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BCOV ring and holomorphic anomaly equation

Shinobu Hosono

Abstract.

We study certain differential rings over the moduli space of Calabi–Yau manifolds. In the case of an elliptic curve, we observe a close relation to the differential ring of quasi-modular forms due to Kaneko–Zagier[23].

§1. Introduction

Since the pioneering work by Candelas, de la Ossa, Green and Parkes [8] in 1991, the theory of variation of Hodge structures has been one of the indispensable tools in the study of mirror symmetry of Calabi–Yau manifolds and its application to Gromov–Witten theory or enumerative geometry on Calabi–Yau manifolds. In particular, in the generalization due to Bershadsky, Cecotti, Ooguri and Vafa (BCOV) to higher genus Gromov–Witten potentials, the theory of variation of Hodge structures was combined with a framework which is called t-t^* geometry [9]. t-t^* geometry is a deformation theory of $N = 2$ supersymmetric quantum field theory in two dimensions, and there is a natural hermitian (real) structure in the space of observables. BCOV identifies this hermitian structure with the hermitian structure over the moduli space of Calabi–Yau manifolds given by the Weil–Petersson metric, and have proposed a profound recursive relation, called *holomorphic anomaly equation*, for higher genus Gromov–Witten potentials.

In case of dimension one, i.e. for elliptic curves, the counting problem and higher genus Gromov–Witten potentials have been determined by Dijkgraaf [11] in 1995, where it was remarked that BCOV theory

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is closely related to the theory of elliptic quasi-modular forms. In the same proceedings volume as [11], Kaneko and Zagier have presented a general theory of quasi-modular forms introducing the (differential) ring of almost holomorphic modular forms. It is also found in [21] that, for an rational elliptic surface, the higher genus Gromov–Witten potentials are expressed by quasi-modular forms, extending the genus zero result by [25], [26].

For Calabi–Yau threefolds, it has been expected that the BCOV holomorphic anomaly equation is defined over a certain differential ring which generalizes the almost holomorphic elliptic modular forms due to Kaneko and Zagier. Recently, in physics literatures, Yamaguchi and Yau [32] and later Alim and Länge [2] have made important developments toward the structure of the expected differential ring of Calabi–Yau threefolds (see also [1], [15]). In this paper, to make a parallel argument to the theory due to Kaneko and Zagier, we introduce three different differential rings $\mathcal{R}^0_{BCOV}$, $\mathcal{R}^\Gamma_{BCOV}$ and $\mathcal{R}^{hol}_{BCOV}$ over the moduli space of Calabi–Yau threefolds. We call these differential rings simply as *BCOV rings.* Our BCOV rings $\mathcal{R}^\Gamma_{BCOV}$ and $\mathcal{R}^{hol}_{BCOV}$, should be regarded as a natural generalization of the ring of almost holomorphic modular forms and quasi-modular forms, respectively, and may be recognized in the original work [5] and more explicitly in recent physics literatures [32], [2], [1], [15]. We will introduce another form of the BCOV ring $\mathcal{R}^0_{BCOV}$ and observe that, with this ring, our parallelism to Kaneko–Zagier theory becomes complete.

Here we summarize briefly the theory of quasi-modular forms due to Kaneko and Zagier. Kaneko–Zagier [23] starts from a ring

$$\mathbf{C}[[\tau]][\frac{1}{\tau-\bar{\tau}}] \ , \tag{1.1}$$

with τ in the upper-half plane, and the standard modular group action, $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. Inside this large ring, one first considers the almost modular forms $\widehat{M}(\Gamma)_k$ of weight k as almost holomorphic functions $F(\tau,\bar{\tau})$ on the upper-half plane which transforms like a modular form of weight k;

$$F(\frac{a\tau+b}{c\tau+d}, \overline{\frac{a\tau+b}{c\tau+d}}) = (c\tau+d)^k F(\tau,\bar{\tau}) \ .$$

