{\mathfrak{n}}_+[t].$$

Let $\mathfrak{h}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ be the dual space of $\mathfrak{h}$. Here, Λ_0 and δ are dual elements of K and D, respectively. For $\lambda \in \mathfrak{h}^*$, the number $\langle \lambda, K \rangle$ is called the *level of* λ.

Let $\bar{\lambda}$ be the restriction of $\lambda \in \mathfrak{h}^*$ to $\bar{\mathfrak{h}}^*$. We refer to $\bar{\lambda}$ as the *classical part* of λ.

Let Δ be the set of roots of $\mathfrak{g}$, Δ_+ the set of positive roots, $\Delta_- = -\Delta_+$. Then, $\Delta = \Delta^{\mathrm{re}} \sqcup \Delta^{\mathrm{im}}$, where Δ^{re} is the set of real roots and Δ^{im} is the set of imaginary roots. Let $\Delta^{\mathrm{re}}_{\pm} = \Delta^{\mathrm{re}} \cap \Delta_{\pm}$ and $\Delta^{\mathrm{im}}_{\pm} = \Delta^{\mathrm{im}} \cap \Delta_{\pm}$. One has

$$\Delta^{\mathrm{re}}_+ = \{\alpha + n\delta; \alpha \in \bar{\Delta}_+,\ n \geq 0\} \sqcup \{-\alpha + n\delta; \alpha \in \bar{\Delta}_+,\ n \geq 1\}.$$

Let Q be the root lattice, $Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0}\alpha \subset Q$. We define a partial ordering $\mu \leq \lambda$ on $\mathfrak{h}^*$ by $\lambda - \mu \in Q_+$.

Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$ generated by the reflections s_α with $\alpha \in \Delta^{\mathrm{re}}$, where $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\lambda \in \mathfrak{h}^*$. The dot action of W on $\mathfrak{h}^*$ is defined by $w \circ \lambda = w(\lambda + \rho) - \rho$, where $\rho = \bar{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$ and $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$.

For $\lambda \in \mathfrak{h}^*$, let

(33) $$\Delta(\lambda) = \{\alpha \in \Delta^{\mathrm{re}}; \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\},$$

(34) $$\Delta_+(\lambda) = \Delta(\lambda) \cap \Delta_+,$$

(35) $$W(\lambda) = \langle s_\alpha; \alpha \in \Delta(\lambda) \rangle \subset W.$$

One knows that $W(\lambda)$ is a Coxeter subgroup of W, and $W(\lambda)$ is called the *integral Weyl group* of $\lambda \in \mathfrak{h}^*$. Let $\ell_\lambda : W(\lambda) \to \mathbb{Z}_{\geq 0}$ be its length function.

For an $\mathfrak{h}$-module M let M^λ be the weight space of weight $\lambda \in \mathfrak{h}^*$:

$$M^\lambda = \{m \in M; am = \lambda(a)m\ \forall a \in \mathfrak{h}\}.$$

We say M admits a weight space decomposition if $M = \bigoplus_\lambda M^\lambda$ and $\dim M^\lambda < \infty$ for all λ. In this case we define the graded dual M^* of M by

(36) $$M^* = \bigoplus_\lambda \mathrm{Hom}_{\mathbb{C}}(M^\lambda, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}).$$

Also, we set[7]

$$M_d = \{m \in M; -Dm = dm\}, \tag{37}$$

$$\mathsf{D}(M) = \bigoplus_{d\in\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(M_d, \mathbb{C}) \tag{38}$$

for a semisimple $\mathbb{C}D$-module M. Note that if M is a $\mathfrak{g}$-module then M_d is a $\bar{\mathfrak{g}}$-submodule of M for any d.

Lemma 3.3.1. *Let M be a $\mathfrak{h}$-module that admits a weight space decomposition. Suppose that M_d is finite-dimensional for all d. Then* $\mathsf{D}(M) = M^*$.

3.4. Universal affine vertex algebras

For $k \in \mathbb{C}$ define the $\mathfrak{g}$-module $V^k(\bar{\mathfrak{g}})$ by

$$V^k(\bar{\mathfrak{g}}) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{g}}[t]\oplus\mathbb{C}K\oplus\mathbb{C}D)} \mathbb{C}_k, \tag{39}$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\bar{\mathfrak{g}}[t]\oplus\mathbb{C}K\oplus\mathbb{C}D$ on which $\bar{\mathfrak{g}}[t]\oplus\mathbb{C}D$ acts trivially and K acts as the multiplication by k.

Define a field $J(z)$ on $V^k(\bar{\mathfrak{g}})$ for $J \in \bar{\mathfrak{g}}$ by

$$J(z) = \sum_{n\in\mathbb{Z}} J(n) z^{-n-1}. \tag{40}$$

There is a unique vertex algebra structure on $V^k(\bar{\mathfrak{g}})$ such that $\mathbf{1} = 1\otimes 1 \in V^k(\bar{\mathfrak{g}})$ is the vacuum vector and $\{J(z); J \in \bar{\mathfrak{g}}\}$ is a set of generating fields. The vertex algebra $V^k(\bar{\mathfrak{g}})$ is called the *universal affine vertex algebra* associated with $\bar{\mathfrak{g}}$ at level k.

3.5. Clifford vertex algebras

Set

$$L\bar{\mathfrak{g}}_{>0} = \bar{\mathfrak{g}}_{>0}\otimes\mathbb{C}[t,t^{-1}], \quad L\bar{\mathfrak{g}}_{<0} = \bar{\mathfrak{g}}_{<0}\otimes\mathbb{C}[t,t^{-1}]. \tag{41}$$

They are nilpotent subalgebras of $\mathfrak{g}$.

Let $\mathcal{C}l$ be the Clifford algebra associated with $L\bar{\mathfrak{g}}_{<0}\oplus L\bar{\mathfrak{g}}_{>0}$ and the restriction of $(\ |\ )$ to $L\bar{\mathfrak{g}}_{<0}\oplus L\bar{\mathfrak{g}}_{>0}$. The Clifford algebra $\mathcal{C}l$ is identified with the superalgebra defined by the following generators and relations:

generators: $\psi_\alpha(n)$ $\quad(\alpha \in \bar{\Delta}_{\neq 0}, n \in \mathbb{Z})$,

relations: $\psi_\alpha(n)$ is odd,

$$[\psi_\alpha(m), \psi_\beta(n)] = \delta_{\alpha+\beta,0}\delta_{m+n,0} \quad (\alpha,\beta \in \bar{\Delta}_{\neq 0}, m,n \in \mathbb{Z}).$$

[7]This differs from the notation in [Ara07].

Here $\bar{\Delta}_{\neq 0} = \bar{\Delta}_{<0} \sqcup \bar{\Delta}_{>0}$ and the bracket $[\ ,\ ]$ denotes the supercommutator.

The algebra $\mathcal{C}l$ contains the Grassmann algebras $\bigwedge(L\bar{\mathfrak{g}}_{<0})$ and $\bigwedge(L\bar{\mathfrak{g}}_{>0})$ as its subalgebras; $\bigwedge(L\bar{\mathfrak{g}}_{<0}) = \langle \psi_\alpha(n); \alpha \in \bar{\Delta}_{<0}, n \in \mathbb{Z}\rangle$, $\bigwedge(L\bar{\mathfrak{g}}_{>0}) = \langle \psi_\alpha(n); \alpha \in \bar{\Delta}_{>0}, n \in \mathbb{Z}\rangle$. One has

$$\mathcal{C}l = \bigwedge(L\bar{\mathfrak{g}}_{>0}) \otimes \bigwedge(L\bar{\mathfrak{g}}_{<0}) \tag{42}$$

as linear spaces.

In view of (42), the adjoint action of $\mathfrak{h}$ on $L\bar{\mathfrak{g}}_{<0} \oplus L\bar{\mathfrak{g}}_{>0}$ induces an action of $\mathfrak{h}$ on $\mathcal{C}l$: $\mathcal{C}l = \bigoplus_{\lambda \in Q} \mathcal{C}l^\lambda$.

Let $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ be the irreducible representation of $\mathcal{C}l$ generated by the vector $\mathbf{1}$ such that

$$\psi_\alpha(n)\mathbf{1} = 0 \quad \text{if } \alpha + n\delta \in \Delta_+^{\text{re}}. \tag{43}$$

Then $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0}) = \bigwedge(L\bar{\mathfrak{g}}_{<0} \cap \mathfrak{n}_-) \otimes \bigwedge(L\bar{\mathfrak{g}}_{>0} \cap \mathfrak{n}_-)$ as linear spaces. We regard $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ as an $\mathfrak{h}$-module under this identification:

$$\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0}) = \bigoplus_{\lambda \in -Q_+} \left(\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})\right)^\lambda.$$

The space $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ is $\mathbb{Z}$-graded by *charge*:

$$\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0}) = \bigoplus_{i \in \mathbb{Z}} \bigwedge^{\frac{\infty}{2}+i}(L\bar{\mathfrak{g}}_{>0}). \tag{44}$$

The charge of $\mathbf{1}$, $\psi_\alpha(n)$ and $\psi_{-\alpha}(n)$ with $\alpha \in \bar{\Delta}_{>0}$ and $n \in \mathbb{Z}$ are 0, -1 and 1, respectively. The $\mathcal{C}l$-module $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ is often called the *space of semi-infinite forms* on $L\bar{\mathfrak{g}}_{>0}$.

There is a unique vertex (super)algebra structure on $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ such that $\mathbf{1}$ is the vacuum vector, and

$$\psi_\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_\alpha(n) z^{-n-1} \quad \text{with } \alpha \in \bar{\Delta}_{>0}, \tag{45}$$

$$\psi_\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_\alpha(n) z^{-n} \quad \text{with } \alpha \in \bar{\Delta}_{<0} \tag{46}$$

are generating fields. The vertex algebra $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ is also called the *Clifford vertex algebra* associated with $L\bar{\mathfrak{g}}_{>0}$.

3.6. The W-algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$

Because both $V^k(\bar{\mathfrak{g}})$ and $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ are vertex algebras, the tensor product

$$C^\bullet = V^k(\bar{\mathfrak{g}}) \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0}) \tag{47}$$

is also a vertex algebra. Set $C^i = V^k(\bar{\mathfrak{g}}) \otimes \bigwedge^{\frac{\infty}{2}+i}(L\bar{\mathfrak{g}}_{>0})$, so that

$$C^\bullet = \bigoplus_{i \in \mathbb{Z}} C^i. \tag{48}$$

Let $Q(z)$ be the odd field on $C^\bullet$ defined by

$$Q(z) = Q^{\mathrm{st}}(z) + \chi(z), \tag{49}$$

where

$$Q^{\mathrm{st}}(z) = \sum_{\alpha \in \bar{\Delta}_{>0}} J_\alpha(z)\psi_{-\alpha}(z) - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \bar{\Delta}_{>0}} c_{\alpha,\beta}^{\gamma} \psi_{-\alpha}(z)\psi_{-\beta}(z)\psi_\gamma(z),$$

$$\chi(z) = \sum_{\alpha \in \bar{\Delta}_{>0}} \bar{\chi}(J_\alpha)\psi_{-\alpha}(z)$$

($\bar{\chi}$ is defined in (24)). One has

$$[Q(z), Q(w)] = 0, \tag{50}$$

and therefore,

$$Q_{(0)}^2 = 0 \tag{51}$$

because $Q(z)$ is odd. (Recall that $Q_{(0)} = \operatorname{Res}_z Q(z)$, see §2.1.)
Since $Q_{(0)} C^i \subset C^{i+1}$, $(C^\bullet, Q_{(0)})$ is a cochain complex.

