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Frobenius morphism and semi-stable bundles

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Abstract.

This article is the expanded version of a talk given at the conference: Algebraic geometry in East Asia 2008. In this notes, I intend to give a brief survey of results on the behavior of semi-stable bundles under the Frobenius pullback and direct images. Some results are new.

§1. Introduction

Let X be a smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by $\mathcal{O}_X \to \mathcal{O}_X, \quad f \mapsto f^p$. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over k. This simple endomorphism of X is of fundamental importance in algebraic geometry over characteristic $p > 0$. One of the themes is to study its action on the geometric objects on X. Here we consider the pull-back F^* and direct image F_* of torsion free sheaves on X. For example, is the semi-stability (resp. stability) of torsion free sheaves preserved by F^* and F_*? Even on curves of genus $g \geq 2$, it is known that F^* does not preserve the semi-stability of torsion free sheaves (cf. [2] for example). However, it is now also know that F_* preserves the stability of torsion free sheaves on curves of genus $g \geq 2$ (cf. [20]). In this paper, we are going to discuss the behavior of semi-stability of torsion free sheaves under F^* and F_*.

Recall that a torsion free sheaf $\mathcal{E}$ is called semi-stable (resp. stable) if $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$ (resp. $\mu(\mathcal{E}') < \mu(\mathcal{E})$) for any nontrivial proper sub-sheaf $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{E}'$ torsion free, where $\mu(\mathcal{E})$ is the slope of $\mathcal{E}$ (See definition in Section 3). Semi-stable sheaves are basic constituents of

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torsion free sheaves in the sense that any torsion free sheaf $\mathcal{E}$ admits a unique filtration

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_{\ell+1}(\mathcal{E}) = \mathcal{E},$$

which is the so called Harder–Narasimhan filtration, such that

(1) $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}) := \mathrm{HN}_i(\mathcal{E})/\mathrm{HN}_{i-1}(\mathcal{E})$ $(1 \le i \le \ell + 1)$ are semistable;
(2) $\mu(\mathrm{gr}_1^{\mathrm{HN}}(\mathcal{E})) > \mu(\mathrm{gr}_2^{\mathrm{HN}}(\mathcal{E})) > \cdots > \mu(\mathrm{gr}_{\ell+1}^{\mathrm{HN}}(\mathcal{E}))$.

The rational number $\mathrm{I}(\mathcal{E}) := \mu(\mathrm{gr}_1^{\mathrm{HN}}(\mathcal{E})) - \mu(\mathrm{gr}_{\ell+1}^{\mathrm{HN}}(\mathcal{E}))$, which measures how far is a torsion free sheaf from being semi-stable, is called the instability of $\mathcal{E}$. It is clear that $\mathcal{E}$ is semi-stable if and only if $\mathrm{I}(\mathcal{E}) = 0$. Thus the main theme of this investigation is to look for upper bound of $\mathrm{I}(F^*\mathcal{E})$ and $\mathrm{I}(F_*\mathcal{E})$.

In Section 2, we recall the notion of connections with p-curvature zero and Cartier's theorem, which simply says that a quasi-coherent sheaf is the Frobenius pullback of a sheaf if and only if it has a connection of p-curvature zero. In particular, a sub-sheaf of $F^*\mathcal{E}$ is the pullback of a sub-sheaf of $\mathcal{E}$ if and only if it is invariant under the action of the canonical connection on $F^*\mathcal{E}$. This is the main tool in Section 3 to find a upper bound of $\mathrm{I}(F^*\mathcal{E})$.

In Section 3, we survey various upper bounds of the instability $\mathrm{I}(F^*\mathcal{E})$ in terms of $\mathrm{I}(\mathcal{E})$ and numerical invariants of Ω_X^1. For curves, the bound is a linear combination of $\mathrm{I}(\mathcal{E})$ and $\mu(\Omega_X^1)$. For higher dimensional varieties X, the difficulty to obtain such a bound lies in the fact that tensor product of two semi-stable sheaves may not be semi-stable in characteristic $p > 0$. A theorem of A. Langer can solve this difficulty in certain sense. He proved in [11] that there is a k_0 for a torsion free sheaf $\mathcal{E}$ such that the Harder–Narasimhan filtration of $F^{k*}\mathcal{E}$ has strongly semi-stable quotients whenever $k \ge k_0$. As a price of it, the upper bound is a linear combination of $\mathrm{I}(\mathcal{E})$ and the limit

$$L_{\max}(\Omega_X^1) = \lim_{k\to\infty} \frac{\mu_{\max}(F^{k*}\Omega_X^1)}{p^k}.$$

It is natural to expect a upper bound in terms of $\mathrm{I}(\mathcal{E})$ and $\mu_{\max}(\Omega_X^1)$ (cf. Remark 3.13), but I do not know any such bound in general.

In Section 4, we discuss the stability of F_*W. The main tool in this section is the canonical filtration (4.5) of $F^*(F_*W)$, which is again induced by the canonical connection on $F^*(F_*W)$. After a brief proof of the main theorem in [20], we reveal some implications in the proof. We show that the proof itself implies that $F_*\mathcal{L}$ and the sheaf B_X^1 of local exact differential 1-forms on X are stable if $\mu(\Omega_X^1) > 0$ and $\mathrm{T}^\ell(\Omega_X^1)$ $(1 \le \ell \le n(p-1))$ are semi-stable. In fact, for $\mathcal{E} \subset F_*\mathcal{L}$ (resp. $B' \subset B_X^1$), we show that $\mu(\mathcal{E}) - \mu(F_*\mathcal{L})$ (resp. $\mu(B') - \mu(B^1_X)$) is bounded by an explicit negative number (cf. the inequalities (4.18) and (4.20)). The work of M. Raynaud have revealed the important relationship between B^1_X and the fundamental group of X. I do not know if the result above has any application in this direction.

§2. Frobenius and connections of p-curvature zero

Let X be a smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by the homomorphism

$$\mathcal{O}_X \to \mathcal{O}_X, \qquad f \mapsto f^p$$

of rings. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over k that satisfies

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X_1 \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(k) \arrow[r, "F_k"] & \mathrm{Spec}(k) \end{tikzcd} .$$

According to a theorem of Cartier, the fact that a quasi-coherent $\mathcal{E}$ on X is the pull-back of a sheaf on X_1 by F is equivalent to the fact that $\mathcal{E}$ has a connection of p-curvature zero. Let me recall briefly the theme from [7] (See Section 5 of [7]).

For a quasi-coherent sheaf $\mathcal{E}$ on X, a connection on $\mathcal{E}$ is a k-linear homomorphism $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ satisfying the Leibniz rule

$$\nabla(f \cdot e) = f\nabla(e) + e \otimes df, \quad \forall\ f \in \mathcal{O}_X,\ e \in \mathcal{E}$$

where df denotes the image of f under $d : \mathcal{O}_X \to \Omega^1_X$. The kernel

$$\mathcal{E}^\nabla := ker(\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X)$$

is an abelian sheaf of the germs of horizontal sections of $(\mathcal{E}, \nabla)$.

Let $\mathrm{Der}(\mathcal{O}_X)$ be the sheaf of derivations, i.e., for any open set $U \subset X$, $\mathrm{Der}(\mathcal{O}_X)(U)$ is the set of derivations $\mathrm{D} : \mathcal{O}_U \to \mathcal{O}_U$. It is a sheaf of k-Lie algebras and it is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ as $\mathcal{O}_X$-modules. A connection ∇ on $\mathcal{E}$ is equivalent to an $\mathcal{O}_X$-linear morphism

$$\nabla : \mathrm{Der}(\mathcal{O}_X) \to \mathrm{End}_k(\mathcal{E})$$

satisfying $\nabla(\mathrm{D})(f \cdot e) = \mathrm{D}(f) \cdot e + f\nabla(\mathrm{D})$ where $\mathrm{End}_k(\mathcal{E})$ is the sheaf of k-linear endomorphisms of $\mathcal{E}$, which is also a sheaf of k-Lie algebras.

A connection $\nabla : \mathrm{Der}(\mathcal{O}_X) \to \mathrm{End}_k(\mathcal{E})$ is integrable if it is a homomorphism of Lie algebras. A morphism between $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \nabla')$ is a morphism $\Phi : \mathcal{E} \to \mathcal{F}$ of quasi-coherent $\mathcal{O}_X$-modules satisfying

$$\Phi(\nabla(\mathrm{D})(e)) = \nabla'(\mathrm{D})(\Phi(e)), \quad \forall\ \mathrm{D} \in \mathrm{Der}(\mathcal{O}_X),\ e \in \mathcal{E}.$$

Then the pairs $(\mathcal{E}, \nabla)$ of quasi-coherent sheaves with integrable connections form an abelian category $MIC(X)$.