Then the ring of the almost holomorphic modular forms $\widehat{M}(\Gamma) = \oplus_{k\geq 0}\widehat{M}(\Gamma)_k$ becomes a differential ring under $D_\tau : \widehat{M}(\Gamma)_k \to \widehat{M}(\Gamma)_{k+2}$, with $D_\tau := \frac{1}{2\pi i}(\frac{\partial}{\partial\tau} + \frac{k}{\tau-\bar{\tau}})$. Elements of $\widehat{M}(\Gamma)$ have an expansion $F = \sum_{m\geq 0} c_m(\tau)\big(\frac{1}{\tau-\bar{\tau}}\big)^m$, and by taking the first coefficient $c_0(\tau)$ we obtain holomorphic objects. Kaneko–Zagier shows that this map defines a (differential) ring isomorphism $\varphi : \widehat{M}(\Gamma) \to \widetilde{M}(\Gamma)$, where $\widetilde{M}(\Gamma) = \mathbf{C}[E_2(\tau), E_4(\tau), E_6(\tau)]$ is the ring of the quasi-modular forms with the differential $\partial_\tau := \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$. Our observation here is that the ring (1.1) is defined by the Kähler geometry on the upper-half plane, and has a natural generalization to the Weil–Petersson geometry on the moduli space of Calabi–Yau manifolds. Based on this, we will introduce our BCOV ring $\mathcal{R}^0_{BCOV}$ in terms of purely geometric data, and subsequently introduce other forms of the ring, $\mathcal{R}^\Gamma_{BCOV}$, $\mathcal{R}^{hol}_{BCOV}$. We may schematically write our parallelism of the BCOV rings to the relevant rings in Kaneko–Zagier theory:

$$\begin{array}{ccccc} \mathbf{C}[[\tau]][\frac{1}{\tau-\bar{\tau}}] & \supset & \widehat{M}(\Gamma) & \xrightarrow{\varphi} & \widetilde{M}(\Gamma) \\ \mathcal{R}^0_{BCOV} & \to & \mathcal{R}^\Gamma_{BCOV} & \xrightarrow{\bar{t}\to\infty} & \mathcal{R}^{hol}_{BCOV} \end{array} .$$

As one see in the arrow $\mathcal{R}^0_{BCOV} \to \mathcal{R}^\Gamma_{BCOV}$, instead of $\supset$, the relations shown in this diagram are not exact correspondences but should be understood simply as parallelism. In fact, in our BCOV ring, following [5], we work with meromorphic sections of certain bundles instead of (almost) holomorphic forms in Kaneko–Zagier theory. Details will be described in the text, however it should be helpful to have this schematic diagram in mind.

The main result of this paper is the introduction of the BCOV ring $\mathcal{R}^0_{BCOV}$ (Definition 3.3, Theorem 3.5) and making the parallelism to Kaneko–Zagier theory of quasi-modular forms complete.

Construction of this paper is as follows. To make the paper self-contained, in Section 2, we review the geometry of the moduli space of Calabi–Yau manifold, which is called special Kähler geometry, and set up our notations. In Section 3, we define our BCOV rings. In subsection (3-1), we introduce the first form of our BCOV (differential) ring $\mathcal{R}^0_{BCOV}$ based on the special Kähler geometry (Definition 3.1, Theorem 3.5). We remark that the BCOV ring $\mathcal{R}^0_{BCOV}$ is infinitely generated, however there is a natural reduction $\mathcal{R}^{0,red}_{BCOV}$ to a finitely generated ring. In (3-2), we consider modular (monodromy) property of the ring and we will define a differential ring $\mathcal{R}^\Gamma_{BCOV}$ as the ring of monodromy invariants. We, then, understand in our framework the process of 'fixing a holomorphic (meromorphic) ambiguities in the propagator functions' given in the Section 6.3 of [5]. In (3-3), BCOV rings $\mathcal{R}^0_{BCOV}$ and $\mathcal{R}^\Gamma_{BCOV}$ will be determined explicitly for an elliptic curve. There we provide a precise relation to the theory of quasi-modular forms (Propositions 3.11, 3.12). In (3-4), the final form $\mathcal{R}^{hol}_{BCOV}$ is defined under a choice of a symplectic basis of the middle dimensional homology group. In Section 4, toward an application to the holomorphic anomaly equation, we introduce the holomorphic anomaly equation in the form appeared in [32], [2]. Considering holomorphic anomaly equation in the reduced ring $\mathcal{R}^{0,red}_{BCOV}$ we note that the equation simplifies to a system of linear differential equation (Proposition 4.3) which is easy to handle. Conclusions and discussion are given in Section 5. There, the equivalence of the modular anomaly equation in [21] to BCOV holomorphic anomaly equation is also announced.