Definition 3.6.1. *The universal affine W-algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ associate with $(\bar{\mathfrak{g}}, f)$ at level k is the zeroth cohomology of the complex $(C^\bullet, Q_{(0)})$:*

$$\mathcal{W}^k(\bar{\mathfrak{g}}, f) = H^0(C^\bullet, Q_{(0)}). \tag{52}$$

The W-algebra $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ inherits the vertex algebra structure from $C^\bullet$.

3.7. The Hamiltonian of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$

Set

$$H = -D - \frac{1}{2}h_0, \tag{53}$$

where h_0 is the element in the $\mathfrak{sl}_2$-triple $\{e, h_0, f\}$ fixed in §3.1. The right-hand-side is considered as an element of $\mathfrak{h}$ which acts diagonally on the complex $\mathcal{C}^\bullet = V^k(\bar{\mathfrak{g}}) \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$.

One knows that H defines a Hamiltonian on $\mathcal{C}^\bullet$ (cf. §4.9 of [Kac98]).

Lemma 3.7.1. *One has* $[H, Q_{(0)}] = 0$*,*

Proof. Obviously $[H, Q^{\mathrm{st}}_{(0)}] = 0$. Also,

$$\alpha(h_0) = 2 \text{ for all } \alpha \text{ such that } \bar{\chi}(J_\alpha) \neq 0. \tag{54}$$

This gives $[H, \chi_{(0)}] = 0$. Therefore $[H, Q_{(0)}] = 0$. Q.E.D.

From Lemma 3.7.1 it follows that H defines a Hamiltonian of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$. One has

$$\mathcal{W}^k(\bar{\mathfrak{g}}, f) = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}} \mathcal{W}^k(\bar{\mathfrak{g}}, f)_\Delta,$$

$$\mathcal{W}^k(\bar{\mathfrak{g}}, f)_\Delta = \{a \in \mathcal{W}^k(\bar{\mathfrak{g}}, f); Ha = \Delta a\}. \tag{55}$$

3.8. Generating fields of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$

Set

$$\widehat{J}_a(z) = \sum_{n\in\mathbb{Z}} \widehat{J}_a(n) z^{-n-1} = J_a(z) - \sum_{\beta,\gamma\in\Delta_{>0}} c^\gamma_{\alpha,\beta} : \psi_{-\beta}(z)\psi_\gamma(z) : \tag{56}$$

for $a \in \bar{I} \sqcup \bar{\Delta}$. One has [KW04, (2.5)] on $\mathcal{C}^\bullet$

$$\begin{aligned} &[\widehat{J}_a(m), \widehat{J}_b(n)] \\ &= \sum_d c^d_{a,b} \widehat{J}_d(m+n) + \left((k+h^\vee)(a|b) - \frac{1}{2}\kappa_{\bar{\mathfrak{g}}_0}(a,b)\right) m\delta_{m+n,0}\,\mathrm{id}, \end{aligned} \tag{57}$$

$$[\widehat{J}_a(m), \psi_\alpha(n)] = \sum_d c^\beta_{a,\alpha} \psi_\beta(m+n) \tag{58}$$

provided that either $a, b \in \bar{\Delta}_{\geq 0} \sqcup \bar{I}$ and $\alpha \in \bar{\Delta}_{>0}$, or $a, b \in \bar{\Delta}_{\leq 0} \sqcup \bar{I}$ and $\alpha \in \bar{\Delta}_{<0}$, where $\kappa_{\bar{\mathfrak{g}}_0}(a,b)$ is the Killing form of $\bar{\mathfrak{g}}_0$.

Let $\mathcal{C}_+^\bullet$ be the vertex subalgebra of $\mathcal{C}^\bullet$ generated by the fields $\widehat{J}_a(z)$ and $\psi_\beta(z)$ with $a \in \bar{I} \sqcup \bar{\Delta}_{\leq 0}$ and $\beta \in \bar{\Delta}_{<0}$. Here $\bar{\Delta}_{\leq 0} = \bigcup_{j \leq 0} \bar{\Delta}_j$. By (57) and (58), $\mathcal{C}_+^\bullet$ is spanned by the vectors

$$\widehat{J}_{a_1}(m_1) \dots \widehat{J}_{a_r}(m_r)\psi_{\beta_1}(n_1) \dots \psi_{\beta_s}(n_s)\mathbf{1}$$

with $a_i \in \bar{I} \sqcup \bar{\Delta}_{\leq 0}$, $\beta_i \in \bar{\Delta}_{<0}$, $m_i, n_i \in \mathbb{Z}$.

Similarly let $\mathcal{C}_-^\bullet$ be the vertex subalgebra of $\mathcal{C}^\bullet$ generated by the fields $\widehat{J}_\alpha(z)$ and $\psi_\beta(z)$ with $\alpha, \beta \in \bar{\Delta}_{>0}$. Then $\mathcal{C}_-^\bullet$ is spanned by the vectors

$$\widehat{J}_{\alpha_1}(m_1) \dots \widehat{J}_{\alpha_r}(m_r)\psi_{\beta_1}(n_1) \dots \psi_{\beta_s}(n_s)\mathbf{1}$$

with $\alpha_i \in \bar{\Delta}_{>0}$, $\beta_i \in \bar{\Delta}_{>0}$, $m_i, n_i \in \mathbb{Z}$.

One has the linear isomorphism

$$\mathcal{C}^\bullet \cong \mathcal{C}_-^\bullet \otimes \mathcal{C}_+^\bullet. \tag{59}$$

Moreover it was shown [KW04] (cf. [dBT94, FBZ04]) that both $\mathcal{C}_\pm^\bullet$ are subcomplexes of $\mathcal{C}^\bullet$, and that

$$H^i(\mathcal{C}_-^\bullet) = \begin{cases} \mathbb{C} & (i = 0) \\ 0 & (i \neq 0) \end{cases}.$$

Therefore by the Künneth theorem

$$H^\bullet(\mathcal{C}^\bullet) = H^\bullet(\mathcal{C}_+^\bullet). \tag{60}$$

It follows that we may identify $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ with the vertex subalgebra $H^0(\mathcal{C}_+^\bullet)$ of $\mathcal{C}^\bullet$ (Note that the cohomological gradation takes only non-negative values on $\mathcal{C}_+^\bullet$.):

$$\mathcal{W}^k(\bar{\mathfrak{g}}, f) = H^0(\mathcal{C}_+^\bullet) \subset \mathcal{C}^\bullet. \tag{61}$$

Let $\bar{\mathfrak{g}}^f_{\mathrm{aff}} = \bar{\mathfrak{g}}^f \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ be the central extension of the Lie algebra $\bar{\mathfrak{g}}^f \otimes \mathbb{C}[t, t^{-1}]$ with respect to the 2-cocycle ϕ_k, defined by $\phi_k(a, b) = (k + h^\vee)(a|b) - \frac{1}{2}\kappa_{\bar{\mathfrak{g}}_0}(a, b)$. Set

$$V^{\phi_k}(\bar{\mathfrak{g}}^f) = U(\bar{\mathfrak{g}}^f_{\mathrm{aff}}) \otimes_{U(\bar{\mathfrak{g}}^f \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}$$

where $\mathbb{C}$ is the $\bar{\mathfrak{g}}^f \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{1}$-module on which $\bar{\mathfrak{g}}^f \otimes \mathbb{C}[t]$ acts trivially and $\mathbf{1}$ acts as 1.

By (57) one can regard $V^{\phi_k}(\bar{\mathfrak{g}}^f)$ as a vertex subalgebra of $V^k(\bar{\mathfrak{g}})$.

Theorem 3.8.1 ([KW04]). *For any $k \in \mathbb{C}$ one has the following.*

(i) *It holds that* $H^i(C^\bullet_+) = 0$ *for all* $i \neq 0$. *Therefore* $H^i(C^\bullet) = 0$ *for all* $i \neq 0$.
(ii) *There exists an exhaustive, separated filtration* $\{F_p\mathcal{W}^k(\bar{\mathfrak{g}}, f)\}$ *of the vertex algebra* $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ *such that*

$$\mathrm{gr}^F\, \mathcal{W}^k(\bar{\mathfrak{g}}, f) \cong V^{\phi_k}(\bar{\mathfrak{g}}^f)$$

as graded vertex algebras.

Remark 3.8.2. The filtration in Theorem 3.8.1 arises from the spectral sequence associated with the filtration of $C^\bullet_+$ defined by

$$F_pC^n_+ = \bigoplus_{\langle\lambda, x_0\rangle \geq p-n} (C^n_+)^\lambda$$

(cf. §4 of [Ara07]).

Because $\bar{\mathfrak{g}}^f$ is preserved by the adjoint action of x_0 and h_0, there exists a basis $\{u_j; j = 1, \dots, \dim \bar{\mathfrak{g}}^f\}$ of $\bar{\mathfrak{g}}^f$ consisting of simultaneous eigenvectors of $\mathrm{ad}\, x_0$ and $\mathrm{ad}\, h_0$. Let $d_j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ be the half of the eigenvalue of $\mathrm{ad}\, h_0$ on u_j:

$$[h_0, u_j] = -2d_j u_j. \tag{62}$$

By Theorem 3.8.1 there exist homogeneous elements $\mathsf{W}^{(j)}$ of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ with $j = 1, \dots, \dim \bar{\mathfrak{g}}^f$ whose symbols are $u_j(-1)\mathbf{1}$, and $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ is (strongly [Kac98]) generated by the fields

$$\mathsf{W}^{(j)}(z) = Y(\mathsf{W}^{(j)}, z) \tag{63}$$

in $C^\bullet$. The vector $\mathsf{W}^{(j)}$ has the conformal weight $1 + d_j$. Thus it follows that $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ is positively graded:

$$\mathcal{W}^k(\bar{\mathfrak{g}}, f) = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{W}^k(\bar{\mathfrak{g}}, f)_\Delta, \quad \mathcal{W}^k(\bar{\mathfrak{g}}, f)_0 = \mathbb{C}\mathbf{1}. \tag{64}$$

§4. Ramond twisted representation of affine W-algebras

4.1. Ramond twisted representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$

Let $\sigma_R : C^\bullet \to C^\bullet$ be the automorphism of order ≤ 2 defined by

$$\sigma_R = e^{\pi i h_0}. \tag{65}$$

By (54), σ_R fixes the vector $Q = Q_{(-1)}\mathbf{1}$. Therefore [KW05] σ_R defines an automorphism of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$.

A σ_R-twisted representation of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ is called a *Ramond twisted representation* of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$.

Note that $\sigma_R = \sigma_H$ (see §2.4 and (53)). Therefore Ramond twisted representations are exactly the H-twisted representations.

Remark 4.1.1. If the nilpotent element f is even then σ_R is trivial. In this case a Ramond twisted representations are usual (non-twisted) representations.

Proposition 4.1.2. *Let M be a σ_R-twisted representation of $C^\bullet$. Then the space*

$$\frac{\ker((Q)^M_{(0)} : M \to M)}{\operatorname{im}((Q)^M_{(0)} : M \to M)}$$

is naturally a Ramond twisted representation of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$.

Proof. By (11) and (51), the square of $(Q)^M_{(0)}$ is equal to zero. Therefore the above space is well-defined. The rest also follows from (11). Q.E.D.

4.2. σ_R-twisted representations of $C^\bullet$

Set

$$\bar{\mathfrak{g}}_j^{\mathrm{Dyn}} = \{x \in \bar{\mathfrak{g}}; [h_0, x] = 2jx\}.$$

Then $\bar{\mathfrak{g}} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \bar{\mathfrak{g}}_j^{\mathrm{Dyn}}$ gives a good grading for f, called the *Dynkin grading* [KRW03].