Since $\mathrm{char}(k) = p > 0$, the p-th iterate D^p of a derivation D is again a derivation. Thus $\mathrm{Der}(\mathcal{O}_X)$ and $\mathrm{End}_k(\mathcal{E})$ are both sheaves of restricted p-Lie algebras. The p-curvature of an integrable connection

$$\nabla : \mathrm{Der}(\mathcal{O}_X) \to \mathrm{End}_k(\mathcal{E})$$

measures how far the homomorphism ∇ is from being a homomorphism of restricted p-Lie algebras. More precisely,

Definition 2.1. *The p-curvature of $\nabla : \mathrm{Der}(\mathcal{O}_X) \to \mathrm{End}_k(\mathcal{E})$ is the morphism of sheaves $\Psi^\nabla : \mathrm{Der}(\mathcal{O}_X) \to \mathrm{End}_k(\mathcal{E})$ defined by*

$$\Psi^\nabla(\mathrm{D}) := (\nabla(\mathrm{D}))^{\mathrm{p}} - \nabla(\mathrm{D}^{\mathrm{p}})$$

which is in fact a morphism $\Psi^\nabla : \mathrm{Der}(\mathcal{O}_X) \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ i.e. $\Psi^\nabla(\mathrm{D})$ is $\mathcal{O}_X$-linear for any $\mathrm{D} \in \mathrm{Der}(\mathcal{O}_X)$.

Let $F : X \to X_1$ be the relative Frobenius morphism. Then, for any quasi-coherent sheaf $\mathcal{F}$ on X_1, there is a unique connection

$$\nabla_{\mathrm{can}} : F^*(\mathcal{F}) \to F^*(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^1_X,$$

which is integrable and of p-curvature zero, such that

$$\mathcal{F} \cong (F^*(\mathcal{F}))^{\nabla_{\mathrm{can}}}.$$

We call ∇_{can} the canonical connection on the pull-back $F^*(\mathcal{F})$. It turns out that a quasi-coherent sheaf $\mathcal{E}$ on X having a connection of p-curvature zero is enough to characterize that $\mathcal{E}$ is a pull-back of a quasi-coherent sheaf on X_1. More precisely, given a $(\mathcal{E}, \nabla)$ of p-curvature zero, the abelian sheaf $\mathcal{E}^\nabla$ is in a natural way a quasi-coherent sheaf on X_1 such that $F^*(\mathcal{E}^\nabla) \cong \mathcal{E}$. Moreover, we have

Theorem 2.1. *(Cartier) Let $F : X \to X_1$ be the relative Frobenius morphism. Then the functor*

$$\mathcal{F} \mapsto (F^*(\mathcal{F}), \nabla_{\mathrm{can}})$$

is an equivalence of categories between the category of quasi-coherent sheaves on X_1 and the full subcategory of $MIC(X)$ consisting of $(\mathcal{E}, \nabla)$ whose p-curvature is zero. The inverse functor is

$$(\mathcal{E}, \nabla) \mapsto \mathcal{E}^{\nabla}.$$

§3. Instability of Frobenius pull-back

Let X be a smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. Fix an ample divisor H on X, for a torsion free sheaf $\mathcal{E}$ on X, the slope of $\mathcal{E}$ is defined as

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot \mathrm{H}^{n-1}}{\mathrm{rk}(\mathcal{E})}$$

where $\mathrm{rk}(\mathcal{E})$ denotes the rank of $\mathcal{E}$. Then

Definition 3.1. *A torsion free sheaf $\mathcal{E}$ on X is called semi-stable (resp. stable) if for any subsheaf $\mathcal{E}' \subset \mathcal{E}$ with $\mathcal{E}/\mathcal{E}'$ torsion free, we have*

$$\mu(\mathcal{E}') \ \leq (\textit{resp.} <) \ \mu(\mathcal{E}).$$

Theorem 3.1. *(Harder–Narasimhan filtration) For any torsion free sheaf $\mathcal{E}$, there is a unique filtration*

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_{\ell+1}(\mathcal{E}) = \mathcal{E},$$

which is the so called Harder–Narasimhan filtration, such that

(1) $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}) := \mathrm{HN}_i(\mathcal{E})/\mathrm{HN}_{i-1}(\mathcal{E})$ $(1 \leq i \leq \ell + 1)$ *are semistable;*
(2) $\mu(\mathrm{gr}_1^{\mathrm{HN}}(\mathcal{E})) > \mu(\mathrm{gr}_2^{\mathrm{HN}}(\mathcal{E})) > \cdots > \mu(\mathrm{gr}_{\ell+1}^{\mathrm{HN}}(\mathcal{E}))$.

Remark 3.2. *In* [4, Theorem 1.3.4], *the proof of existence of the filtration is given in terms of Gieseker stability. In particular, $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E})$ are Gieseker semi-stable, thus they are μ-semistable torsion free sheaves.*

By using this unique filtration of $\mathcal{E}$, we can introduce an invariant $\mathrm{I}(\mathcal{E})$ of $\mathcal{E}$, which we call the instability of $\mathcal{E}$. It is a rational number and measures how far is $\mathcal{E}$ from being semi-stable.

Definition 3.2. *Let $\mu_{\max}(\mathcal{E}) = \mu(\mathrm{gr}_1^{\mathrm{HN}}(\mathcal{E}))$, $\mu_{\min}(\mathcal{E}) = \mu(\mathrm{gr}_{\ell+1}^{\mathrm{HN}}(\mathcal{E}))$. Then the instability of $\mathcal{E}$ is defined to be*

$$\mathrm{I}(\mathcal{E}) := \mu_{\max}(\mathcal{E}) - \mu_{\min}(\mathcal{E}).$$

It is easy to see that a torsion free sheaf $\mathcal{E}$ is semi-stable if and only if $\mathrm{I}(\mathcal{E}) = 0$. We collect some elementary facts.

Proposition 3.3. *Let* $\mathrm{HN}_\bullet(\mathcal{E})$ *be the Harder–Narasimhan filtration of length* ℓ *and* $\mu_i = \mu(\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}))$ $(i = 1, \ldots, \ell + 1)$*. Then*

(1) $\mu_{\max}(\mathcal{E}/\mathrm{HN}_i(\mathcal{E})) = \mu_{i+1}, \quad \mu_{\min}(\mathrm{HN}_i(\mathcal{E})) = \mu_i$

(2) $\mu(\mathrm{HN}_1(\mathcal{E})) > \mu(\mathrm{HN}_2(\mathcal{E})) > \cdots > \mu(\mathrm{HN}_\ell(\mathcal{E})) > \mu(\mathcal{E})$

(3) *For any torsion free quotient* $\mathcal{E} \to \mathcal{Q} \to 0$ *and any subsheaf* $\mathcal{E}' \subset \mathcal{E}$*,*

$$\mu(\mathcal{Q}) \geq \mu_{\min}(\mathcal{E}), \quad \mu(\mathcal{E}') \leq \mu_{\max}(\mathcal{E})$$

(4) *For any torsion free sheaves* $\mathcal{F}$*,* $\mathcal{E}$*, if* $\mu_{\min}(\mathcal{F}) > \mu_{\max}(\mathcal{E})$*, then*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0.$$

Proof. (1) follows the definition. (2) was proved in [3, Lemma 1.3.11] for curves, but the proof there works also for higher dimensional varieties. The sub-sheaf case in (3) follows from [4, Lemma 1.3.5]. To see that $\mu(\mathcal{Q}) \geq \mu_{\min}(\mathcal{E})$, by Theorem 3.1, we can replace $\mathcal{Q}$ by the last grade quotient of $\mathrm{HN}_\bullet(\mathcal{Q})$, thus we can assume that $\mathcal{Q}$ is semi-stable. Then the quotient morphism induces a non-trivial morphism $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}) \to \mathcal{Q}$. Thus $\mu(\mathcal{Q}) \geq \mu_i \geq \mu_{\min}(\mathcal{E})$. (4) follows from (3). Q.E.D.

In this section, we discuss the behavior of $\mathrm{I}(\mathcal{E})$ under the Frobenius pull-back. We start it by introducing some discrete invariants of a torsion free sheaf and its Frobenius pull-back. A sub-sheaf $\mathcal{F} \subset F^*\mathcal{E}$ is called ∇_{can}-invariant if $\nabla_{\mathrm{can}}(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1$, where ∇_{can} is the canonical connection on $F^*\mathcal{E}$.

Definition 3.3. *Let* $\ell(\mathcal{E}) = \ell$ *be the length of the Harder–Narasimhan filtration* $\mathrm{HN}_\bullet(\mathcal{E})$ *of* $\mathcal{E}$ *and* $s(X, \mathcal{E})$ *be the number of* ∇_{can}*-invariant subsheaves* $\mathrm{HN}_i(F^*\mathcal{E}) \subset F^*\mathcal{E}$ *that appear in* $\mathrm{HN}_\bullet(F^*\mathcal{E})$*.*

Our goal is to bound $\mathrm{I}(F^*\mathcal{E})$ in terms of $\mathrm{I}(\mathcal{E})$, $\ell(F^*\mathcal{E})$, $s(X, \mathcal{E})$ and some invariants of X. The lower bound of $\mathrm{I}(F^*\mathcal{E})$

$$\mathrm{I}(\mathcal{E}) \leq \frac{1}{p}\, \mathrm{I}(F^*\mathcal{E})$$

is trivial by using Proposition 3.3 (3).