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§2. Special Kähler geometry on deformation spaces

(2-1) Period integrals. Let us consider a family of Calabi–Yau 3-folds $\mathcal{Y} = \{Y_x\}$ over a small complex domain B with fibers Y_x ($x \in B$). As such a family, we will consider hypersurfaces or complete intersections in projective toric varieties, and assume that the local family eventually extends to a family over a toric variety $\mathcal{M}$ ($= \mathbf{P}_{Sec(\Sigma)}$ with the secondary fan, see eg. [12]). We also assume that $\dim\mathcal{M} = \dim H^{2,1}(Y_{x_0})$ for a smooth Y_{x_0}.

We fix a smooth $Y_{x_0}(x_0 \in B)$ as above. We denote the cohomology $H^3(Y_{x_0}, \mathbf{Z})$ by H_{x_0} and write the symplectic form there by

$$(2.1) \qquad \langle u, v \rangle := \sqrt{-1} \int_{Y_{x_0}} u \cup v \ .$$

We choose a symplectic basis $\{\alpha_I, \beta^J\}_{0 \le I,J \le r}$ satisfying $\langle \alpha_I, \beta^J \rangle = \delta_I^{\ J}$, $\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0$, and we denote its dual homology basis by $\{A^I, B_J\}_{0 \le I,J \le r}$ ($r := \dim H^{2,1}(Y_{x_0})$). With respect to this basis, we write the symplectic form

$$Q = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} ,$$

where $E = E_{r+1}$ represents the unit matrix of size $(r+1)$. We define the period domain

$$\mathcal{D} = \{ [\omega] \in \mathbf{P}(H_{x_0} \otimes \mathbf{C}) \,|\, \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0 \} .$$

We choose a holomorphic three form of the fiber Y_x $(x \in B)$ and denote it by $\Omega_x := \Omega(Y_x)$. Then the period map $\mathcal{P}_0 : B \to \mathcal{D}$ is defined by

$$\mathcal{P}_0(x) = [\sum_I \left(\int_{A^I} \Omega_x\right) \alpha_I + \sum_J \left(\int_{B_J} \Omega_x\right) \beta^J] \ .$$

Using path-dependent identification $H_3(Y_x, \mathbf{Z}) \cong H_3(Y_{x_0}, \mathbf{Z})$, we may globalize this period map on B to $\mathcal{M}$ by introducing a covering space $\tilde{\mathcal{M}}$. We denote the resulting period map $\mathcal{P} : \tilde{\mathcal{M}} \to \mathcal{D}$ and assume $\Gamma \subset Sp(2r+2, \mathbf{Z})$ as the covering group. In this paper, we will write the period map $\mathcal{P}(x) = [\vec{\omega}(x)]$ (or $\mathcal{P}_0(x) = [\vec{\omega}(x)]$ $(x \in B)$) with the notations for the period integrals,

$$\vec{\omega}(x) = \sum_I X^I(x)\alpha_I + \sum_J P_J(x)\beta^J \ .$$

We write by $\mathcal{U}$ the restriction to $\mathcal{D}$ of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}(H_{x_0} \otimes \mathbf{C})$. We then set $\mathcal{L} = \mathcal{P}^*\mathcal{U}$, i.e., the pullback to $\tilde{\mathcal{M}}$. Complex conjugate of $\mathcal{L}$ will be denoted by $\overline{\mathcal{L}}$. The sections of $\mathcal{L}^{\otimes n} \otimes \overline{\mathcal{L}}^{\otimes m}$ will be often referred to as 'sections of weight (n, m)'. The period integral $\vec{\omega}(x)$ may be considered as a section of $\mathcal{L}$, and thus has weight $(1, 0)$.