Let

$$\mathfrak{g}^R = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \bar{\mathfrak{g}}_{\bar{j}}^{\mathrm{Dyn}} \otimes \mathbb{C}t^j \oplus \mathbb{C}K \oplus \mathbb{C}D$$

be the σ_R-twisted affine Lie algebra [Kac90], where $\bar{\mathfrak{g}}_{\bar{j}}^{\mathrm{Dyn}} = \bigoplus_{\substack{r \in \frac{1}{2}\mathbb{Z} \\ r \equiv j \pmod{\mathbb{Z}}}} \bar{\mathfrak{g}}_r^{\mathrm{Dyn}}$, and the commutation relations are given by the same formula as $\mathfrak{g}$.

The nilpotent element f is called *even* if $\bar{\mathfrak{g}}_i^{\mathrm{Dyn}} = 0$ for all $i \in 1/2 + \mathbb{Z}$. In such a case the twisted affine Lie algebra $\mathfrak{g}^R$ coincides with the non-twisted one.

We write $J(n)^R$ for $J \otimes t^n \in \mathfrak{g}^R$. Also, to avoid confusion we write K^R and D^R for K and D in $\mathfrak{g}^R$, respectively.

Lemma 4.2.1. *Let M be a vector space. Defining a σ_R-twisted $V^k(\mathfrak{g})$-module structure on M is equivalent to defining a $\mathfrak{g}^R$-module structure on M of level k such that $J(n)^R m = 0$ for all $m \in M$ and $n \gg 0$.*

Proof. By (12), given a σ_R-twisted module structure on M one has

$$(66)\qquad \begin{aligned}&[Y^M(J(-1)\mathbf{1},z),Y^M(J'(-1)\mathbf{1},w)]\\&=Y^M([J,J'](-1)\mathbf{1},w)\delta_j(z-w)+k(J|J')\,\mathrm{id}_M\,\partial_w\delta_j(z-w)\end{aligned}$$

for $J\in\bar{\mathfrak{g}}^{\mathrm{Dyn}}_{\bar{j}}$, $J'\in\bar{\mathfrak{g}}$. It follows that the correspondence $J(n)^R\mapsto(J(-1)\mathbf{1})^M_{(n)}$ define a representation of $\mathfrak{g}^R$ on M of level k with $J(n)^R m=0$ for $m\in M$ and $n\gg 0$.

Conversely, suppose that we are given a $\mathfrak{g}^R$-module structure on M of level k such that $J(n)^R m=0$ for $m\in M$ and $n\gg 0$. Define a 2-twisted filed $J(z)^R$ on M by

$$(67)\qquad J(z)^R=\sum_{n\in j+\mathbb{Z}}J(n)^R z^{-n-1}\quad\text{for } J\in\bar{\mathfrak{g}}^{\mathrm{Dyn}}_{\bar{j}}.$$

These fields satisfy the same formula as (66):

$$(68)\qquad [J(z)^R,J'(w)^R]=[J,J'](w)^R\delta_j(z-w)+k(J|J')\,\mathrm{id}_M\,\partial_w\delta_j(z-w).$$

for $J\in\bar{\mathfrak{g}}^{\mathrm{Dyn}}_{\bar{j}}$ and $J'\in\bar{\mathfrak{g}}$.

Let V be a vertex algebra generated by $J(z)^R$ with $J\in\bar{\mathfrak{g}}$ in the space of 2-twisted fields on M in the sense of Li [Li96]. By (68) it follows that the correspondence $J(-1)\mathbf{1}\mapsto J(z)^R$ defines a vertex algebra homomorphism from $V^k(\bar{\mathfrak{g}})$ to V (cf. (13)). Thanks to Proposition 3.17 of [Li96], this completes the proof.

Q.E.D.

Let $\mathcal{C}l^R$ be the superalgebra generated by the odd fields $\psi_\alpha(n)^R$ ($\alpha\in\bar{\Delta}_{\neq 0}:=\bar{\Delta}_{>0}\sqcup\bar{\Delta}_{<0}$, $n\in\alpha(h_0)/2+\mathbb{Z}$) with the relations $[\psi_\alpha(m)^R,\psi_\beta(n)^R]=\delta_{m+n,0}\delta_{\alpha+\beta,0}$.

The proof of the following assertion is the same as that of Lemma 4.2.1.

Lemma 4.2.2. *Let M be a $\mathcal{C}l^R$-module such that $\psi_\alpha(n)^R m=0$ for all $m\in M$, $\alpha\in\bar{\Delta}_{\neq 0}$ and $n\gg 0$. Then the formulas*

$$Y^M(\psi_\alpha(-1)\mathbf{1},z)=\psi_\alpha(z)^R=\sum_{n\in\alpha(h_0)/2+\mathbb{Z}}\psi_\alpha(n)^R z^{-n-1}\quad(\alpha\in\bar{\Delta}_{>0}),$$

$$Y^M(\psi_\alpha(0)\mathbf{1},z)=\psi_\alpha(z)^R=\sum_{n\in\alpha(h_0)/2+\mathbb{Z}}\psi_\alpha(n)^R z^{-n}\quad(\alpha\in\bar{\Delta}_{<0})$$

defines a σ_R-twisted $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$-module structure on M.

Set $U_k(\mathfrak{g}^R) = U(\mathfrak{g}^R)/\langle K - k1\rangle$. Let M be a $U_k(\mathfrak{g}^R)\otimes\mathcal{C}l^R$-module such that $J(n)^R m = \psi_\alpha(n)^R m = 0$ for $n \gg 0$, $m \in M$, $J \in \bar{\mathfrak{g}}$ and $\alpha \in \bar{\Delta}_{\neq 0}$. Then by Lemmas 4.2.1 and 4.2.2, M can be naturally considered as a σ_R-twisted representation of $\mathcal{C}^\bullet$. By Proposition 4.1.2, the space $\ker(Q)^M_{(0)}/\operatorname{im}(Q)^M_{(0)}$ is a Ramond twisted representation of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$. One has

$$(Q)^M_{(0)} = (Q^{\mathrm{st}})^M_{(0)} + \chi^M_{(0)}, \tag{69}$$

where $(Q^{\mathrm{st}})^M_{(0)}$ and $\chi^M_{(0)}$ are explicitly expressed as follows:

$$\begin{aligned}(Q^{\mathrm{st}})^M_{(0)} = &\sum_{\substack{\alpha\in\bar{\Delta}_{>0}\\ n\in\alpha(h_0)/2+\mathbb{Z}}} J_\alpha(n)^M\psi_{-\alpha}(-n)^M\\ &-\frac{1}{2}\sum_{\substack{\alpha,\beta,\gamma\in\bar{\Delta}_{>0}\\ k\in\alpha(h_0)/2+\mathbb{Z}, l\in\beta(h_0)/2+\mathbb{Z}}} c^\gamma_{\alpha,\beta}\psi_{-\alpha}(-k)^M\psi_{-\beta}(-l)^M\psi_\gamma(k+l)^M,\\ \chi^M_{(0)} = &\sum_{\alpha\in\bar{\Delta}_{>0}} \bar{\chi}(J_\alpha)\psi_{-\alpha}(1)^M.\end{aligned}$$

4.3. Identification with non-twisted representations

The superalgebra $U(\mathfrak{g}^R)\otimes\mathcal{C}l^R$ is isomorphic to $U(\mathfrak{g})\otimes\mathcal{C}l$ [KW08]: the isomorphism is given by:

$$\begin{aligned}\widehat{t}_{-\frac{1}{2}h_0} : &J_\alpha(n)^R \mapsto J_\alpha(n+\alpha(h_0)/2) && (\alpha\in\bar{\Delta}),\\ &J_i(n)^R \mapsto J_i(n) + \frac{1}{2}\delta_{n,0}(h_0|J_i)K && (i\in\bar{I},\ n\in\mathbb{Z}),\\ &K^R \mapsto K,\\ &D^R \mapsto D - \frac{1}{2}h_0(0),\\ &\psi_\alpha(n)^R \mapsto \psi_\alpha(n+\alpha(h_0)/2) && (\alpha\in\bar{\Delta}_{\neq 0}, n\in\mathbb{Z}),\end{aligned}$$

Set $U_k(\mathfrak{g}) = U(\mathfrak{g})/\langle K - k\rangle$ for $k \in \mathbb{C}$. Let $\widehat{w}_0 \in \operatorname{Aut}(U_k(\mathfrak{g})\otimes\mathcal{C}l)$ be a lift of the longest element w_0 of the Weyl group $\bar{W}$ of $\bar{\mathfrak{g}}$ such that $\widehat{w}_0(J_\alpha(n)) = c_{w_0(\alpha)}J_{w_0(\alpha)}(n)$ $(\alpha \in \bar{\Delta})$, $\widehat{w}_0(\psi_\alpha(n)) = c_{w_0(\alpha)}\psi_{w_0(\alpha)}(n)$ $(\alpha \in \bar{\Delta}_{>0})$, $\widehat{w}_0(\psi_{-\alpha}(n)) = c^{-1}_{w_0(\alpha)}\psi_{-w_0(\alpha)}(n)$ $(\alpha \in \bar{\Delta}_{>0})$ with $c_\alpha \in \mathbb{C}^*$. Set

$$\widehat{y}_0 = \widehat{w}_0\widehat{t}_{-\frac{1}{2}h_0}. \tag{70}$$

Then $\widehat{y}_0$ defines an isomorphism $U_k(\mathfrak{g}^R)\otimes\mathcal{C}l^R \xrightarrow{\sim} U_k(\mathfrak{g})\otimes\mathcal{C}l$.

Let M be a (non-twisted) positive energy representation of $V^k(\bar{\mathfrak{g}})$. Then the space $M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ can be regarded as a σ_R-twisted representation of $U_k(\bar{\mathfrak{g}}) \otimes \mathcal{C}l$, by the action

$$u \cdot m = \widehat{y}_0(u)m \tag{71}$$

for $m \in M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ and $u \in U_k(\mathfrak{g}^R) \otimes \mathcal{C}l^R$. We have

$$(Q)_{(0)}^{M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})} = \mathbf{Q}_- = \mathbf{Q}_-^{\mathrm{st}} + \chi_-,$$

where

$$\mathbf{Q}_-^{\mathrm{st}} = \sum_{\substack{n \in \mathbb{Z} \\ \alpha \in \bar{\Delta}_{<0}}} J_\alpha(-n)\psi_{-\alpha}(n) - \frac{1}{2} \sum_{\substack{k,l \in \mathbb{Z} \\ \alpha,\beta,\gamma \in \bar{\Delta}_{<0}}} c_{\alpha,\beta}^{\gamma} \psi_{-\alpha}(-k)\psi_{-\beta}(-l)\psi_\gamma(k+l), \tag{72}$$

$$\chi_- = \sum_{\alpha \in \bar{\Delta}_{<0}} c_\alpha^{-1} \bar{\chi}(J_{-\alpha})\psi_{-\alpha}(0). \tag{73}$$

One has

$$\mathbf{Q}_-(M \otimes \bigwedge^{\frac{\infty}{2}+i}(L\bar{\mathfrak{g}}_{>0})) \subset M \otimes \bigwedge^{\frac{\infty}{2}+i-1}(L\bar{\mathfrak{g}}_{>0}).$$

It follows that by Proposition 4.1.2 the homology space

$$H_\bullet^{\mathrm{BRST}}(M) := H_\bullet(M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0}), \mathbf{Q}_-)$$

can be considered as a Ramond twisted representation of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$.