When X is a curve of genus $g \geq 1$ and $\mathcal{E}$ is semi-stable, a upper bound of $\mathrm{I}(F^*\mathcal{E})$ has been found (See [17], [18] and [19]). One of the main observations in the proof of [19, Theorem 3.1] is

$$\mathrm{I}(F^*\mathcal{E}) = \sum_{i=1}^{\ell} \{\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) - \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E}))\} \tag{3.1}$$

where $\ell = \ell(F^*\mathcal{E})$. Then, when $\mathcal{E}$ is semi-stable, all of the sub-sheaves $\mathrm{HN}_i(F^*\mathcal{E})$ $(1 \leq i \leq \ell)$ are not ∇_{can}-invariant. Thus ∇_{can} induces non-trivial $\mathcal{O}_X$-homomorphisms

$$\mathrm{HN}_i(F^*\mathcal{E}) \to \frac{F^*\mathcal{E}}{\mathrm{HN}_i(F^*\mathcal{E})} \otimes \Omega^1_X \quad (1 \leq i \leq \ell)$$

which, by Proposition 3.3, imply

$$\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) \leq \mu_{\max}(\frac{F^*\mathcal{E}}{\mathrm{HN}_i(F^*\mathcal{E})} \otimes \Omega^1_X) \quad (1 \leq i \leq \ell). \tag{3.2}$$

When Ω^1_X has rank one, we have, for all $1 \leq i \leq \ell$,

$$\mu_{\max}(\frac{F^*\mathcal{E}}{\mathrm{HN}_i(F^*\mathcal{E})} \otimes \Omega^1_X) = \mu_{\max}(\frac{F^*\mathcal{E}}{\mathrm{HN}_i(F^*\mathcal{E})}) + \mu(\Omega^1_X) \tag{3.3}$$

which implies immediately

$$\mathrm{I}(F^*\mathcal{E}) \leq \ell \cdot (2g-2) \leq (\mathrm{rk}(\mathcal{E}) - 1)(2g-2).$$

In a more general version, we have

Theorem 3.4. *Let X be a smooth projective curve of genus $g \geq 1$ and $\mathcal{E}$ a vector bundle on X. Let $\ell(F^*\mathcal{E}) = \ell$, $s(X, \mathcal{E}) = s$. Then*

$$p \cdot \mathrm{I}(\mathcal{E}) \leq \mathrm{I}(F^*\mathcal{E}) \leq (\ell - s)(2g-2) + p \cdot s \cdot \mathrm{I}(\mathcal{E}).$$

Proof. Let S be the set of numbers $1 \leq i_k \leq \ell$ such that $\mathrm{HN}_{i_k}(F^*\mathcal{E})$ is a ∇_{can}-invariant sub-sheaf of $F^*\mathcal{E}$. Let $\mu_i = \mu(\mathrm{gr}_i^{\mathrm{HN}}(F^*\mathcal{E}))$, notice $\mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) = \mu_{i+1}$, $\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) = \mu_i$, we have

$$\begin{aligned}\mathrm{I}(F^*\mathcal{E}) &= \mu_1 - \mu_{\ell+1} = \sum_{i=1}^{\ell}(\mu_i - \mu_{i+1}) \\ &= \sum_{i=1}^{\ell}\{\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) - \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E}))\}.\end{aligned}$$

When $i \notin S$, $\mathrm{HN}_i(F^*\mathcal{E})$ is not ∇_{can}-invariant, which means that

$$\mathrm{HN}_i(F^*\mathcal{E}) \xrightarrow{\nabla_{\mathrm{can}}} (F^*\mathcal{E}) \otimes \Omega^1_X \to F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E}) \otimes \Omega^1_X$$

is a nontrivial $\mathcal{O}_X$-homomorphism. By Proposition 3.3 (4), we have

$$\begin{aligned}\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) &\leq \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E}) \otimes \Omega^1_X) \\ &= \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) + 2g - 2.\end{aligned}$$

Thus, for $i \notin S$, we have

$$\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) - \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) \leq 2g - 2.$$

When $i \in S$, by Theorem 2.1, there is a sub-sheaf $\mathcal{E}_i \subset \mathcal{E}$ such that $\mathrm{HN}_i(F^*\mathcal{E}) = F^*\mathcal{E}_i$ and $F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E}) = F^*(\mathcal{E}/\mathcal{E}_i)$. Then

$$\mathrm{I}(F^*\mathcal{E}) \leq (\ell - s)(2g-2) + \sum_{i\in S}(\mu_{\min}(F^*\mathcal{E}_i) - \mu_{\max}(F^*(\mathcal{E}/\mathcal{E}_i))).$$

Notice that $\mu_{\min}(F^*\mathcal{E}_i) \leq \mu(F^*\mathcal{E}_i)$, $\mu_{\max}(F^*(\mathcal{E}/\mathcal{E}_i)) \geq \mu(F^*(\mathcal{E}/\mathcal{E}_i))$ and $\mu(\mathcal{E}_i) \leq \mu_{\max}(\mathcal{E})$, $\mu(\mathcal{E}/\mathcal{E}_i) \geq \mu_{\min}(\mathcal{E})$. Therefore we have

$$\mu_{\min}(F^*\mathcal{E}_i) - \mu_{\max}(F^*(\mathcal{E}/\mathcal{E}_i)) \leq p\,\mathrm{I}(\mathcal{E}).$$

Thus

$$p \cdot \mathrm{I}(\mathcal{E}) \leq \mathrm{I}(F^*\mathcal{E}) \leq (\ell - s)(2g-2) + p \cdot s \cdot \mathrm{I}(\mathcal{E}).$$

Q.E.D.

When $\dim(X) > 1$ and $\mathcal{E}$ is semi-stable, an upper bound on $\mathrm{I}(F^*\mathcal{E})$ was given in [11, Corollary 6.2] by A. Langer. Before the discussion of his result, let us make some remarks. It is easy to see that all of the arguments above go through except the equation (3.3) does not hold in general. Thus one can ask the following question

Question 3.5. *What is the constant $a_i(\mathcal{E}, X)$ such that*

$$\mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E}) \otimes \Omega_X^1) = \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) + a_i(\mathcal{E}, X)\ ?$$

More general, what is the upper bound of

$$\mu_{\max}(\mathcal{E}_1 \otimes \mathcal{E}_2) - \mu_{\max}(\mathcal{E}_1) - \mu_{\max}(\mathcal{E}_2)$$

for any torsion free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$?

Remark 3.6. *Let $a_i(\mathcal{E}, X)$ be the constants in Question 3.5 and $a(\mathcal{E}, X)$ be the maximal one of $a_i(\mathcal{E}, X)$ ($1 \leq i \leq \ell$). Then, for any torsion free sheaf $\mathcal{E}$ on a smooth projective variety X, the proof of Theorem 3.4 implies the following inequalities*

$$p \cdot \mathrm{I}(\mathcal{E}) \leq \mathrm{I}(F^*\mathcal{E}) \leq (\ell - s) \cdot a(\mathcal{E}, X) + p \cdot s \cdot \mathrm{I}(\mathcal{E})$$

where ℓ is the length of the Harder–Narasimhan filtration $\mathrm{HN}_\bullet(F^*\mathcal{E})$ and s is the number of ∇_{can}-invariant sub-sheaves $\mathrm{HN}_i(F^*\mathcal{E})$.

The difficulty in answering Question 3.5 lies in the fact that tensor product of two semi-stable sheaves may not be semi-stable in the case of positive characteristic (such examples are easy to construct, see Remark 4.10). However, the following theorem was known by many people (see [11, Theorem 6.1], where it is referred to a special case of [14, Theorem 3.23]).

Theorem 3.7. *A sheaf is called strongly semi-stable (resp. stable) if its pullback by k-th power F^k of Frobenius is semi-stable (resp. stable) for any $k \geq 0$. Then a tensor product of two strongly semi-stable sheaves is a strongly semi-stable sheaf.*

One of theorems proved by A. Langer in his celebrated paper [11] is the following

Theorem 3.8. *For any torsion free sheaf $\mathcal{E}$, there exists an k_0 such that all of quotients $\mathrm{gr}_i^{\mathrm{HN}}(F^{k*}\mathcal{E})$ in the Harder–Narasimhan of $F^{k*}\mathcal{E}$ are strongly semi-stable whenever $k \geq k_0$.*

Proposition 3.9. *If all quotients $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}_1)$, $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}_2)$ in the Harder–Narasimhan filtration of $\mathcal{E}_1$ and $\mathcal{E}_2$ are strongly semi-stable, then*

$$\mu_{\max}(\mathcal{E}_1 \otimes \mathcal{E}_2) \leq \mu_{\max}(\mathcal{E}_1) + \mu_{\max}(\mathcal{E}_2).$$

In particular, if all $\mathrm{gr}_i^{\mathrm{HN}}(F^*\mathcal{E})$ are strongly semi-stable, then

$$p \cdot \mathrm{I}(\mathcal{E}) \leq \mathrm{I}(F^*\mathcal{E}) \leq (\ell - s) \cdot \mu_{\max}(\Omega_X^1) + p \cdot s \cdot \mathrm{I}(\mathcal{E}) \tag{3.4}$$

where ℓ is the length of the Harder–Narasimhan filtration $\mathrm{HN}_\bullet(F^*\mathcal{E})$ and s is the number of ∇_{can}-invariant sub-sheaves $\mathrm{HN}_i(F^*\mathcal{E})$.