(2-2) Prepotential. The symplectic form (2.1) naturally induces one form θ on $\mathcal{D}$ by $\theta := \langle d\omega, \omega \rangle$. With this one form, $(\mathcal{D}, \theta)$ becomes a holomorphic contact manifold of dimension $2r+1$. Since locally, the period map $\mathcal{P}_0 : B \to \mathcal{D}$ is an embedding [6], [30], [31] and also $\theta|_{\mathcal{P}_0(B)} = 0$ due to Griffiths transversality, we know that the image of the period map $\mathcal{P}_0$ is a Legendre submanifold. Combining this with Gauss correspondence in projective geometry, it is found in general [7] that the image of the period map $\mathcal{P}_0$ can be recovered by the half of the period integrals $X^I(x) = \int_{A^I} \Omega_x$. More concretely, it is known that:

1) The map $x \mapsto [X^0(x), \cdots, X^r(x)] \in \mathbf{P}^r$ is an local isomorphism $B \to \mathbf{P}^r$,
2) Integrating $\theta|_{\mathcal{P}_0(B)} = 0$ on B, we can write the other half of the period integrals,

$$P_J(x) = \frac{\partial \mathcal{F}(X^I)}{\partial X^J} \ \ (J = 0, 1, \cdots, r),$$

in terms of a holomorphic function $\mathcal{F}(X)$ called *prepotential.*

The function $\mathcal{F}(X)$ is an holomorphic function of $X^0(x), \cdots, X^r(x)$ and has the following homogeneous property

$$\sum_{I=0}^{r} X^I(x) \frac{\partial \mathcal{F}}{\partial X^I} = 2\mathcal{F}(X) \ .$$

This potential function exists locally for the small domain B. When globalizing the above local arguments to $\tilde{\mathcal{M}}$, we naturally see that the monodromy group Γ plays a role for the definition of $\mathcal{F}(X)$ (see below). The group action of Γ is referred to as *duality transformation* in physics (see e.g. [10] and references therein). Here we remark that the holomorphic prepotential has a simple relation to the so-called Griffiths–Yukawa coupling of the family $\{Y_x\}_{x\in B}$;

$$(2.2) \quad -\int_{Y_x} \Omega_x \cup \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\Omega_x = \sum_{I,J,K=0}^{r} \frac{\partial X^I}{\partial x^i}\frac{\partial X^J}{\partial x^j}\frac{\partial X^K}{\partial x^k}\frac{\partial^3 \mathcal{F}(X)}{\partial X^I \partial X^J \partial X^K} \ ,$$

where the l.h.s. is will be written by $C_{ijk}(x)$ hereafter. We denote $\overline{C}_{\bar{i}\bar{j}\bar{k}}$ the complex conjugate of $C_{ijk}(x)$.

(2-3) Period matrix (1). The most important object to introduce the BCOV anomaly equation is the classical period matrix. For simplicity, let us assume that our family $\{Y_x\}$ is given by a family of hypersurfaces in a (smooth) toric variety $\mathbf{P}^4_\Sigma$, with the parameter $x = (x^1, x^2, \cdots, x^r)$ compactified to a toric variety $\mathcal{M}$. We then consider $\mathcal{A}^q_k$, the set of rational q forms on $\mathbf{P}^4_\Sigma$ with a pole order less than k along Y_x, and set the cohomology group

$$\mathcal{H}_k = \mathcal{A}^4_k / d\mathcal{A}^3_{k-1} \ .$$

Obviously we have $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots$, and, in fact, this stabilizes at $\mathcal{H}_4$ $(= \mathcal{H}_5 = \cdots)$ to the rational 4 forms $\mathcal{H}$ of poles along Y_x. We note that, for 3-folds, the 'tubular' map $\tau : H_3(Y_x, \mathbf{Z}) \tilde{\rightarrow} H_4(\mathbf{P}_\Sigma \setminus Y_x, \mathbf{Z})$ is an isomorphism ([Gr, Proposition 3.5]), and is related to the Poincaré residue map $R : \mathcal{H} \rightarrow H^3(Y_x, \mathbf{C})$ along Y_x by

$$\int_{\tau(\gamma)} \omega = \int_\gamma R(\omega) \ \ (\omega \in \mathcal{H}_k).$$