The σ_R-twisted representation $M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ of $U_k(\bar{\mathfrak{g}}) \otimes \mathcal{C}l$ is graded by the Hamiltonian $-D$, which acts on it diagonally. Obviously D commutes with $\mathbf{Q}_-$, and hence $H_\bullet^{\mathrm{BRST}}(M)$ is graded by the Hamiltonian $-D$. It follows that we have obtained the functor

$$V^k(\bar{\mathfrak{g}})\text{-}\mathfrak{Mod} \to \mathcal{W}^k(\bar{\mathfrak{g}}, f)\text{-}\mathfrak{Mod}, \quad M \mapsto H_0^{\mathrm{BRST}}(M). \tag{74}$$

Remark 4.3.1. One has $H_\bullet^{\mathrm{BRST}}(M) = H_{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{<0}, M \otimes \mathbb{C}_{\chi_-})$, where the right-hand-side is the Feigin's semi-infinite $L\bar{\mathfrak{g}}_{<0}$-homology [Feĭ84] with the coefficient in the $L\bar{\mathfrak{g}}_{<0}$-module $M \otimes \mathbb{C}_{\chi_-}$, and χ_- is identified with the character of $L\bar{\mathfrak{g}}_{<0}$ such that $\chi_-(J_{-\alpha}(n)) = \delta_{n,0} c_\alpha^{-1} \bar{\chi}(J_\alpha)$.

4.4. Finite W-algebras as H-twisted Zhu algebras

Let $M \in \mathcal{W}^k(\bar{\mathfrak{g}}, f)\text{-}\mathfrak{Mod}_{\sigma_R}$ and suppose that M_{top} is concentrated in one degree: $M_{\text{top}} = M_{d_0}$. Then

$$H^{\text{BRST}}_{\bullet}(M) = \bigoplus_{d \in d_0 + \mathbb{Z}_{\geq 0}} H^{\text{BRST}}_{\bullet}(M)_d,$$

and therefore,

$$H^{\text{BRST}}_{\bullet}(M)_{\text{top}} = H^{\text{BRST}}_{\bullet}(M)_{d_0}, \tag{75}$$

provided that $H^{\text{BRST}}_{\bullet}(M)_{d_0} \neq 0$.

In this case $H^{\text{BRST}}_{\bullet}(M)_{\text{top}}$ is easily described as follows. Identify the Grassmann algebra $\bigwedge^{\bullet}(\bar{\mathfrak{g}}_{<0})$ of $\bar{\mathfrak{g}}_{<0}$ with the subalgebra of $\mathcal{C}l$ generated by $\psi_\alpha(0)$ with $\alpha \in \bar{\Delta}_{<0}$. Then $\bigwedge^{\bullet}(\bar{\mathfrak{g}}_{<0})$ is also identified with the subspace $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})_{\text{top}}$ of $\bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$. One has

$$H^{\text{BRST}}_{0}(M)_{\text{top}} = H_0(M_{\text{top}} \otimes \bigwedge^{\bullet}(\bar{\mathfrak{g}}_{<0}), \mathbf{Q}_-). \tag{76}$$

One sees that the operator $\mathbf{Q}_-$ acts on $M_{\text{top}} \otimes \bigwedge^{\bullet}(\bar{\mathfrak{g}}_{<0})$ as

$$\begin{aligned} \bar{\mathbf{Q}}_- = & \sum_{\alpha \in \bar{\Delta}_{<0}} (J_\alpha(0) + c_\alpha^{-1} \chi(J_{-\alpha}))\psi_{-\alpha}(0) \\ & - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \bar{\Delta}_{<0}} c^{\gamma}_{\alpha,\beta} \psi_{-\alpha}(0)\psi_{-\beta}(0)\psi_{\gamma}(0). \end{aligned} \tag{77}$$

From this formula it follows that the complex $(M_{\text{top}} \otimes \bigwedge^{\bullet}(\bar{\mathfrak{g}}_{<0}), \mathbf{Q}_-)$ is identical to the Chevalley–Eilenberg complex which defines the Lie algebra $\bar{\mathfrak{g}}_{<0}$-homology $H^{\text{Lie}}_{\bullet}(\bar{\mathfrak{g}}_{<0}, M_{\text{top}} \otimes \mathbb{C}_{\bar{\chi}_-})$ with the coefficient in the $\bar{\mathfrak{g}}_{<0}$-module $M \otimes \mathbb{C}_{\bar{\chi}_-}$, where $\mathbb{C}_{\bar{\chi}_-} = U(\bar{\mathfrak{g}}_{<0}) / \ker \bar{\chi}_-$ and $\bar{\chi}_-$ is the character of $\bar{\mathfrak{g}}_{<0}$ defined by

$$\bar{\chi}_-(J_\alpha) = c_\alpha^{-1} \chi(J_{-\alpha}).$$

Thus one has

$$H^{\text{BRST}}_{\bullet}(M)_{\text{top}} = H^{\text{Lie}}_{\bullet}(\bar{\mathfrak{g}}_{<0}, M_{\text{top}} \otimes \mathbb{C}_{\chi_-}). \tag{78}$$

This in particular means that $H^{\text{Lie}}_{\bullet}(\bar{\mathfrak{n}}_-, M_{\text{top}} \otimes \mathbb{C}_{\chi_-})$ is a module over $\text{Zh}_H(\mathcal{W}^k(\bar{\mathfrak{g}}, f))$.

Recall [DSK06] that

$$\text{Zh}_H(\mathcal{W}^k(\bar{\mathfrak{g}}, f)) \cong \mathcal{W}^{\text{fin}}(\bar{\mathfrak{g}}, f), \tag{79}$$

where $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$ is the *finite W-algebra* associated with $(\bar{\mathfrak{g}}, f)$. The finite W-algebra $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$ may be defined by means of the quantum BRST reduction [KS87, DDCDS+06]: Let $\overline{\mathcal{C}l}$ be the Clifford algebra associated with $\bar{\mathfrak{g}}_{<0} \oplus \bar{\mathfrak{g}}_{>0}$ and $(\ |\ )_{|\bar{\mathfrak{g}}_{<0} \oplus \bar{\mathfrak{g}}_{>0}}$. We identify $\overline{\mathcal{C}l}$ with the subalgebra of $\mathcal{C}l$ generated by $\psi_\alpha = \psi_\alpha(0)$ with $\alpha \in \bar{\Delta}_{\neq 0}$. One has the subalgebra $U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$ in $U_k(\mathfrak{g}) \otimes \mathcal{C}l$, and $\bar{\mathbf{Q}}_-$ can be considered as an odd element of $U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$. One has $(\bar{\mathbf{Q}}_-)^2 = 0$, and thus

$$(\operatorname{ad} \bar{\mathbf{Q}}_-)^2 = 0.$$

Therefore $(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}, \operatorname{ad} \bar{\mathbf{Q}}_-)$ is a chain complex (with respect the grading by charge). The corresponding homology

$$H_\bullet(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}) = H_\bullet(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}, \operatorname{ad} \bar{\mathbf{Q}}_-) \tag{80}$$

is naturally a $\mathbb{Z}$-graded superalgebra.

Theorem 4.4.1 ([DDCDS+06], cf. Theorem 2.4.2 of [Ara07]).

(i) *It holds that* $H_i(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}) = 0$ *for all* $i \neq 0$.

(ii) *There is an algebra isomorphism* $H_0(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}) \cong \mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$.

For a $\bar{\mathfrak{g}}$-module M, $M \otimes \Lambda(\bar{\mathfrak{g}}_{<0})$ is naturally a $U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$-module. Therefore the algebra $H_0(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l})$ naturally acts on $H_\bullet^{\mathrm{Lie}}(\bar{\mathfrak{g}}, M \otimes \mathbb{C}_{\bar{\chi}_-})$. As in the same manner as [Ara07], it follows that the action of $\mathrm{Zh}_H(\mathcal{W}^k(\bar{\mathfrak{g}}, f))$ on $H_\bullet^{\mathrm{BRST}}(M)_{\mathrm{top}}$ coincides with the action of $H_0(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l})$ on the space $H_\bullet^{\mathrm{Lie}}(\bar{\mathfrak{g}}_{<0}, M_{\mathrm{top}} \otimes \mathbb{C}_{\bar{\chi}_-})$, via the isomorphisms (78) and (ii) of Theorem 4.4.1.

§5. Representation theory of affine W-algebras via the BRST cohomology functor

5.1. The vanishing of the Lie algebra homology

Recall the notation from §3.1 and §3.2.

Let $\bar{L}(\bar{\lambda})$ be the irreducible highest weight representation of $\bar{\mathfrak{g}}$ with highest weight $\bar{\lambda} \in \bar{\mathfrak{h}}^*$.

Let $\mathcal{O}_0(\bar{\mathfrak{g}})$ be the full subcategory of the category of finitely generated left $\bar{\mathfrak{g}}$-modules consisting of objects M such that (1) $\dim U(\bar{\mathfrak{n}}_+)m < \infty$ for all $m \in M$, (2) $\bar{\mathfrak{h}}$ acts semisimply on M, (3) M is a direct sum of finite-dimensional $\bar{\mathfrak{g}}_0$-modules.

Set

$$\bar{P}_0^+ = \{\bar{\lambda} \in \bar{\mathfrak{h}}^*; \langle \bar{\lambda}, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \bar{\Delta}_{0,+}\}. \tag{81}$$

For $\bar{\lambda} \in \bar{P}_0^+$ put $\bar{M}_0(\bar{\lambda}) = U(\bar{\mathfrak{g}}) \otimes_{U(\bar{\mathfrak{g}}_{\geq 0})} \bar{E}(\bar{\lambda})$, where $\bar{E}(\bar{\lambda})$ is the irreducible finite-dimensional representation of $\bar{\mathfrak{g}}_0$ with highest weight $\bar{\lambda}$, considered as a $\bar{\mathfrak{g}}_{\geq 0}$-module on which $\bar{\mathfrak{g}}_{>0}$ acts trivially. The $\bar{M}_0(\bar{\lambda})$ has $\bar{L}(\bar{\lambda})$ as its unique simple quotient. Every simple object of $\mathcal{O}_0(\bar{\mathfrak{g}})$ is isomorphic to exactly one of the $\bar{L}(\bar{\lambda})$ with $\bar{\lambda} \in \bar{P}_0^+$.

For a finitely generated $\bar{\mathfrak{g}}$-module M let $\operatorname{Dim} M$ be the Gelfand–Kirillov dimension of M. By (26), one has

$$\operatorname{Dim} M \leq d_{\bar{\chi}} \tag{82}$$

for all $M \in \mathcal{O}_0(\bar{\mathfrak{g}})$.

Set

$$H_\bullet^{\mathrm{Lie}}(M) = H_\bullet^{\mathrm{Lie}}(\bar{\mathfrak{g}}_{<0}, M \otimes \mathbb{C}_{\bar{\chi}_-}). \tag{83}$$

One sees that $H_0^{\mathrm{Lie}}(M)$ is finite-dimensional for any object M of $\mathcal{O}_0(\bar{\mathfrak{g}})$ as in Lemma 2.5.1 of [Ara07]. From §4.4 it follows that the correspondence $M \mapsto H_0^{\mathrm{Lie}}(M)$ defines a functor from $\mathcal{O}_0(\bar{\mathfrak{g}})$ to $\mathfrak{Fin}(\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f))$, the category of finite-dimensional representations of $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$.

The following assertion is essentially proved by Matumoto [Mat90a] (see also [Mat90b]).

Theorem 5.1.1.