Proof. Since $\mathcal{E}_1 \otimes \mathcal{E}_2$ has at most torsion of dimension $n-2$, without loss of generality, we can assume that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is torsion free. Let

$$\mathcal{F} = \mathrm{HN}_1(\mathcal{E}_1 \otimes \mathcal{E}_2) \subset \mathcal{E}_1 \otimes \mathcal{E}_2, \quad \mu(\mathcal{F}) = \mu_{\max}(\mathcal{E}_1 \otimes \mathcal{E}_2).$$

By Theorem 3.8, there exists an k_0 such that for all $k \geq k_0$

$$\mathcal{F}_k := \mathrm{HN}_1(F^{k*}(\mathcal{E}_1 \otimes \mathcal{E}_2)) \subset F^{k*}(\mathcal{E}_1 \otimes \mathcal{E}_2) = F^{k*}\mathcal{E}_1 \otimes F^{k*}\mathcal{E}$$

are strongly semi-stable. By Proposition 3.3, the nontrivial homomorphism $(F^{k*}\mathcal{E}_1)^\vee \otimes \mathcal{F}_k \to F^{k*}\mathcal{E}_2$ implies

$$\mu_{\min}((F^{k*}\mathcal{E}_1)^\vee \otimes \mathcal{F}_k) \leq \mu_{\max}(F^{k*}\mathcal{E}_2).$$

Since $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}_1)$, $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}_2)$, $\mathcal{F}_k$ are strongly semi-stable, by Theorem 3.7, we have $\mu(\mathcal{F}) \leq \mu_{\max}(\mathcal{E}_1) + \mu_{\max}(\mathcal{E}_2)$.

To show (3.4), it is enough to show

$$\mu_{\min}(\mathrm{HN}_i(F^*\mathcal{E})) - \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) \leq \mu_{\max}(\Omega^1_X) \tag{3.5}$$

when $\mathrm{HN}_i(F^*\mathcal{E}))$ is not ∇_{can}-invariant. In this case, there is a nontrivial homomorphism $T_X \to (F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) \otimes \mathrm{HN}_i(F^*\mathcal{E})^\vee$. Then

$$\mu_{\min}(T_X) \leq \mu_{\max}(F^*\mathcal{E}/\mathrm{HN}_i(F^*\mathcal{E})) + \mu_{\max}(\mathrm{HN}_i(F^*\mathcal{E})^\vee)$$

since all $\mathrm{gr}_i^{\mathrm{HN}}(F^*\mathcal{E})$ are strongly semi-stable. Q.E.D.

The inequality (3.4) has the following corollary, which was first proved by Mehta and Ramanathan (See [13, Theorem 2.1]).

Corollary 3.10. *If $\mu_{\max}(\Omega^1_X) \leq 0$, then all semi-stable sheaves on X are strongly semi-stable. If $\mu_{\max}(\Omega^1_X) < 0$, then all stable sheaves on X are strongly stable.*

Proof. Let $\mathcal{E}$ be a semi-stable sheaf of rank r and assume the corollary true for all semi-stable sheaves of rank smaller than r. Then, if $F^*\mathcal{E}$ is not semi-stable, all $\mathrm{gr}_i^{\mathrm{HN}}(F^*\mathcal{E})$ are strongly semi-stable by the assumption. Thus, by inequality (3.4), $F^*\mathcal{E}$ must be semi-stable.

If $\mu_{\max}(\Omega^1_X) < 0$ and $\mathcal{E}$ is stable, then for any proper sub-sheaf $\mathcal{F} \subset F^*\mathcal{E}$, $\mu(\mathcal{F}) \leq \mu(F^*\mathcal{E})$. If $\mu(\mathcal{F}) = \mu(F^*\mathcal{E})$, then $\mathcal{F}$ is not a pullback of a sub-sheaf of $\mathcal{E}$ since $\mathcal{E}$ is stable. Thus the $\mathcal{O}_X$-homomorphism

$$\mathcal{F} \xrightarrow{\nabla_{\mathrm{can}}} F^*\mathcal{E} \otimes \Omega^1_X \to F^*\mathcal{E}/\mathcal{F} \otimes \Omega^1_X$$

is non-trivial, which implies $\mu_{\max}(\Omega^1_X) \geq 0$ since $\mathcal{F}$, $F^*\mathcal{E}/\mathcal{F}$ are strongly semi-stable with the same slope. Q.E.D.

Now it becomes clear, since $p^{k-1}\mathrm{I}(F^*\mathcal{E}) \leq \mathrm{I}(F^{k*}\mathcal{E})$, one can bound

$$\frac{\mathrm{I}(F^{k*}\mathcal{E})}{p^k}, \quad k \geq k_0$$

where the difficult in Question 3.5 vanishes by Proposition 3.9. Indeed, A. Langer made the following definition in [11]:

$$L_{\max}(\mathcal{E}) := \lim_{k\to\infty} \frac{\mu_{\max}(F^{k*}\mathcal{E})}{p^k}, \quad L_{\min}(\mathcal{E}) := \lim_{k\to\infty} \frac{\mu_{\min}(F^{k*}\mathcal{E})}{p^k}.$$

Then he proved the following (See [11, Corollary 6.2]).

Theorem 3.11. *Let $\mathcal{E}$ be a semi-stable torsion free sheaf. Then*

$$L_{\max}(\mathcal{E}) - L_{\min}(\mathcal{E}) \leq \frac{\mathrm{rk}(\mathcal{E}) - 1}{p} \cdot \max\{\, 0,\ L_{\max}(\Omega^1_X)\,\}.$$

In particular, $\mathrm{I}(F^*\mathcal{E}) \leq (\mathrm{rk}(\mathcal{E}) - 1) \cdot \max\{\, 0,\ L_{\max}(\Omega^1_X)\,\}$.

For a torsion free sheaf $\mathcal{E}$ of rank r, by Theorem 3.8, there is a k_0 such that all of quotients $\mathrm{gr}_i^{\mathrm{HN}}(F^{k*}\mathcal{E})$ in the Harder–Narasimhan of $F^{k*}\mathcal{E}$ are strongly semi-stable whenever $k \geq k_0$. We choose k_0 to be the minimal integer such that all quotients $\mathrm{gr}_i^{\mathrm{HN}}(F^{k_0*}\mathcal{E})$ in

$$0 \subset \mathrm{HN}_1(F^{k_0*}\mathcal{E}) \subset \cdots \subset \mathrm{HN}_\ell(F^{k_0*}\mathcal{E}) \subset \mathrm{HN}_{\ell+1}(F^{k_0*}\mathcal{E}) = F^{k_0*}\mathcal{E}$$

are strongly semi-stable. For each $\mathrm{HN}_i(F^{k_0*}\mathcal{E})$ $(1 \leq i \leq \ell)$, there is a $0 \leq k_i \leq k_0$ and a sub-sheaf $\mathcal{E}_i \subset F^{k_i*}\mathcal{E}$ such that

$$\mathrm{HN}_i(F^{k_0*}\mathcal{E}) = F^{k_0-k_i*}\mathcal{E}_i, \quad \nabla_{\mathrm{can}}(\mathcal{E}_i) \not\subseteq \mathcal{E}_i \otimes \Omega_X^1 \text{ if } k_i > 0. \tag{3.6}$$

Let $S = \{\, 1 \leq i \leq k_0 \mid k_i = 0 \,\}$. Then, for $i \in S$,

$$\mu_{\min}(\mathrm{HN}_i(F^{k_0*}\mathcal{E})) - \mu_{\max}\left(\frac{F^{k_0*}\mathcal{E}}{\mathrm{HN}_i(F^{k_0*}\mathcal{E})}\right) \leq p^{k_0}\mathrm{I}(\mathcal{E}). \tag{3.7}$$