This residue map is an isomorphism, and more precisely, maps the filtration $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \mathcal{H}_4$ to the Hodge filtration

$$F^{3,0} \subset F^{3,1} \subset F^{3,2} \subset F^{3,3} \ .$$

The holomorphic three form $\Omega_x = R(\omega_0)$ may then be given by a basis ω_0 of $\mathcal{H}_1 \cong F^{3,0}$. We take a basis $\omega_0, \omega_1, \cdots, \omega_r$ of $\mathcal{H}_2$, and define the period matrix

$$(2.3) \quad \mathbf{\Omega} = \begin{pmatrix} \int_{\tau(A_0)} \omega_0 & \cdots & \int_{\tau(A_r)} \omega_0 & \int_{\tau(B_0)} \omega_0 & \cdots & \int_{\tau(B_r)} \omega_0 \\ & \vdots & & & \vdots & \\ \int_{\tau(A_0)} \omega_r & \cdots & \int_{\tau(A_r)} \omega_r & \int_{\tau(B_0)} \omega_r & \cdots & \int_{\tau(B_r)} \omega_r \end{pmatrix}$$

as $(r+1, 2r+2)$ matrix. The first row of this matrix coincides with the period integral $\vec{\omega}(x) = \sum_I X^I(x)\alpha_I + \sum_J P_J(x)\beta^J$, and the monodromy group Γ acts on this period matrix from the right.

The following properties of $\mathbf{\Omega}$ are consequences of the filtration (2) and the Hodge–Riemann bilinear relations;

$$(2.4) \quad \begin{aligned} &1) \quad \mathbf{\Omega}\, Q\, {}^t\mathbf{\Omega} = 0 \\ &2) \quad \sqrt{-1}\mathbf{\Omega}\, Q\, {}^t\overline{\mathbf{\Omega}} > 0\ . \end{aligned}$$

Here 2) means that when we decompose the $(r+1) \times (r+1)$ hermitian matrix $\sqrt{-1}\mathbf{\Omega} Q {}^t\overline{\mathbf{\Omega}}$ into the block form compatible with the filtration $\mathcal{H}_1 \subset \mathcal{H}_2$, then the first diagonal block (1×1 matrix) is positive definite and the whole $(r+1)\times(r+1)$ hermitian matrix has 1 positive eigenvalue and r negative eigenvalues.

In our case of hypersurfaces (or complete intersections) in toric varieties $\mathbf{P}^4_\Sigma$, the basis ω_0 for the rational differential $\mathcal{H}_1$ can be given explicitly by the defining equation of Y_x with the deformations $x^1, \cdots, x^r$. Then our assumption hereafter for the family $\{Y_x\}_{x\in B}$ is that the derivatives

$$(2.5) \qquad \partial_1\omega_0, \cdots, \partial_r\omega_0 \quad (\partial_i\omega_0 := \frac{\partial}{\partial x^i}\omega_0)$$

span $\mathcal{H}_2$ together with the basis ω_0 of $\mathcal{H}_1$ for each small complex domain $B \subset \mathcal{M}$. It will be useful to define a notation $\partial_0\omega_0 \equiv \omega_0$. Now using $\int_{\tau(\gamma)} \partial_i\omega_0 = \int_\gamma R(\partial_i\omega_0) = \partial_i \int_\gamma R(\omega_0)$, we can write the period matrix simply by

$$(2.6) \qquad \mathbf{\Omega} = \begin{pmatrix} \partial_i X^I & \partial_i P_J \end{pmatrix} = \begin{pmatrix} \partial_i X^I & \partial_i X^J \tau_{IJ} \end{pmatrix}_{0 \le i, I \le r}$$