(i) *The functor* $H_0^{\mathrm{Lie}}(?) : \mathcal{O}_0(\bar{\mathfrak{g}}) \to \mathfrak{Fin}(\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f))$ *is exact.*

(ii) *Let* M *be an object of* $\mathcal{O}_0(\bar{\mathfrak{g}})$. *One has* $H_0^{\mathrm{Lie}}(M) \neq 0$ *if and only if* $\operatorname{Dim} M = d_{\bar{\chi}}$.

Proof. (i) follows from [Mat90a, Corollary 3.3.3] by using the argument of [Kos78, Theorem 4.3], see [BK08, Lemma 8.20]. (ii) follows from (i) and [Mat90a, Corollary 3.3.2]. Q.E.D.

Because every projective object of $\mathcal{O}_0(\bar{\mathfrak{g}})$ is free over $U(\bar{\mathfrak{g}}_{<0})$, the following assertion follows from (i) of Theorem 5.1.1 in the same manner as Theorem 2.5.6 of [Ara07].

Theorem 5.1.2. *One has* $H_i^{\mathrm{Lie}}(M) = 0$ *for all* $i \neq 0$ *and for all* $M \in \mathcal{O}_0(\bar{\mathfrak{g}})$.

5.2. Representations of finite W-algebras in type A

In [BK08], Brundan and Kleshchev gave a complete description of irreducible finite-dimensional representations of $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$ in type A, as we recall below:

Let $\bar{\mathfrak{g}} = \mathfrak{sl}_n(\mathbb{C})$. As usual, we write $\bar{\Delta} = \{\alpha_{i,j}; 1 \leq i, j \leq n\}$, $\bar{\Delta}_+ = \{\alpha_{i,j}; 1 \leq i < j \leq n\}$.

Let Y_f be the partition $(p_1 \leq p_2 \leq \cdots \leq p_r)$ of n corresponding to the nilpotent element f. Following [BK08], we identify Y_f with the Young diagram with p_i boxes in the ith row, and number the boxes of Y_f by 1, 2, ..., n down columns from left to right. The corresponding good grading is defined so that

(84)
$$\bar{\Delta}_0 = \{\alpha_{i,j}; \text{the } i\text{th and the } j\text{th boxes belong to the same column}\}$$

(see [EK05, BK08] for details). Let

$$\bar{\Delta}^f = \{\alpha \in \bar{\Delta}; \alpha(h) = 0 \ \forall h \in \bar{\mathfrak{h}}^f\}, \tag{85}$$

$\bar{\Delta}^f_+ = \bar{\Delta}^f \cap \bar{\Delta}_+$. It is easy to see that

$$\bar{\Delta}^f = \{\alpha_{i,j} \in \bar{\Delta}; \text{the } i\text{th and the } j\text{th boxes belong to the same row}\}.$$

Let

$$\bar{W}^f = \{w \in \bar{W}; w(h) = h \ \forall h \in \bar{\mathfrak{h}}^f\}. \tag{86}$$

Then $\bar{W}^f$ is the subgroup of $\bar{W} = \mathfrak{S}_n$ generated by s_α with $\alpha \in \bar{\Delta}^f$.

Theorem 5.2.1 (Brundan and Kleshchev [BK08], $\bar{\mathfrak{g}} = \mathfrak{sl}_n(\mathbb{C})$).

(i) *For $\bar{\lambda} \in \bar{P}^+_0$, $H^{\mathrm{Lie}}_0(\bar{L}(\bar{\lambda})) \neq 0$ if and only if $\langle \bar{\lambda} + \bar{\rho}, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \bar{\Delta}^f_+$. In this case $H^{\mathrm{Lie}}_0(\bar{L}(\bar{\lambda}))$ is irreducible. Further, any irreducible finite-dimensional representation of $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$ arises in this way.*

(ii) *Nonzero $H^{\mathrm{Lie}}_0(\bar{L}(\bar{\lambda}))$ and $H^{\mathrm{Lie}}_0(\bar{L}(\bar{\mu}))$, with $\bar{\lambda}, \bar{\mu} \in \bar{P}^+_0$, are isomorphic if and only if $\bar{\mu} + \bar{\rho} \in \bar{W}^f(\bar{\lambda} + \bar{\rho})$.*

5.3. The category $\mathcal{O}_{0,k}$ of $\mathfrak{g}$-modules

Recall the notation from §3.3.

For $\lambda \in \mathfrak{h}^*$ let $L(\lambda)$ be the irreducible representation of $\mathfrak{g}$ with highest weight λ.

Let $\mathcal{O}_{0,k}$ be the full subcategory of the category of left $\mathfrak{g}$-modules consisting of objects M such that the following hold:

- K acts as the multiplication by k on M;
- M admits a weight space decomposition;
- there exists a finite subset $\{\mu_1, \dots, \mu_n\}$ of $\mathfrak{h}^*_k$ such that $M = \bigoplus_{\mu \in \bigcup_i \mu_i - Q_+} M^\mu$;
- for each $d \in \mathbb{C}$, M_d is an objects of $\mathcal{O}_0(\bar{\mathfrak{g}})$ as $\bar{\mathfrak{g}}$-modules.

Set

(87) $$P_{0,k}^{+} = \{\lambda \in \mathfrak{h}_k^*; \bar{\lambda} \in \bar{P}_0^{+},\ \langle \lambda, K\rangle = k\}.$$

For $\lambda \in P_{0,k}^{+}$, let

(88) $$M_0(\lambda) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{g}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \bar{M}_0(\bar{\lambda}),$$

where $\bar{M}_0(\bar{\lambda})$ is considered as a $\bar{\mathfrak{g}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$-module on which $\bar{\mathfrak{g}}[t]t$ acts trivially, and K and D act the multiplication by $\langle \lambda, K\rangle$ and $\langle \lambda, D\rangle$, respectively. The $M_0(\lambda)$ is an object of $\mathcal{O}_{0,k}$, and has $L(\lambda)$ as its unique simple quotient. Every irreducible object of $\mathcal{O}_k$ is isomorphic to exactly one of the $L(\lambda)$ with $\lambda \in P_{0,k}^{+}$.

The correspondence $M \mapsto M^*$ defines a duality functor on $\mathcal{O}_{0,k}$. Here, $\mathfrak{g}$ acts on M^* by

(89) $$(Xf)(v) = f(X^t v)$$

where $X \mapsto X^t$ is the anti-automorphism of $\mathfrak{g}$ define by $K^t = K$, $D^t = D$ and $J(n)^t = J^t(-n)$ for $J \in \bar{\mathfrak{g}}$, $n \in \mathbb{Z}$.

Clearly, $(M^*)^* = M$ for $M \in \mathcal{O}_{0,k}$. It follows that $L(\lambda)^* = L(\lambda)$.

Let $\mathcal{O}_{0,k}^{\triangle}$ be the full subcategory of $\mathcal{O}_{0,k}$ consisting of objects M that admit a finite filtration $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ such that each successive subquotient M_i/M_{i+1} is isomorphic to some generalized Verma module $M_0(\lambda_i)$ with $\lambda_i \in P_{0,k}^{+}$. Dually, let $\mathcal{O}_{0,k}^{\nabla}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects M such that $M^* \in Obj\mathcal{O}_k^{\triangle}$.

For $\lambda \in P_{0,k}^{+}$, let $\mathcal{O}_{0,k}^{\leq\lambda}$ be the Serre full subcategory of $\mathcal{O}_{0,k}$ consisting of objects M such that $M = \bigoplus_{\mu \leq \lambda} M^{\mu}$. It is well-known [RCW82] that every $L(\mu)$ that lies in $\mathcal{O}_{0,k}^{\leq\lambda}$ admits the indecomposable projective cover $P_{\leq\lambda}(\mu)$ in $\mathcal{O}_{0,k}^{\leq\lambda}$, and hence, every finitely generated object in $\mathcal{O}_{0,k}^{\leq\lambda}$ is an image of a projective object of the form $\bigoplus_{i=1}^{r} P_{\leq\lambda}(\mu_i)$. The $P_{\leq\lambda}(\mu)$ is an object of $\mathcal{O}_{0,k}^{\triangle}$. Dually, $I_{\leq\lambda}(\mu) = P_{\leq\lambda}(\mu)^*$ is the injective envelope of $L(\mu)$ in $\mathcal{O}_{0,k}^{\leq\lambda}$.

5.4. The "Top" part of the BRST cohomology

Let M be an object of $\mathcal{O}_{0,k}$. Clearly, M_{top} is a $\bar{\mathfrak{g}}$-submodule of M. By Theorem 5.1.2, $H_i^{\mathrm{Lie}}(M_{\mathrm{top}}) = 0$ for all $i > 0$, and $H_0^{\mathrm{Lie}}(M_{\mathrm{top}})$ is a finite-dimensional $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$-module.

The following assertion follows from (78), (83) and Theorems 4.4.1 and 5.1.2.

Lemma 5.4.1. *Let M be an object of $\mathcal{O}_{0,k}$. Assume that $H_\bullet^{\mathrm{Lie}}(M_{\mathrm{top}}) \neq 0$. Then one has the following isomorphism of $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$-modules:*

$$H_i^{\mathrm{BRST}}(M)_{\mathrm{top}} \cong \begin{cases} H_0^{\mathrm{Lie}}(M_{\mathrm{top}}) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

The following assertion follows from Theorems 5.1.1, 5.1.2 and Lemma 5.4.1.

Proposition 5.4.2. *One has*

$$H_i^{\mathrm{BRST}}(M_0(\lambda))_{\mathrm{top}} \cong \begin{cases} H_0^{\mathrm{Lie}}(\bar{M}_0(\bar{\lambda})) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0, \end{cases}$$

$$H_i^{\mathrm{BRST}}(M_0(\lambda)^*)_{\mathrm{top}} \cong \begin{cases} H_0^{\mathrm{Lie}}(\bar{M}_0(\bar{\lambda})^*) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0, \end{cases}$$

and if $\mathrm{Dim}\, \bar{L}(\bar{\lambda}) = d_{\bar{\chi}}$, *then*

$$H_i^{\mathrm{BRST}}(L(\lambda))_{\mathrm{top}} \cong \begin{cases} H_0^{\mathrm{Lie}}(\bar{L}(\bar{\lambda})) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

5.5. The vanishing and the almost irreduciblity

Theorem 5.5.1. *Let M be an object of $\mathcal{O}_{0,k}$. Then $H_\bullet^{\mathrm{BRST}}(M)_d$ is finite-dimensional for all d. If M is an object of $\mathcal{O}_{0,k}^{\leq\lambda}$ then $H_i^{\mathrm{BRST}}(M)_d = 0$ unless $|i| \leq d - \langle \lambda, D \rangle$.*

Proof. By Theorem 5.1.2 one has $H_i^{\mathrm{Lie}}(M|_{\bar{\mathfrak{g}}}) = 0$ for all $i \neq 0$. Therefore by considering the Hochschild–Serre spectral sequence for $\bar{\mathfrak{g}}_{<0} \subset L\bar{\mathfrak{g}}_{<0}$, the assertion follows in the same manner as Theorem 7.4.2 of [Ara07]. Q.E.D.

Theorem 5.5.1 in particular implies that $H_\bullet^{\mathrm{BRST}}(M)$ is an ordinary representations for all $M \in \mathcal{O}_{0,k}$. It follows that one has the functor

$$\mathcal{O}_{0,k} \to \mathcal{W}^k(\bar{\mathfrak{g}}, f)\text{-}\mathfrak{mod}_{\sigma_R}, \quad M \to H_0^{\mathrm{BRST}}(M). \tag{90}$$

Theorem 5.5.2 ([KW04]). *For $\lambda \in P_{0,+}^k$ one has the following:*

(i) $H_i^{\mathrm{BRST}}(M_0(\lambda)) = 0$ *for all* $i \neq 0$.
(ii) $H_0^{\mathrm{BRST}}(M_0(\lambda))$ *is almost highest weight.*

(The proof of Theorem 5.5.2 is the same as that of Theorem 3.8.1.)