For $i \notin S$, there is a nontrivial $\mathcal{O}_X$-homomorphism

$$\mathrm{HN}_i(F^{k_0*}\mathcal{E}) \to \frac{F^{k_0*}\mathcal{E}}{\mathrm{HN}_i(F^{k_0*}\mathcal{E})} \otimes F^{k_0-k_i*}\Omega_X^1$$

which is the pullback of $\mathcal{E}_i \xrightarrow{\nabla_{\mathrm{can}}} F^{k_i*}\mathcal{E} \otimes \Omega_X^1 \to \frac{F^{k_i*}\mathcal{E}}{\mathcal{E}_i} \otimes \Omega_X^1$. Thus

$$\mu_{\min}(\mathrm{HN}_i(F^{k_0*}\mathcal{E})) - \mu_{\max}\left(\frac{F^{k_0*}\mathcal{E}}{\mathrm{HN}_i(F^{k_0*}\mathcal{E})}\right) \leq \mu_{\max}(F^{k_0-k_i*}\Omega_X^1). \tag{3.8}$$

Notice that $p^{k_i}\mu_{\max}(F^{k_0-k_i*}\Omega_X^1) \leq \mu_{\max}(F^{k_0*}\Omega_X^1)$, we have

$$\mathrm{I}(F^{k_0*}\mathcal{E}) \leq \frac{\ell - s}{p}\mu_{\max}(F^{k_0*}\Omega_X^1) + s \cdot p^{k_0}\mathrm{I}(\mathcal{E}) \tag{3.9}$$

where $s = |S|$ is number of elements in S. Since, for any $k \geq k_0$, $\mathrm{I}(F^{k*}\mathcal{E}) = p^{k-k_0}\mathrm{I}(F^{k_0*}\mathcal{E})$, we have

$$\frac{\mathrm{I}(F^{k*}\mathcal{E})}{p^k} \leq \frac{\ell - s}{p} \cdot \frac{\mu_{\max}(F^{k*}\Omega_X^1)}{p^k} + s \cdot \mathrm{I}(\mathcal{E}). \tag{3.10}$$

By Corollary 3.10, to study $\mathrm{I}(F^*\mathcal{E})$, it is enough to consider varieties X with $\mu_{\max}(\Omega_X^1) > 0$. Then we can formulate above discussions as

Theorem 3.12. *Let X be a smooth projective variety of $\mu_{\max}(\Omega_X^1) > 0$. Then, for any torsion free sheaf $\mathcal{E}$ of rank r, we have*

$$L_{\max}(\mathcal{E}) - L_{\min}(\mathcal{E}) \leq \frac{\ell - s}{p} \cdot L_{\max}(\Omega_X^1) + s \cdot \mathrm{I}(\mathcal{E}).$$

In particular, $\mathrm{I}(F^*\mathcal{E}) \leq (r-1)(L_{\max}(\Omega_X^1) + \mathrm{I}(\mathcal{E}))$.

Remark 3.13. *It is clear that* $L_{\max}(\mathcal{E}) - L_{\min}(\mathcal{E}) = \frac{\mathrm{I}(F^{k_0*}\mathcal{E})}{p^{k_0}}$ *and*

$$\mathrm{I}(F^*\mathcal{E}) \leq (\ell - s) \cdot \frac{\mu_{\max}(F^{k_0*}\Omega^1_X)}{p^{k_0}} + s \cdot \mathrm{I}(\mathcal{E}).$$

One may make the following conjecture that

$$\mathrm{I}(F^*\mathcal{E}) \leq (r-1)\mu_{\max}(\Omega^1_X) + (r-1)\mathrm{I}(\mathcal{E}). \tag{3.11}$$

§4. Instability of Frobenius direct images

In this section, we study the instability of direct image F_*W for a torsion free sheaf W on X. For example, is F_*W semi-stable when W is semi-stable? Compare with the case of characteristic zero, for a Galois G-cover $\pi : Y \to X$, the locally free sheaf $\pi_*\mathcal{O}_Y$ is not semi-stable if π is not ètale. However, if π is ètale, then π_*W is semi-stable whenever W is semi-stable. The proof of this fact is based on a decomposition

$$\pi^*(\pi_*W) = \bigoplus_{\sigma \in G} W^\sigma. \tag{4.1}$$

To imitate this idea, we need a similar "decomposition" of $V = F^*(F_*W)$ for $F : X \to X_1$. In general, we can not expect to have a real decomposition of $V = F^*(F_*W)$. Instead of, we will have a filtration

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V \tag{4.2}$$

such that $V_\ell/V_{\ell+1} \cong W \otimes_{\mathcal{O}_X} \mathrm{T}^\ell(\Omega^1_X)$.

The filtration (4.2) was defined and studied in [6] for curves. Its definition can be generalized straightforwardly by using the canonical connection $\nabla_{\mathrm{can}} : V \to V \otimes \Omega^1_X$. The study of its graded quotients are much involved (cf. [20, Section 3]).

Definition 4.1. *Let* $V_0 := V = F^*(F_*W)$, $V_1 = \ker(F^*(F_*W) \twoheadrightarrow W)$

$$V_{\ell+1} := \ker\{V_\ell \xrightarrow{\nabla} V \otimes_{\mathcal{O}_X} \Omega^1_X \to (V/V_\ell) \otimes_{\mathcal{O}_X} \Omega^1_X\} \tag{4.3}$$

where $\nabla := \nabla_{\mathrm{can}}$ *is the canonical connection.*

In order to describe the filtration, we recall a $\mathrm{GL}(n)$-representation $\mathrm{T}^\ell(V) \subset V^{\otimes \ell}$ where V is the standard representation of $\mathrm{GL}(n)$. Let S_ℓ be the symmetric group of ℓ elements with the action on $V^{\otimes \ell}$ by

$(v_1 \otimes \cdots \otimes v_\ell) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(\ell)}$ for $v_i \in V$ and $\sigma \in \mathrm{S}_\ell$. Let $e_1, \ldots, e_n$ be a basis of V, for $k_i \geq 0$ with $k_1 + \cdots + k_n = \ell$ define

$$v(k_1, \ldots, k_n) = \sum_{\sigma \in \mathrm{S}_\ell} (e_1^{\otimes k_1} \otimes \cdots \otimes e_n^{\otimes k_n}) \cdot \sigma. \tag{4.4}$$

Definition 4.2. *Let $\mathrm{T}^\ell(V) \subset V^{\otimes \ell}$ be the linear subspace generated by all vectors $v(k_1, \ldots, k_n)$ for all $k_i \geq 0$ satisfying $k_1 + \cdots + k_n = \ell$. It is a representation of $\mathrm{GL}(V)$. If $\mathcal{V}$ is a vector bundle of rank n, the subbundle $\mathrm{T}^\ell(\mathcal{V}) \subset \mathcal{V}^{\otimes \ell}$ is defined to be the associated bundle of the frame bundle of $\mathcal{V}$ (which is a principal $\mathrm{GL}(n)$-bundle) through the representation $\mathrm{T}^\ell(V)$.*

Then the following theorem was proved in [20, Theorem 3.7].

Theorem 4.1. *The filtration defined in Definition 4.1 is*

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W) \tag{4.5}$$

which has the following properties

(i) $\nabla(V_{\ell+1}) \subset V_\ell \otimes \Omega^1_X$ *for* $\ell \geq 1$, *and* $V_0/V_1 \cong W$.

(ii) $V_\ell/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_\ell) \otimes \Omega^1_X$ *are injective for* $1 \leq \ell \leq n(p-1)$, *which induced isomorphisms*

$$\nabla^\ell : V_\ell/V_{\ell+1} \cong W \otimes_{\mathcal{O}_X} \mathrm{T}^\ell(\Omega^1_X), \quad 0 \leq \ell \leq n(p-1).$$

The vector bundle $\mathrm{T}^\ell(\Omega^1_X)$ is suited in the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Sym}^{\ell-\ell(p)\cdot p}(\Omega^1_X) \otimes F^*\Omega^{\ell(p)}_X \xrightarrow{\phi} \mathrm{Sym}^{\ell-(\ell(p)-1)\cdot p}(\Omega^1_X) \otimes F^*\Omega^{\ell(p)-1}_X \\ &\to \cdots \to \mathrm{Sym}^{\ell-q\cdot p}(\Omega^1_X) \otimes F^*\Omega^q_X \xrightarrow{\phi} \mathrm{Sym}^{\ell-(q-1)\cdot p}(\Omega^1_X) \otimes F^*\Omega^{q-1}_X \\ &\to \cdots \to \mathrm{Sym}^{\ell-p}(\Omega^1_X) \otimes F^*\Omega^1_X \xrightarrow{\phi} \mathrm{Sym}^\ell(\Omega^1_X) \to \mathrm{T}^\ell(\Omega^1_X) \to 0 \end{aligned}$$

where $\ell(p) \geq 0$ is the integer such that $\ell - \ell(p) \cdot p < p$.