where we define $\tau_{IJ} = \frac{\partial^2}{\partial X^I \partial X^J}\mathcal{F}(X)$, and set $\partial_0 X^I \equiv X^I, \partial_0 P_J \equiv P_J$. In the above formula, and also hereafter, the repeated indices are assumed to be summed over (Einstein's convention) unless otherwise mentioned. Using the property 1) in the previous paragraph (2-2), we may assume

$$\det(\partial_i X^I)_{0\le i,I\le r} = (X^0)^{r+1} \det\left(\partial_i\left(\frac{X^I}{X^0}\right)\right)_{1\le i,I\le r} \neq 0$$

for x such that $X^0(x) \neq 0$. Hence, at least locally, we can normalize the period matrix in the form

$$(\partial_i X^I)^{-1}\mathbf{\Omega} = (E\ \tau) \ .$$

In this normalized form, the bilinear relation 1) in (2.4) is trivial since $\tau_{IJ} = \tau_{JI}$, while 2) entails

$$\sqrt{-1}(\overline{\tau} - \tau) > 0 \ ,$$

which means the matrix $\mathrm{Im}\,\tau$ has one positive and r negative eigenvalues. Here we note a similarity to the period matrix of genus g curves, however the mixed property of the eigenvalues is a new feature in higher dimensions.

Finally, we note that the monodromy group Γ acts on the normalized period matrix from the right, and for $\left(\begin{smallmatrix} D & B \\ C & A\end{smallmatrix}\right) \in \Gamma$ we have

$$\tau \mapsto (C\tau + D)^{-1}(A\tau + B) \ . \tag{2.7}$$

Since $\frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}\mathcal{F} = \tau_{IJ}$, this describes the transformation property of the prepotential which is defined locally for the family over B.

(2-4) Period matrix (2). Period matrix (2.6) has been defined entirely in the holomorphic category, since it is based on the Hodge filtration. One may modify the Hodge filtration to Hodge decomposition if we incorporate a hermitian structure coming from the Kähler geometry on $\mathcal{M}$. Let us first note that over the small domain B, and hence on $\mathcal{M}$ except the degeneration loci, there exists a Kähler metric $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(x,\bar{x})$ $(1 \le i, \bar{j} \le r)$ with the Kähler potential

$$K(x,\bar{x}) = -\log\{\langle\Omega_x, \overline{\Omega}_x\rangle\} = -\log\{\langle R(\omega_0), \overline{R(\omega_0)}\rangle\} \ .$$

This Kähler metric is called Weil–Petersson metric on $\mathcal{M}$. The bases $\partial_0\omega_0 \equiv \omega_0; \partial_1\omega_0, \cdots, \partial_r\omega_0$ which are compatible with the filtration $\mathcal{H}_1 \subset \mathcal{H}_2$, or the Hodge filtration $F^{3,0} \subset F^{3,1}$, may now be modified to

$$D_0\omega_0 \equiv \omega_0; D_1\omega_0, \cdots, D_r\omega_0 \ \ (D_i = \partial_i + K_i, i = 1, \cdots, r),$$

where $K_i = \partial_i K(x,\bar{x})$. One should observe that, since $\langle R(\omega_0), R(D_i\omega_0)\rangle = 0$ holds for $i = 1, \cdots, r$, these bases are compatible to the Hodge decomposition $H^{3,0}(Y_x)\oplus H^{2,1}(Y_x)$. Correspondingly, the period matrix (2.6) may be modified to

$$\boldsymbol{\Omega} = \begin{pmatrix} D_iX^I & D_iP_J \end{pmatrix} = \begin{pmatrix} D_iX^I & D_iX^J\tau_{IJ} \end{pmatrix}_{0\le i,I\le r} , \tag{2.8}$$

where we set $D_0X^I \equiv X^I, D_0P_J \equiv P_J$. Since we have $\det(D_iX^I)_{0\le i,I\le r} = \det(\partial_iX^I)_{0\le i,I\le r} \neq 0$, the normalized period integral has a similar form as before;

$$(D_iX^I)^{-1}\boldsymbol{\Omega} = \begin{pmatrix} E & \tau \end{pmatrix} .$$

The same monodromy group Γ acts from the right on the period matrix. In contrast to this, the left action shows a nice connection to the Kähler geometry on $\mathcal{M}$.