Theorem 5.5.3. *For $\lambda \in P_{0,+}^k$ one has the following:*

(i) $H_i^{\mathrm{BRST}}(M_0(\lambda)^*) = 0$ *for all* $i \neq 0$.
(ii) $H_0^{\mathrm{BRST}}(M_0(\lambda)^*)$ *is almost co-highest weight.*

The proof of Theorem 5.5.3 is given in Section 6.

Though our formulation is slightly different from that of [KW08], the following assertion essentially confirms Conjecture B of [KW08], partially (cf. Theorems 5.7.1, 5.8.1 and 5.9.2 below).

Theorem 5.5.4 (The main result). *Let k be any complex number.*

(i) *Let M be an object of $\mathcal{O}_{0,k}$. Then $H_i^{\mathrm{BRST}}(M) = 0$ for all $i \neq 0$. In particular the functor $H_0^{\mathrm{BRST}}(?) : \mathcal{O}_{0,k} \to \mathcal{W}^k(\bar{\mathfrak{g}}, f)$-$\mathfrak{mod}_{\sigma_R}$ is exact.*
(ii) *For $\lambda \in P_{0,+}^k$, $H_0^{\mathrm{BRST}}(L(\lambda))$ is zero or almost irreducible. Further, one has $H_0^{\mathrm{BRST}}(L(\lambda)) \neq 0$ if and only if $\operatorname{Dim} \bar{L}(\bar{\lambda}) = d_{\bar{\chi}}$.*

Proof. We give only the sketch of the proof because it is essentially the same as those of Theorems 7.6.1 and 7.6.3 of [Ara07].

From Theorem 5.5.2 (i) it follows that $H_i^{\mathrm{BRST}}(M) = 0$ for all $i \neq 0$ and $M \in \mathcal{O}_{0,k}^{\Delta}$, and hence $H_i^{\mathrm{BRST}}(P_{\leq \lambda}(\mu)) = 0$ for all $i \neq 0$ and all $\mu \leq \lambda$ in $P_{0,+}^k$. This together with Theorem 5.5.1 gives the vanishing of $H_i^{\mathrm{BRST}}(M)$ for all $i > 0$ and all $M \in \mathcal{O}_{0,k}$. Likewise, Theorem 5.5.3 (i) gives $H_i^{\mathrm{BRST}}(M) = 0$ for all $i < 0$ and all $M \in \mathcal{O}_{0,k}$. This shows (i). (ii) follows from (i) using Theorem 5.1.1 (ii), Theorem 5.5.2 (ii) and Theorem 5.5.3 (ii). Q.E.D.

Corollary 5.5.5. *Let $\lambda \in P_{0,+}^k$ with $k \in \mathbb{C}$. The representation $H_0^{\mathrm{BRST}}(L(\lambda))$ is irreducible over $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ if and only if $H_0^{\mathrm{Lie}}(\bar{L}(\bar{\lambda}))$ is irreducible over $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$.*

Proof. The assertion follows immediately from Proposition 5.4.2 and Theorem 5.5.4 (ii). Q.E.D.

5.6. The Character of $H_0^{\mathrm{BRST}}(L(\lambda))$

Let $\operatorname{ch} L(\lambda)$ be the character of $L(\lambda)$: $\operatorname{ch} L(\lambda) = \sum_{\mu} e^{\mu} \dim L(\lambda)^{\mu}$. One has

$$\operatorname{ch} L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} c_{\lambda,\mu} \frac{e^{\mu}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_{\alpha}}}$$

with some $c_{\lambda,\mu} \in \mathbb{Z}$. The coefficient $c_{\lambda,\mu}$ is known by Kashiwara and Tanisaki [KT00] (in terms of the Kazhdan–Lusztig polynomials) provided that k is not critical (for any simple summand of $\bar{\mathfrak{g}}$).

Recall [KW04, KW08] that the "Cartan subalgebra" of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ is given by

$$\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathbb{C}D, \quad \text{where } \bar{\mathfrak{t}} = \bar{\mathfrak{h}}^f. \tag{91}$$

Because it commutes with $\mathbf{Q}_-$, $\mathfrak{t}$ acts on the space $H^{\mathrm{BRST}}_\bullet(M)$.

Let

$$\operatorname{ch} H^{\mathrm{BRST}}_\bullet(L(\lambda)) = \sum_{\xi\in\mathfrak{t}^*} e^\xi \dim H^{\mathrm{BRST}}_\bullet(L(\lambda))_\xi,$$

where $H^{\mathrm{BRST}}_\bullet(L(\lambda))_\xi = \{c \in H^{\mathrm{BRST}}_\bullet(L(\lambda)); tc = \xi(t)c \ \forall t \in \mathfrak{t}\}$.

Set

$$\chi_{H^{\mathrm{BRST}}_\bullet(L(\lambda))} = \sum_{i=-\infty}^{\infty} (-1)^i \operatorname{ch} H^{\mathrm{BRST}}_i(L(\lambda)). \tag{92}$$

By the Euler–Poincaré principle one has [FKW92, KRW03, KW08]

$$\chi_{H^{\mathrm{BRST}}_\bullet(L(\lambda))} = \frac{\sum_\mu c_{\lambda,\mu} e^{\mu|_\mathfrak{t}}}{\prod_{j\geq 1}(1-e^{-j\delta|_\mathfrak{t}})^{\operatorname{rank}\bar{\mathfrak{g}}} \prod_{\alpha\in\Delta^{\mathrm{re}}_{0,+}}(1-e^{-\alpha|_\mathfrak{t}})}, \tag{93}$$

where $\Delta^{\mathrm{re}}_{0,+} = \{\alpha \in \Delta^{\mathrm{re}}_+; \bar{\alpha} \in \bar{\Delta}_0\}$.

The following assertion follows immediately from Theorem 5.5.4.

Theorem 5.6.1. *For $\lambda \in P^k_{0,+}$ one has*

$$\operatorname{ch} H^{\mathrm{BRST}}_0(L(\lambda)) = \chi_{H^{\mathrm{BRST}}_\bullet(L(\lambda))}.$$

5.7. Type A case

In type A, the following assertion follows immediately from (79), Theorems 4.4.1, 5.2.1 and 5.5.4 (in the notation of §5.2).

Theorem 5.7.1 ($\bar{\mathfrak{g}} = \mathfrak{sl}_n$). *Let k be any complex number.*

(i) *One has $H^{\mathrm{BRST}}_i(M) = 0$ for all $i \neq 0$ and all $M \in \mathcal{O}_{0,k}$.*

(ii) *For $\lambda \in P^k_{0,+}$, $H^{\mathrm{BRST}}_0(L(\lambda)) \neq 0$ if and only if $\langle \bar{\lambda} + \bar{\rho}, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \bar{\Delta}^f_+$. In this case $H^{\mathrm{BRST}}_0(L(\lambda))$ is irreducible. Further, any irreducible ordinary Ramond twisted representation of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ arises in this way.*

(iii) *Nonzero $H^{\mathrm{BRST}}_0(L(\lambda))$ and $H^{\mathrm{BRST}}_0(L(\mu))$ with $\lambda, \mu \in P^k_{0,+}$ are isomorphic if and only of $\bar{\mu} + \bar{\rho} \in \bar{W}^f(\bar{\lambda} + \bar{\rho})$.*

Theorems 5.6.1 and 5.7.1 determine[8] the characters of all irreducible ordinary Ramond twisted representations of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ for all nilpotent elements f at all non-critical levels k.

[8] In the case of f is a principal nilpotent element the characters of all irreducible positive energy representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ was previously determined in [Ara07] (for all $\bar{\mathfrak{g}}$ and all $k \in \mathbb{C}$). Also, in the case f is a minimal nilpotent element the characters of all irreducible (non-twisted) positive energy representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ was previously determined in [Ara05] (for all $\bar{\mathfrak{g}}$ and all non-critical k).

Remark 5.7.2. If $\bar{\mathfrak{g}}$ is not of type A, it not true that nonzero $H_0^{\mathrm{BRST}}(L(\lambda))$ is always irreducible, see Theorem 3.6.3 of [Mat90a]. However it is likely that $H_0^{\mathrm{BRST}}(L(\lambda))$ is a direct sum of irreducible modules.

5.8. Irreducibility of the images of principal admissible representations

Let Pr^k be the set of principal admissible weights [KW89, KW08] of $\mathfrak{g}$ of level k. For $\lambda \in Pr^k$ one has [KW88]

(94)
$$\operatorname{ch} L(\lambda) = \sum_{w \in W(\lambda)} (-1)^{\ell_\lambda(w)} \frac{e^{w \circ \lambda}}{\prod_{j \geq 1}(1-e^{-j\delta})^{\operatorname{rank}\bar{\mathfrak{g}}} \prod_{\alpha \in \Delta_+^{\mathrm{re}}}(1-e^{-\alpha})}.$$

Let $\bar{\Delta}(\lambda) = \Delta(\lambda) \cap \bar{\Delta}$, and let $\bar{W}(\bar{\lambda}) \subset \bar{W}$ be the integral Weyl group of $\bar{\lambda} \in \bar{\mathfrak{h}}^*$ generated by s_α with $\alpha \in \bar{\Delta}(\lambda)$. The formula (94) in particular implies that

$$\operatorname{ch} \bar{L}(\bar{\lambda}) = \sum_{w \in \bar{W}(\bar{\lambda})} (-1)^{\ell_{\bar{\lambda}}(\bar{\lambda})} \frac{e^{w \circ \lambda}}{\prod_{\alpha \in \bar{\Delta}_+}(1-e^{-\alpha})}. \tag{95}$$

We remark that an element λ of Pr^k done not necessarily belong to $P_{0,+}^k$. However the Euler–Poincaré character $\chi_{H_0^{\mathrm{BRST}}(L(\lambda))}$ makes sense for all $\lambda \in Pr^k$ [KW08], and coincides with the right-hand-side of (93). Thus it has the form

$$\chi_{H_0^{\mathrm{BRST}}(L(\lambda))} = e^{\langle \lambda, D \rangle \delta|_{\mathfrak{t}}} \sum_{j \in \mathbb{Z}_{\geq 0}} e^{-j\delta|_{\mathfrak{t}}} \varphi_{\lambda,j} \tag{96}$$

with

$$\varphi_{\lambda,0} = \frac{\sum_{w \in \bar{W}(\bar{\lambda})} (-1)^{\ell_\lambda(w)} e^{w \circ \lambda|_{\mathfrak{t}}}}{\prod_{\alpha \in \bar{\Delta}_{0,+}}(1-e^{-\alpha|_{\mathfrak{t}}})}. \tag{97}$$

Note that $\varphi_{\lambda,0}$ is the Euler–Poincaré character of $H_\bullet^{\mathrm{Lie}}(\bar{L}(\bar{\lambda}))$.

The Euler–Poincaré character $\chi_{H_\bullet^{\mathrm{BRST}}(L(\lambda))}$ is called *almost convergent* [KW08] if $\lim_{z \to 0} \varphi_{\lambda,0}(z)$ $(z \in \bar{\mathfrak{t}})$ exists and is non-zero. Set

$$\widetilde{M}_k = \{\lambda \in Pr^k; \chi_{H_\bullet^{\mathrm{BRST}}(L(\lambda))} \text{ is almost convergent}\}, \tag{98}$$

$$M_k = \widetilde{M}_k \cap P_{0,+}^k. \tag{99}$$

Theorem 5.8.1 ($\bar{\mathfrak{g}}$ arbitrary). *Let $\lambda \in M_k$. Then $H_\bullet^{\mathrm{BRST}}(L(\lambda))$ is irreducible.*

Proof. By Corollary 5.5.5 it is sufficient to show that $H_0^{\mathrm{Lie}}(\bar{L}(\bar{\lambda}))$ is irreducible over $\mathcal{W}^{\mathrm{fin}}(\bar{\mathfrak{g}}, f)$.