It is this filtration that we used in [20] to find a upper bound of $\mathrm{I}(F_*W)$. To state the results, let X be an irreducible smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. For any torsion free sheaf W on X, let

$$\mathrm{I}(W, X) = \max\{\mathrm{I}(W \otimes \mathrm{T}^\ell(\Omega^1_X)) \mid 0 \leq \ell \leq n(p-1)\}$$

be the maximal value of instabilities $\mathrm{I}(W \otimes \mathrm{T}^\ell(\Omega^1_X))$. Then we have

Theorem 4.2. *When $K_X \cdot \mathrm{H}^{n-1} \geq 0$, we have, for any $\mathcal{E} \subset F_*W$,*

$$\mu(F_*W) - \mu(\mathcal{E}) \geq -\frac{\mathrm{I}(W,X)}{p}. \tag{4.6}$$

In particular, if $W \otimes \mathrm{T}^\ell(\Omega^1_X)$, $0 \leq \ell \leq n(p-1)$, are semistable, then F_*W is semistable. Moreover, if $K_X \cdot \mathrm{H}^{n-1} > 0$, the stability of the bundles $W \otimes \mathrm{T}^\ell(\Omega^1_X)$, $0 \leq \ell \leq n(p-1)$, implies the stability of F_*W.

Corollary 4.3. *Let X be a smooth projective variety of* $\dim(X) = n$*, whose canonical divisor K_X satisfies $K_X \cdot \mathrm{H}^{n-1} \geq 0$. Then*

$$\mathrm{I}(W) \leq \mathrm{I}(F_*W) \leq p^{n-1}\mathrm{rk}(W)\,\mathrm{I}(W,X).$$

Proof. The lower bound is trivial, the upper bound is Theorem 4.2 plus the following trivial remark: For any vector bundle E, if there is a constant λ satisfying $\mu(E') - \mu(E) \leq \lambda$ for any $E' \subset E$. Then $\mathrm{I}(E) \leq \mathrm{rk}(E)\lambda$. Q.E.D.

When $\dim(X) = 1$, we have the following corollary, which was proved in [10] when W is a line bundle. The fact that semi-stability of W implies semi-stability of F_*W was also proved in [12] by a different method. However, the method in [12] was not able to prove that stability of W implies stability of F_*W.

Corollary 4.4. *When $g \geq 1$, $F_*(W)$ is semi-stable if and only if W is semi-stable. Moreover, if $g \geq 2$, then $F_*(W)$ is stable if and only if W is stable.*

Proof. When $\dim(X) = 1$, $W \otimes \mathrm{T}^\ell(\Omega^1_X) = W \otimes {\Omega^1_X}^{\otimes \ell}$ is semi-stable (resp. stable) whenever W is semi-stable (resp. stable). Thus F_*W is semi-stable (resp. stable).

Q.E.D.

Let $\mathcal{E} \subset F_*W$ be a nontrivial subsheaf, the canonical filtration (4.5) induces the filtration (we assume $V_m \cap F^*\mathcal{E} \neq 0$)

$$0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}. \tag{4.7}$$

Let

$$\mathcal{F}_\ell := \frac{V_\ell \cap F^*\mathcal{E}}{V_{\ell+1} \cap F^*\mathcal{E}} \subset \frac{V_\ell}{V_{\ell+1}}, \qquad r_\ell = \mathrm{rk}(\mathcal{F}_\ell).$$

Then $\mu(F^*\mathcal{E}) = \frac{1}{\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^m r_\ell \cdot \mu(\mathcal{F}_\ell)$ and

$$\mu(\mathcal{E}) - \mu(F_*W) = \frac{1}{p \cdot \mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^m r_\ell\left(\mu(\mathcal{F}_\ell) - \mu(F^*F_*W)\right). \tag{4.8}$$

Lemma 4.5. *With the same notation in Theorem 4.1, we have*

$$\mu(F^*F_*W) = p\cdot\mu(F_*W) = \frac{p-1}{2}K_X\cdot \mathrm{H}^{n-1} + \mu(W), \tag{4.9}$$

$$\mu(V_\ell/V_{\ell+1}) = \mu(W\otimes \mathrm{T}^\ell(\Omega^1_X)) = \frac{\ell}{n}K_X\cdot \mathrm{H}^{n-1} + \mu(W).$$

By using above lemma (see [20] for the proof), we have

$$\mu(\mathcal{E}) - \mu(F_*W) = \sum_{\ell=0}^{m} r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p\cdot \mathrm{rk}(\mathcal{E})} \tag{4.10}$$

$$-\frac{\mu(\Omega^1_X)}{p\cdot\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^{m}(\frac{n(p-1)}{2} - \ell)r_\ell.$$

It is clear that $\mu(\mathcal{F}_\ell)-\mu(V_\ell/V_{\ell+1}) \leq \mathrm{I}(V_\ell/V_{\ell+1}) = \mathrm{I}(W\otimes \mathrm{T}^\ell(\Omega^1_X))$. Thus the proof of Theorem 4.2 will be completed if one can prove

Lemma 4.6. *The ranks r_ℓ of $\mathcal{F}_\ell\subset V_\ell/V_{\ell+1}$ $(0\leq \ell\leq m)$ satisfy*

$$\sum_{\ell=0}^{m}(\frac{n(p-1)}{2} - \ell)r_\ell \geq 0.$$

When $m\leq \frac{n(p-1)}{2}$, the lemma is clear. In fact, we have

$$\sum_{\ell=0}^{m}(\frac{n(p-1)}{2} - \ell)r_\ell \geq \frac{n(p-1)}{2}r_0 \geq \frac{n(p-1)}{2}. \tag{4.11}$$

When $m > \frac{n(p-1)}{2}$, we can write

$$\sum_{\ell=0}^{m}(\frac{n(p-1)}{2} - \ell)r_\ell = \sum_{\ell=m+1}^{n(p-1)}(\ell - \frac{n(p-1)}{2})r_{n(p-1)-\ell} \tag{4.12}$$

$$+\sum_{\ell>\frac{n(p-1)}{2}}^{m}(\ell - \frac{n(p-1)}{2})(r_{n(p-1)-\ell} - r_\ell).$$

The numbers r_ℓ $(0\leq \ell\leq m)$ are related by the following fact that $V_\ell/V_{\ell+1}\xrightarrow{\nabla}(V_{\ell-1}/V_\ell)\otimes\Omega^1_X$ induce injective $\mathcal{O}_X$-homomorphisms

$$\mathcal{F}_\ell \xrightarrow{\nabla} \mathcal{F}_{\ell-1}\otimes\Omega^1_X \quad (1\leq\ell\leq m). \tag{4.13}$$

Using this fact, we proved in [20] the following inequalities

$$r_{n(p-1)-\ell} - r_\ell \geq 0 \qquad (\ell > \frac{n(p-1)}{2})$$

which complete the proof of Lemma 4.6.

The proof of Theorem 4.2 has more implications than the theorem itself. Recall that the sheaf B^1_X of locally exact differential forms on X is defined by exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B^1_X \to 0. \tag{4.14}$$

Theorem 4.7. *Let $\mathcal{L}$ be a torsion free sheaf of rank* 1. *Then, for any nontrivial $\mathcal{E} \subset F_*\mathcal{L}$ with* $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(F_*\mathcal{L})$, *we have*

$$\mu(\mathcal{E}) - \mu(F_*\mathcal{L}) \le \frac{\mathrm{I}(\mathcal{L}, X)}{p} - \frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \cdot \frac{n(p-1)}{2}. \tag{4.15}$$

In particular, when $\mu(\Omega^1_X) > 0$ and $\mathrm{T}^\ell(\Omega^1_X)$ $(1 \le \ell < n(p-1))$ are semi-stable, then $F_*\mathcal{L}$ and B^1_X are stable.

Proof. Since $\mu(\mathcal{F}_\ell) - \mu(V_\ell/V_{\ell+1}) \le \mathrm{I}(\mathcal{L} \otimes \mathrm{T}^\ell(\Omega^1_X)) = \mathrm{I}(\mathrm{T}^\ell(\Omega^1_X))$ and $\mathrm{I}(\mathcal{L}, X) = \max\{\ \mathrm{I}(\mathrm{T}^\ell(\Omega^1_X)) \mid 1 \le \ell < n(p-1)\ \}$, by (4.10), we only have to show

$$\sum_{\ell=0}^{m} (\frac{n(p-1)}{2} - \ell) r_\ell \ge \frac{n(p-1)}{2}.$$

From (4.11) and (4.12), we have

$$\sum_{\ell=0}^{m} (\frac{n(p-1)}{2} - \ell) r_\ell \ge \frac{n(p-1)}{2} r_0 \quad \text{if } m \ne n(p-1).$$

Thus it is enough to show $m \ne n(p-1)$ when $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(F_*\mathcal{L}) = p^n$. More general, we can show the following inequality

$$r_\ell \ge r_{n(p-1)} \cdot \mathrm{rk}(\mathrm{T}^{n(p-1)-\ell}(\Omega^1_X)) \quad \text{when } m = n(p-1), \tag{4.16}$$

which implies the following inequality

$$\mathrm{rk}(\mathcal{E}) = \sum_{\ell=0}^{m} r_\ell \ge r_{n(p-1)} \sum_{\ell=0}^{m} \mathrm{rk}(\mathrm{T}^{n(p-1)-\ell}(\Omega^1_X)) = r_{n(p-1)} \cdot p^n$$

if $m = n(p-1)$. Thus $m \ne n(p-1)$ when $\mathrm{rk}(\mathcal{E}) < p^n$.

To show (4.16) is a local problem. Let $K = K(X)$ be the function field of X and consider the K-algebra

$$R = \frac{K[\alpha_1, \cdots, \alpha_n]}{(\alpha_1^p, \cdots, \alpha_n^p)} = \bigoplus_{\ell=0}^{n(p-1)} R^\ell,$$

where R^ℓ is the K-linear space generated by

$$\{\,\alpha_1^{k_1}\cdots\alpha_n^{k_n}\mid k_1+\cdots+k_n=\ell,\quad 0\le k_i\le p-1\,\}.$$

The quotients in the filtration (4.5) can be described locally

$$V_\ell/V_{\ell+1}=W\otimes_K R^\ell$$

as K-vector spaces. If $K=k(x_1,...,x_n)$, then the homomorphism

$$\nabla: W\otimes_K R^\ell\to W\otimes_K R^{\ell-1}\otimes_K\Omega^1_{K/k}$$

in Theorem 4.1 (ii) is locally the k-linear homomorphism defined by

$$\nabla(w\otimes\alpha_1^{k_1}\cdots\alpha_n^{k_n})=-w\otimes\sum_{i=1}^n k_i(\alpha_1^{k_1}\cdots\alpha_i^{k_i-1}\cdots\alpha_n^{k_n})\otimes_K \mathrm{d}x_i.$$

Then the fact that $\mathcal{F}_\ell\xrightarrow{\nabla}\mathcal{F}_{\ell-1}\otimes\Omega^1_X$ for $\mathcal{F}_\ell\subset W\otimes R^\ell$ is equivalent to

$$\text{(4.17)}\quad \forall\ \sum_j w_j\otimes f_j\in\mathcal{F}_\ell\ \Rightarrow\ \sum_j w_j\otimes\frac{\partial f_j}{\partial\alpha_i}\in\mathcal{F}_{\ell-1}\quad(1\le i\le n).$$

The polynomial ring $\mathrm{P}=K[\partial_{\alpha_1},\cdots,\partial_{\alpha_n}]$ acts on R through partial derivations, which induces a D-module structure on R, where

$$\mathrm{D}=\frac{K[\partial_{\alpha_1},\cdots,\partial_{\alpha_n}]}{(\partial_{\alpha_1}^p,\cdots,\partial_{\alpha_n}^p)}=\bigoplus_{\ell=0}^{n(p-1)}\mathrm{D}_\ell$$

and D_ℓ is the linear space of degree ℓ homogeneous elements. In particular, $W\otimes R$ has the induced D-module structure with D acts on W trivially. Use this notation, (4.17) is equivalent to $\mathrm{D}_1\cdot\mathcal{F}_\ell\subset\mathcal{F}_{\ell-1}$.

Since $R^{n(p-1)}$ is of dimension 1, for any subspace

$$\mathcal{F}_{n(p-1)}\subset W\otimes R^{n(p-1)},$$

there is a subspace $W'\subset W$ of dimension $r_{n(p-1)}$ such that

$$\mathcal{F}_{n(p-1)}=W'\otimes R^{n(p-1)}.$$

Thus $\mathrm{D}_\ell\cdot\mathcal{F}_{n(p-1)}=W'\otimes\mathrm{D}_\ell\cdot R^{n(p-1)}=W'\otimes R^{n(p-1)-\ell}\subset\mathcal{F}_{n(p-1)-\ell}$ for all $0\le\ell\le n(p-1)$, which proves (4.16).

If $\mathrm{T}^\ell(\Omega^1_X)$ $(1\le\ell<n(p-1))$ are semi-stable, then $\mathrm{I}(\mathcal{L},X)=0$ and

$$\text{(4.18)}\qquad \mu(\mathcal{E})-\mu(F_*\mathcal{L})\le-\frac{\mu(\Omega^1_X)}{p\cdot\mathrm{rk}(\mathcal{E})}\cdot\frac{n(p-1)}{2},$$

which implies clearly the stability of $F_*\mathcal{L}$ if $\mu(\Omega^1_X) > 0$.

To show that (4.18) implies the stability of B^1_X, for any nontrivial subsheaf $B' \subset B^1_X$ of rank $r < \mathrm{rk}(B^1_X)$, let $\mathcal{E} \subset F_*\mathcal{O}_X$ be the subsheaf of rank $r+1$ such that we have exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to B' \to 0.$$

Substitute (4.18) to $\mu(B') - \mu(B^1_X) = \frac{r+1}{r}\mu(\mathcal{E}) - \frac{p^n}{p^n-1}\mu(F_*\mathcal{O}_X)$, we have

$$\mu(B') - \mu(B^1_X) \le \frac{p^n - 1 - r}{r(p^n-1)}\mu(F_*\mathcal{O}_X) - \frac{n(p-1)}{2rp}\mu(\Omega^1_X). \tag{4.19}$$

By (4.9) in Lemma 4.5, we have $\mu(F_*\mathcal{O}_X) = \frac{n(p-1)}{2p}\mu(\Omega^1_X)$. Thus

$$\mu(B') - \mu(B^1_X) \le -\frac{\mu(\Omega^1_X)}{p\cdot(p^n-1)} \cdot \frac{n(p-1)}{2}. \tag{4.20}$$

Q.E.D.

Remark 4.8. *When* $\dim(X) = 1$, *the quotients* $V_\ell/V_{\ell+1} = \mathcal{L}\otimes\omega^\ell_X$ *are line bundles and thus* $r_\ell = 1$ $(0 \le \ell \le m)$ *in* (4.10). *Then we can rewrite* (4.10) *(notice* $\mathrm{rk}(\mathcal{E}) = m+1$*):*

$$\mu(\mathcal{E}) - \mu(F_*\mathcal{L}) = \sum_{\ell=0}^{m}\frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p\cdot\mathrm{rk}(\mathcal{E})} - \frac{(p-\mathrm{rk}(\mathcal{E}))(g-1)}{p},$$

which impiles the following inequality

$$\mu(\mathcal{E}) - \mu(F_*\mathcal{L}) \le -\frac{(p-\mathrm{rk}(\mathcal{E}))(g-1)}{p}$$

and the equality holds if and only if $\mathcal{F}_\ell = V_\ell/V_{\ell+1}$. *Thus*

$$\mu(B') - \mu(B^1_X) \le -\frac{p-1-\mathrm{rk}(B')}{p}(g-1).$$

When X *is a curve of genus* $g \ge 2$, *the stability of* $F_*\mathcal{L}$ *was proved in* [10], *the semi-stability of* B^1_X *was proved by M. Raynaud in* [15], *its stability, which is related with a question of M. Raynaud in* [16], *was proved by K. Joshi in* [5]. *When* X *is a surface with* $\mu(\Omega^1_X) > 0$, *if* Ω^1_X *is semi-stable (which implies that* $\mathrm{T}^\ell(\Omega^1_X)$ $(1 \le \ell \le 2(p-1)$ *are semi-stable), thus* $F_*\mathcal{L}$ *and* B^1_X *are stable. The semi-stability of* B^1_X *was proved by Y. Kitadai and H. Sumihiro in* [9].

In the proof of Theorem 4.7, for a sub-sheaf $\mathcal{E} \subset F_*W$, we see that

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \sum_{\ell=0}^{m} r_\ell \frac{\mathrm{I}(W \otimes \mathrm{T}^\ell(\Omega^1_X))}{p \cdot \mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \frac{n(p-1)}{2} \tag{4.21}$$

if $m \neq n(p-1)$. Otherwise there is a sub-sheaf $W' \subset W$ of rank $r_{n(p-1)}$ such that $\mathcal{F}_{n(p-1)} = W' \otimes \mathrm{T}^{n(p-1)}(\Omega^1_X)$ and $W' \otimes \mathrm{T}^\ell(\Omega^1_X) \subset \mathcal{F}_\ell$. Let

$$0 \to W' \otimes \mathrm{T}^\ell(\Omega^1_X) \to \mathcal{F}_\ell \to \mathcal{F}'_\ell \to 0$$

be the induced exact sequence with $\mathcal{F}'_\ell \subset W/W' \otimes \mathrm{T}^\ell(\Omega^1_X)$. Then

$$\begin{aligned}\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}}) \leq & \frac{r_{n(p-1)}(\mathrm{rk}(\frac{V_\ell}{V_{\ell+1}}) - r_\ell)}{r_\ell \cdot \mathrm{rk}(W)}(\mu(W') - \mu(W/W')) \\ & + \frac{r'_\ell}{r_\ell} \cdot \mathrm{I}(W/W' \otimes \mathrm{T}^\ell(\Omega^1_X))\end{aligned}$$

where $r'_\ell := \mathrm{rk}(\mathcal{F}'_\ell)$. Substituting it to the equality (4.10), we have

$$\begin{aligned}\mu(\mathcal{E}) - \mu(F_*W) \leq & \sum_{\ell=0}^{n(p-1)} r'_\ell \cdot \frac{\mathrm{I}(W/W' \otimes \mathrm{T}^\ell(\Omega^1_X))}{p \cdot \mathrm{rk}(\mathcal{E})} + \\ & \frac{r_{n(p-1)}(\mathrm{rk}(F_*W) - \mathrm{rk}(\mathcal{E}))}{p \cdot \mathrm{rk}(\mathcal{E}) \cdot \mathrm{rk}(W)}(\mu(W') - \mu(W/W')).\end{aligned} \tag{4.22}$$