By Corollary 2.2 (or its proof) of [KW08] one has

$$|\bar{\Delta}(\lambda)| = |\bar{\Delta}_0|.$$

(In our setting $\Delta^0 \sqcup \Delta^{1/2}$ in [KW08] is identified with $\bar{\Delta}_0$, see [BG07].) Because $\lambda \in P^k_{0,+}$, $\bar{\Delta}_0 \subset \bar{\Delta}_+(\lambda)$, and hence $\bar{\Delta}(\lambda) = \bar{\Delta}_0$. This implies

$$\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}, \quad \forall \alpha \in \bar{\Delta}_{>0}. \tag{100}$$

Thanks to Theorem 3.4.4 of [Mat90a], this gives the irreduciblity of $H_0^{\mathrm{Lie}}(\bar{L}(\bar{\lambda}))$. Q.E.D.

Remark 5.8.2. Let $\lambda \in P^k_{0,+}$. From Theorem 5.5.4 it follows that $\chi_{H_0^{\mathrm{BRST}}(L(\lambda))}$ is almost convergent if and only if $\operatorname{Dim} \bar{L}(\bar{\lambda}) = d_\chi$.

5.9. Modular invariant representations in type A

Recall [KW08] that the pair (k, f) is called *exceptional* if the Euler–Poincare character $\chi_{H_\bullet^{\mathrm{BRST}}(L(\lambda))}$ is almost convergent for some $\lambda \in Pr^k$, and is either zero or almost convergent for all $\lambda \in Pr^k$.

The exceptional pairs are classified in [KW08] in type A: Each admissible number [KW89] k of $\mathfrak{sl}_n$ is written as

$$k + n = \frac{p}{q}, \quad p \geq n, \quad q \geq 1, \quad (p, q) = 1. \tag{101}$$

For such a k the pair (k, f) is exceptional if and only if f is the nilpotent element corresponding to the partition $(s, q, q, \dots, q)$ ($s \equiv n \pmod{q}$, $0 \leq s < q$).

The following assertion was implicitly proved[9] in [KW08].

Proposition 5.9.1. *Let (k, f) be an exceptional pair for $\mathfrak{sl}_n$. There is an bijection*

$$\bar{W}^f \times M_k \xrightarrow{\sim} \widetilde{M_k}, \quad (w, \lambda) \mapsto w \circ \lambda.$$

Proof. By Theorem 2.3 of [KW08],

$$\widetilde{M_k} = \{\lambda \in Pr^k ; \bar{\Delta}(\lambda) \subset \bar{\Delta} \backslash \bar{\Delta}^f\}. \tag{102}$$

[9]In the case that f is a principal nilpotent element (= the case that $q \geq n$, $\bar{\Delta}_0 = \emptyset$ and $\bar{\Delta}^f = \bar{\Delta}$) Proposition 5.9.1 was proved in [FKW92].

Let $\lambda \in \widetilde{M}_k$, $w \in \bar{W}^f$. Since $\Delta_+^{\mathrm{re}} \cap w^{-1}(\Delta_-^{\mathrm{re}}) \subset \bar{\Delta}_+^f$, (102) gives $\Delta_+^{\mathrm{re}}(\lambda) \cap w^{-1}(\Delta_-^{\mathrm{re}}) = \emptyset$, or equivalently, $w \circ \lambda \in Pr^k$. Because

$$(103) \qquad \chi_{H_\bullet^{\mathrm{BRST}}(L(\lambda))} = \chi_{H_\bullet^{\mathrm{BRST}}(L(w\circ\lambda))}, \quad \forall w \in \bar{W}^f,$$

the element $w \circ \lambda$ belongs to $\widetilde{M}_k$. Therefore the shifted action of $\bar{W}^f$ preserves $\widetilde{M}_k$. Further, again by (102), it follows that this action of $\bar{W}^f$ on $\widetilde{M}_k$ is faithful, and that $M_k \cap (\bar{W}^f \circ \lambda) = \{\lambda\}$ for $\lambda \in M_k$.

Next let k be as in (101). By Lemma 3.1 of [KW08] one has

$$(104) \qquad \operatorname{rank} \bar{\Delta}(\lambda) \geq \min(n-q, 0) = \operatorname{rank} \bar{\Delta}_0, \quad \forall \lambda \in Pr^k.$$

According to (the proof of) Propositions 3.2 and 3.3 of [KW08], the rank of any root subsystem in $\bar{\Delta}\backslash\bar{\Delta}^f$ is equal to or smaller than $\operatorname{rank}\bar{\Delta}_0$, and is equal to $\operatorname{rank}\bar{\Delta}_0$ if and only if it is $\bar{W}^f$-conjugate to $\bar{\Delta}_0$. Thus for $\lambda \in \widetilde{M}_k$ there exists $w \in \bar{W}^f$ such that $\bar{\Delta}(\lambda) = w(\bar{\Delta}_0)$, and thus $w^{-1} \circ \lambda \in M_k$. This completes the proof. Q.E.D.

According to [KW08], Theorem 5.7.1 and Proposition 5.9.1 give the following assertion[10].

Theorem 5.9.2 (Conjectured by Kac and Wakimoto [KW08]). *Let (k, f) be an exceptional pair for $\mathfrak{sl}_n$. The linear span of the normalized characters of irreducible ordinary Ramond twisted representations $H_0^{\mathrm{BRST}}(L(\lambda))$ of $\mathcal{W}^k(\mathfrak{sl}_n, f)$, with $\lambda \in M_k$, are closed under the natural action of $SL_2(\mathbb{Z})$.*

§6. Proof of Theorem 5.5.3

The proof of Theorem 5.5.3 is essentially the repetition of the argument of §7 of [Ara07]. Therefore we give only the sketch of the proof.

6.1. Step 1

Let

$$(105) \qquad C^\bullet(M_0(\lambda)) := M_0(\lambda) \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0}).$$

As in §8.2 of [Ara07], we identify $M_0(\lambda)^* \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{g}}_{>0})$ with $C^\bullet(M_0(\lambda))^*$ ($*$ is defined in (36)):

$$(106) \qquad H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*) = H_\bullet(C^\bullet(M_0(\lambda))^*, \mathbf{Q}_-).$$

[10]However the rationality of the simple quotient of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ still remains to be an open problem.

The differential $\mathbf{Q}_-$ acts on $C^\bullet(M_0(\lambda))^*$ by

$$(\mathbf{Q}_-\phi)(c) = \phi(\mathbf{Q}_+c) \tag{107}$$

for $\phi \in C^\bullet(M_0(\lambda))^*$, $c \in C^\bullet(M_0(\lambda))$, where

$$\mathbf{Q}_+ = (Q_+^{\mathrm{st}})_{(0)} + \chi'_+, \quad \chi'_+ = \sum_{\alpha \in \bar{\Delta}_{\geq 1}} c_\alpha^{-1} \bar{\chi}(x_\alpha)\psi_{-\alpha}(0). \tag{108}$$

Below we twist the action of $\mathcal{C}^\bullet$ on $C^\bullet(M_0(\lambda))$ by the automorphism defined by

$$J_\alpha(n) \mapsto -c_{w_0(\alpha)} J_{-w_0(\alpha)}(n) \quad (\alpha \in \bar{\Delta}), \tag{109}$$

(110)

$$\psi_\alpha(n) \mapsto -c_{w_0(\alpha)}\psi_{-w_0(\alpha)}(n),\ \psi_{-\alpha}(n) \mapsto -c_{w_0(\alpha)}^{-1}\psi_{w_0(\alpha)}(n) \quad (\alpha \in \bar{\Delta}_{>0}).$$

Let $C_+^\bullet(\lambda)$ be the $\mathcal{C}_+^\bullet$-submodule of $C^\bullet(M_0(\lambda))$ spanned by the vectors

$$\widehat{J}_{a_1}(m_1)\dots\widehat{J}_{a_r}(m_r)\psi_{\beta_1}(n_1)\dots\psi_{\beta_s}(n_s)v_\lambda \tag{111}$$

with $a_i \in \bar{\Delta}_{\leq 0} \sqcup \bar{I}$ and $\beta_i \in \bar{\Delta}_{<0}$, where v_λ is the highest weight vector of $C^\bullet(M_0(\lambda))$. As in §3.8, it follows that $C_+^\bullet(\lambda)$ is a subcomplex of $C^\bullet(M_0(\lambda))$.

The graded dual space $C_+^\bullet(\lambda)^*$ of $C_+^\bullet(\lambda)$ is a quotient complex of $C^\bullet(M_0(\lambda))^*$. Thus there is a natural map

$$H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*) \to H_\bullet(C_+^\bullet(\lambda)^*), \tag{112}$$

which is a homomorphism of Ramond twisted representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$. The action of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$ on $H_\bullet(C_+^\bullet(\lambda)^*)$ is described as follows. The action (109,110) gives a σ_R-twisted $\mathcal{C}^\bullet$-module structure on $C^\bullet(M_0(\lambda))$ via the map $\widehat{t}_{-\frac{1}{2}h_0}$ defined in §4.3. This gives a σ_R-twisted $\mathcal{C}_+^\bullet$-module structure on $C_+^\bullet(\lambda)$, which gives a σ_R-twisted $\mathcal{C}_+^\bullet$-module structure on $C_+^\bullet(\lambda)^*$ by

$$(\widehat{J}_a(m)^R f)(c) = f(-\widehat{J}_a(-m)^R c), \quad (\psi_\alpha(m)^R f)(c) = f(-\psi_\alpha(-m)^R c).$$

It is easily seen that this action induces an action of $\mathcal{W}^k(\bar{\mathfrak{g}}, f) \subset \mathcal{C}_+^\bullet$ on $H_\bullet(C_+^\bullet(\lambda)^*)$.

One has the following assertion (cf. Proposition 8.3.4 of [Ara07]):

Proposition 6.1.1. *The map (112) gives the isomorphism*

$$H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*) \cong H_\bullet(C_+^\bullet(\lambda)^*)$$

of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$-modules.