In the case of positive characteristic, it is well-known that tensor product of two semi-stable sheaves may not be semi-stable. Thus, even if W and $\mathrm{T}^\ell(\Omega^1_X)$ are semi-stable, Theorem 4.2 does not imply the semi-stability of F_*W. However the inequalities (4.21) and (4.22) indicate that it may be possible in some special cases that semi-stability of W and $\mathrm{T}^\ell(\Omega^1_X)$ can imply the semi-stability of F_*W. As an example, we prove a slightly generalized version of [9, Theorem 3.1].

Theorem 4.9. *Let X be a smooth projective surface with $\mu(\Omega^1_X) > 0$. Assume that Ω^1_X is semi-stable. Then $F_*(\mathcal{L} \otimes \Omega^1_X)$ is semi-stable for any line bundle $\mathcal{L}$ on X. Moreover, if Ω^1_X is stable, then $F_*(\mathcal{L} \otimes \Omega^1_X)$ is stable.*

Proof. When $\dim(X) = 2$, we have (cf. Proposition 3.5 of [20])

$$\mathrm{T}^\ell(\Omega^1_X) = \begin{cases} \mathrm{Sym}^\ell(\Omega^1_X) & \text{when } \ell < p \\ \mathrm{Sym}^{2(p-1)-\ell}(\Omega^1_X) \otimes \omega_X^{\ell-(p-1)} & \text{when } \ell \geq p \end{cases}$$

where $\omega_X = \mathcal{O}_X(K_X)$ is the canonical line bundle of X. Thus $\mathrm{T}^\ell(\Omega^1_X)$ are semi-stable whenever Ω^1_X is semi-stable.

For any nontrivial sub-sheaf $\mathcal{E} \subset F_*(\mathcal{L} \otimes \Omega^1_X)$, consider the induced filtration $0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}$ and

$$\mathcal{F}_\ell := \frac{V_\ell \cap F^*\mathcal{E}}{V_{\ell+1} \cap F^*\mathcal{E}} \subset \frac{V_\ell}{V_{\ell+1}}, \qquad r_\ell = \mathrm{rk}(\mathcal{F}_\ell).$$

If $m = 2(p-1)$, by using (4.22) for $W = \Omega^1_X$, we have

$$\mu(\mathcal{E}) - \mu(F_*W) \leq 0.$$

If $W = \Omega^1_X$ is stable, then $\mu(W') - \mu(W/W') < 0$ in (4.22) and

$$\mu(\mathcal{E}) - \mu(F_*W) < 0.$$

If $m \neq 2(p-1)$, we have

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \sum_{\ell=0}^{m} r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \cdot (p-1).$$

On the other hand, by a theorem of Ilangovan–Mehta–Parameswaran (cf. Section 6 of [11] for the precise statement): If E_1, E_2 are semi-stable bundles with $\mathrm{rk}(E_1) + \mathrm{rk}(E_2) \leq p+1$, then $E_1 \otimes E_2$ is semi-stable. We see that $V_\ell/V_{\ell+1} = \mathcal{L} \otimes \Omega^1_X \otimes \mathrm{T}^\ell(\Omega^1_X)$ are semi-stable except that

$$V_{p-1}/V_p = \mathcal{L} \otimes \Omega^1_X \otimes \mathrm{Sym}^{p-1}(\Omega^1_X)$$

may not be semi-stable. Thus we have

$$\mu(\mathcal{E}) - \mu(F_*W) \leq r_{p-1} \frac{\mu(\mathcal{F}_{p-1}) - \mu(\frac{V_{p-1}}{V_p})}{p \cdot \mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \cdot (p-1).$$

If $r_{p-1} = 0$, there is nothing to prove. If $r_{p-1} > 0$, we will prove

$$r_{p-1} \cdot (\mu(\mathcal{F}_{p-1}) - \mu(V_{p-1}/V_p)) \leq \mu(\Omega^1_X),$$

by using of the following two exact sequences

$$0 \to \mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L} \to V_{p-1}/V_p \to \mathrm{Sym}^p(\Omega^1_X) \otimes \mathcal{L} \to 0$$

$$0 \to \mathcal{L} \otimes F^*\Omega^1_X \to \mathrm{Sym}^p(\Omega^1_X) \otimes \mathcal{L} \to \mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L} \to 0$$

where all of the bundles have the same slope $p \cdot \mu(\Omega^1_X) + c_1(\mathcal{L}) \cdot \mathrm{H}$.

For $\mathcal{F}_{p-1} \subset V_{p-1}/V_p$, the first exact sequence above induces an exact sequence $0 \to \mathcal{F}'_{p-1} \to \mathcal{F}_{p-1} \to \mathcal{F}''_{p-1} \to 0$, where

$$\mathcal{F}'_{p-1} \subset \mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L}, \quad \mathcal{F}''_{p-1} \subset \mathrm{Sym}^p(\Omega^1_X) \otimes \mathcal{L}.$$

If $\mathcal{F}''_{p-1}$ is trivial, then we are done since $\mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L}$ is semi-stable with slope $\mu(V_{p-1}/V_p)$. If $\mathcal{F}''_{p-1} \neq 0$, we claim

$$r_{p-1} \cdot (\mu(\mathcal{F}_{p-1}) - \mu(V_{p-1}/V_p)) \leq \mathrm{rk}(\mathcal{F}''_{p-1}) \cdot (\mu(\mathcal{F}''_{p-1}) - \mu(V_{p-1}/V_p)).$$

Indeed, if $\mathcal{F}'_{p-1} = 0$, it is clear. If $\mathcal{F}'_{p-1} \neq 0$, we have

$$\mu(\mathcal{F}_{p-1}) = \frac{\mathrm{rk}(\mathcal{F}'_{p-1})}{r_{p-1}} \mu(\mathcal{F}'_{p-1}) + \frac{\mathrm{rk}(\mathcal{F}''_{p-1})}{r_{p-1}} \mu(\mathcal{F}''_{p-1})$$

and $\mu(\mathcal{F}'_{p-1}) \leq \mu(\mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L}) = \mu(V_{p-1}/V_p)$. Put all together, we have the claimed inequality. Thus it is enough to show

$$\mathrm{rk}(\mathcal{F}''_{p-1}) \cdot (\mu(\mathcal{F}''_{p-1}) - \mu(V_{p-1}/V_p)) \leq \mu(\Omega^1_X).$$

The second exact sequence induces an exact sequence

$$0 \to E_1 \to \mathcal{F}''_{p-1} \to E_2 \to 0$$

where $E_1 \subset \mathcal{L} \otimes F^*\Omega^1_X$, $E_2 \subset \mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L}$. If $E_1 = 0$, it is clearly done since $\mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes \mathcal{L}$ is semi-stable of slope $\mu(V_{p-1}/V_p)$. If $E_1 \neq 0$, by the same argument, we have

$$\mathrm{rk}(\mathcal{F}''_{p-1}) \cdot (\mu(\mathcal{F}''_{p-1}) - \mu(V_{p-1}/V_p)) \leq \mathrm{rk}(E_1)(\mu(E_1) - \mu(\mathcal{L} \otimes F^*\Omega^1_X)).$$

If $\mathrm{rk}(E_1) = 2$, then $E_1 = \mathcal{L} \otimes F^*\Omega^1_X$ and we clearly have

$$\mathrm{rk}(E_1)(\mu(E_1) - \mu(\mathcal{L} \otimes F^*\Omega^1_X)) = 0 < \mu(\Omega^1_X).$$

If $\mathrm{rk}(E_1) = 1$, then $\mu(E_1) - \mu(\mathcal{L} \otimes F^*\Omega^1_X) \leq \mu_{\max}(\Omega^1_X) = \mu(\Omega^1_X)$ is a special case of Proposition 3.9, and it is a strict inequality if Ω^1_X is stable. To sum up, what we have proved for $W = \mathcal{L} \otimes \Omega^1_X$ is

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \begin{cases} 0 & \text{when } m = 2(p-1) \\ -\frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \cdot (p-2) & \text{when } m < 2(p-1) \end{cases}$$

which is a strict inequality if Ω^1_X is stable. Q.E.D.

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