6.2. Step 2

One has

$$C_+^\bullet(\lambda) = \bigoplus_{d\in -\langle\lambda,D\rangle+\mathbb{Z}_{\geq 0}} C_+^\bullet(\lambda)_d, \quad \dim C_+^\bullet(\lambda)_d = \infty.$$

Note that the subspace $C_+^\bullet(\lambda)_{\mathrm{top}} = C_+^\bullet(\lambda)_{-\langle\lambda,D\rangle}$ is the subcomplex of $(C_+^\bullet(\lambda), \mathbf{Q}_+)$ spanned by the vectors

$$\widehat{J}_{a_1}(0)\dots\widehat{J}_{a_r}(0)\psi_{\beta_1}(0)\dots\psi_{\beta_s}(0)v_\lambda \tag{113}$$

with $a_i \in \bar{\Delta}_{\leq 0}\sqcup\bar{I}$, $\beta_i \in \bar{\Delta}_{<0}$, and hence,

$$C_+^\bullet(\lambda)_{\mathrm{top}} = \bar{M}_0(\bar{\lambda})\otimes \bigwedge^\bullet(\bar{\mathfrak{g}}_{>0}^*). \tag{114}$$

One has the weight space decomposition

$$C_+^\bullet(\lambda)_{\mathrm{top}} = \bigoplus_{\substack{\mu\in\mathfrak{h}^*\\ \langle\lambda-\mu,x_0\rangle\geq 0}} C_+^\bullet(\lambda)_{\mathrm{top}}^\mu.$$

Define a decreasing filtration

$$C_+^\bullet(\lambda)_{\mathrm{top}} = F^0C_+^\bullet(\lambda)_{\mathrm{top}} \supset F^1C_+^\bullet(\lambda)_{\mathrm{top}} \supset \dots$$

of $C_+^\bullet(\lambda)_{\mathrm{top}}$ by

$$F^pC_+^\bullet(\lambda)_{\mathrm{top}} = \bigoplus_{\substack{\mu\in\mathfrak{h}^*\\ \langle\lambda-\mu,x_0\rangle\geq p}} C_+^\bullet(\lambda)_{\mathrm{top}}^\mu. \tag{115}$$

Then

$$(Q_+^{\mathrm{st}})_{(0)}\cdot F^pC_+^\bullet(\lambda)_{\mathrm{top}} \subset F^pC_+^\bullet(\lambda)_{\mathrm{top}}, \tag{116}$$

$$\chi_+'\cdot F^pC_+^\bullet(\lambda)_{\mathrm{top}} \subset F^{p+1}C_+^\bullet(\lambda)_{\mathrm{top}}. \tag{117}$$

Let $F^pC_+^\bullet(\lambda)$ be the subspace of $C_+^\bullet(\lambda)$ generated by $F^pC_+^\bullet(\lambda)_{\mathrm{top}}$ over $C_+^\bullet$. One has

$$C_+^\bullet(\lambda) = F^0C_+^\bullet(\lambda) \supset F^1C_+^\bullet(\lambda) \supset \dots, \tag{118}$$

$$\bigcap_p F^pC_+^\bullet(\lambda) = 0, \tag{119}$$

$$\mathbf{Q}_+F^pC_+^\bullet(\lambda) \subset F^pC_+^\bullet(\lambda), \tag{120}$$

$$a_{(n)}\cdot F^pC_+^\bullet(\lambda) \subset F^pC_+^\bullet(\lambda) \quad (a\in C_+^\bullet, n\in\mathbb{Z}) \tag{121}$$

(cf. Proposition 8.5.3 of [Ara07]).

Let $({}^{\vee}E_r^{p,q}, d_r)$ be the corresponding spectral sequence:

$$ {}^{\vee}E_0^{p,q} = F^pC_+^{p+q}(\lambda)/F^{p+1}C_+^{p+q}(\lambda), \tag{122} $$

$$ {}^{\vee}E_1^{p,q} = H^{p+q}({}^{\vee}E_0^{p,\bullet}). \tag{123} $$

We do not claim that this spectral sequence converges to $H^\bullet(C_+^\bullet(\lambda))$. We will show in Proposition 6.4.2 below that ${}^{\vee}E_r$ converges to the dual $\mathsf{D}(H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*))$ of $H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*)$.

6.3. Step 3

Set

$$ F_pC_+^\bullet(\lambda)^* = (C_+^\bullet(\lambda)/F^pC_+^\bullet(\lambda))^* \subset C_+^\bullet(\lambda)^*. \tag{124} $$

Then $\{F_pC_+^\bullet(\lambda)^*\}$ defines an exhaustive, increasing filtration of the chain complex $\{C_+^\bullet(\lambda)^*\}$ which is obviously bounded below (cf. Lemma 8.5.4 and Proposition 8.5.5 of [Ara07]). It follows that one has the corresponding converging spectral sequence

$$ E^r \Rightarrow H_\bullet(C_+^\bullet(\lambda)^*) = H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*). \tag{125} $$

Let $\{F_pH_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*)\}$ be the corresponding increasing filtration of $H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*)$.

Because the filtration is compatible with the action of the Hamiltonian $-D$, each $E_{p,q}^r$ decomposes into eigenspaces of $-D$ as complexes:

$$ E_{p,q}^r = \bigoplus_{d\in -\langle\lambda,D\rangle+\mathbb{Z}_{\geq 0}} (E_{p,q}^r)_d. \tag{126} $$

It follows that

$$ E_{p,q}^\infty = \bigoplus_{d\in -\langle\lambda,D\rangle+\mathbb{Z}_{\geq 0}} (E_{p,q}^\infty)_d, \tag{127} $$

and each $(E^r)_d$ converges to $(E^\infty)_d$. In particular one has

$$ \bigoplus_{p+q=n} (E_{p,q}^\infty)_{\mathrm{top}} = \begin{cases} \mathrm{gr}_F\, H_0^{\mathrm{BRST}}(M_0(\lambda)^*)_{\mathrm{top}} & \text{if } p+q=0, \\ 0 & \text{if } p+q\neq 0. \end{cases} \tag{128} $$

by Proposition 5.4.2.

Also by (121) this filtration is compatible with the σ_R-twisted action of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$. Hence each $E_{p,q}^r$ is a Ramond twisted representation of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$, and the differential d^r is a morphism in $\mathcal{W}^k(\bar{\mathfrak{g}}, f)\text{-}\mathfrak{Mod}_{\sigma_R}$.

Therefore $\{F_p H_\bullet^{\mathrm{BRST}}(M_0(\lambda)^*)\}$ is a filtration of Ramond twisted representations of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)$, and the corresponding graded space

$$\mathrm{gr}^F\, H_0^{\mathrm{BRST}}(M_0(\lambda)^*) = \bigoplus_{p+q=0} E_{p,q}^{\infty}$$

is also an object of $\mathcal{W}^k(\bar{\mathfrak{g}}, f)\text{-}\mathfrak{Mod}_{\sigma_R}$.

6.4. Step 4

Consider the subcomplex

$$\begin{aligned}({}^{\vee}E_0^{p,q})_{\mathrm{top}} = ({}^{\vee}E_0^{p,q})_{\langle\lambda,D\rangle} &= F^p C_+^{p+q}(\lambda)_{\mathrm{top}}/F^{p+1}C_+^{p+q}(\lambda)_{\mathrm{top}} \\ &\cong \bigoplus_{\langle\lambda-\mu,x_0\rangle=p} C_+^{p+q}(\lambda)_{\mathrm{top}}^{\mu}\end{aligned}$$

of ${}^{\vee}E_0^{p,q}$. By (117) one has

$$((^{\vee}E_0^{p,\bullet})_{\mathrm{top}}, \mathbf{Q}_+) \cong \bigoplus_{\langle\lambda-\mu,x_0\rangle=p} \left(C_+^{p+q}(\lambda)_{\mathrm{top}}^{\mu}, (Q_+^{\mathrm{st}})_{(0)}\right) \tag{129}$$

as complexes.

By definition ${}^{\vee}E_0^{p,\bullet}$ is spanned by the vectors

$$\widehat{J}_{a_1}(m_1)\dots\widehat{J}_{a_r}(m_r)\psi_{\beta_1}(n_1)\dots\psi_{\beta_s}(n_s)c \tag{130}$$

with $c \in ({}^{\vee}E_0^{p,\bullet})_{\mathrm{top}}$, $a_i \in \bar{\Delta}_{<0} \sqcup \bar{I}$, $\beta_i \in \bar{\Delta}_{<0}$, and $m_i, n_i < 0$. It follows that each D-eigenspace $({}^{\vee}E_0^{p,\bullet})_d$ is finite-dimensional. Thus by Lemma 3.3.1,

$$E_{p,q}^0 (= ({}^{\vee}E_0^{p-1,q+1})^*) = \mathsf{D}({}^{\vee}E_0^{p-1,q+1}). \tag{131}$$

The following assertion follows immediately from (131).

Proposition 6.4.1. *One has* $E_{p,q}^1 = \mathsf{D}({}^{\vee}E_1^{p-1,q+1})$, *or equivalently,* ${}^{\vee}E_1^{p,q} = \mathsf{D}(E_{p+1,q-1}^1)$.

The following assertion follows from Proposition 6.4.1 by the inductive argument.

Proposition 6.4.2. *The spectral sequence* ${}^{\vee}E_r$ *converges to* $\mathsf{D}(E^{\infty})$.

The proof of the following assertion is the same as that of Theorem 3.8.1.

Proposition 6.4.3. *One has* ${}^{\vee}E_1^{p,q}=0$ *for* $p+q\neq 0$ *and there is a linear isomorphism*

$$U(\bar{\mathfrak{g}}^f[t^{-1}]t^{-1})\otimes({}^{\vee}E_1^{p,-p})_{\mathrm{top}}\xrightarrow{\sim}{}^{\vee}E_1^{p,-p}$$

of the form

$$u_{i_1}(-n_1)\dots u_{i_r}(-n_r)\otimes v\mapsto \mathsf{W}^{(i_1)}_{-n_1}\dots\mathsf{W}^{(i_r)}_{-n_r}v \tag{132}$$

with a fixed PBW basis $\{u_{i_1}(-n_1)\dots u_{i_r}(-n_r)\}$ *of* $U(\bar{\mathfrak{g}}^f\otimes\mathbb{C}[t^{-1}]t^{-1})$. *Here the action of* $\mathcal{W}^k(\bar{\mathfrak{g}},f)$ *on* ${}^{\vee}E_1^{p,-p}$ *induced by the* σ_R*-twisted action of* $C^\bullet_+$ *on* $C^\bullet_+(\lambda)$.

Thanks to Proposition 6.4.3 the following assertion follows by induction.

Proposition 6.4.4. *There exist isomorphisms of chain complexes*

$$({}^{\vee}E_r^{p,q},d_r)\cong(U(\bar{\mathfrak{g}}^f[t^{-1}]t^{-1})\otimes({}^{\vee}E_r^{p,q})_{\mathrm{top}},1\otimes d^r)$$

of the form (132) *with* $v\in({}^{\vee}E_r^{p,q})_{\mathrm{top}}$ *for all* $r\geq 1$. *Therefore one has the linear isomorphism*

$${}^{\vee}E_\infty^{p,q}\cong U(\bar{\mathfrak{g}}^f[t^{-1}]t^{-1})\otimes({}^{\vee}E_\infty^{p,q})_{\mathrm{top}}$$

of the form (132) *with* $v\in({}^{\vee}E_\infty^{p,q})_{\mathrm{top}}$.

By (128) and Proposition 6.4.1 one has

$$({}^{\vee}E_\infty^{p,q})_{\mathrm{top}}=\mathsf{D}((E^\infty_{p+1,q-1})_{\mathrm{top}})=0\quad\text{if } p+q\neq 0.$$

By Proposition 6.4.4 this gives ${}^{\vee}E_\infty^{p,q}=0$ if $p+q\neq 0$, or equivalently,

$$E^\infty_{p,q}=0\quad\text{if } p+q\neq 0. \tag{133}$$

This gives that $H_n^{\mathrm{BRST}}(M_0(\lambda)^*)=0$ for all $n\neq 0$.

Also, from Proposition 6.4.4 if follows that each ${}^{\vee}E_\infty^{p,-p}$ is almost highest weight. Therefore $E^\infty_{p,-p}=\mathrm{gr}_p\,H_0^{\mathrm{BRST}}(M_0(\lambda)^*)=\mathsf{D}(E_\infty^{p-1,-p+1})$ is almost co-highest weight with $(E^\infty_{p,-p})_{\mathrm{top}}=(E^\infty_{p,-p})_{-\langle\lambda,D\rangle}$. Hence $H_0^{\mathrm{BRST}}(M_0(\lambda)^*)$ is also co-highest weight.

This completes the proof of (ii) of Theorem 5.5.3. Q.E.D.

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