(2-5) Special Kähler geometry on $\mathcal{M}$. After some algebra, we have for the Kähler metric

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(x,\bar{x}) = -e^{K(x,\bar{x})}\langle R(D_i\omega_0), \overline{R(D_j\omega_0)}\rangle.$$

With respect to this, we introduce the metric connections

$$\Gamma^k_{ij} = g^{k\bar{k}}\partial_i g_{j\bar{k}} \ , \ \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = g^{\bar{k}k}\partial_{\bar{i}} g_{\bar{j}k} ,$$

for the holomorphic tangent bundle $T\mathcal{M}$ and the anti-holomorphic tangent bundle $T'\mathcal{M}$, respectively. The curvature tensor for these connections are given by

$$R_{i\bar{j}}{}^k{}_l = -\partial_{\bar{j}}\Gamma^k_{il} \ , \ R_{i\bar{j}}{}^{\bar{k}}{}_{\bar{l}} = \partial_i\Gamma^{\bar{k}}_{\bar{j}\bar{l}} .$$

In addition to these, we have the Griffiths–Yukawa couplings $C_{ijk}(x)$ and $\overline{C}_{\bar{i}\bar{j}\bar{k}}(\bar{x})$ on the moduli space $\mathcal{M}$. These tensors define the so-called *special Kähler geometry* on $\mathcal{M}$, which can be summarized into a property of the period matrix (2.8).

Let us introduce the following $(2r+2)\times(2r+2)$ matrix

$$\begin{pmatrix} \boldsymbol{\Omega} \\ \overline{\boldsymbol{\Omega}} \end{pmatrix} = \begin{pmatrix} D_iX^I & D_iP_J \\ \overline{D_iX^I} & \overline{D_iP_J} \end{pmatrix}_{0\le i,I,J\le r} . \tag{2.9}$$

By definition of the period matrix, the row vectors represent the Hodge decomposition $H^{3,0}(Y_x)\oplus H^{2,1}(Y_x)\oplus H^{1,2}(Y_x)\oplus H^{0,3}(Y_x)$ in terms of the bases

$$R(\omega_0),\ R(D_i\omega_0)\ ,\ \overline{R(D_i\omega_0)}\ ,\ \overline{R(\omega_0)}\ . \tag{2.10}$$

By simply writing (2.9), we implicitly understand the row vectors are ordered according to the Hodge decomposition above. With these explicit bases in mind, we introduce the covariant derivatives acting on the column vectors $\mathbf{\Omega}$ and $\overline{\mathbf{\Omega}}$;

$$D_i = \begin{cases} \partial_i + K_i - \Gamma^*_{i*} & \text{on } \mathbf{\Omega} \\ \partial_i & \text{on } \overline{\mathbf{\Omega}} \end{cases}, \quad \overline{D}_{\bar{i}} = \begin{cases} \partial_{\bar{i}} & \text{on } \mathbf{\Omega} \\ \partial_{\bar{i}} + K_{\bar{i}} - \Gamma^*_{\bar{i}*} & \text{on } \overline{\mathbf{\Omega}} \end{cases},$$

where Γ^*_{i*} and $\Gamma^*_{\bar{i}*}$ represents the conventional form of the contraction via the metric connection.

Theorem 2.1. *The period matrix satisfies*

$$(2.11) \qquad (D_i + \mathcal{A}_i)\begin{pmatrix} \mathbf{\Omega} \\ \overline{\mathbf{\Omega}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (\overline{D}_{\bar{i}} + \bar{\mathcal{A}}_{\bar{i}})\begin{pmatrix} \mathbf{\Omega} \\ \overline{\mathbf{\Omega}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

with

$$\mathcal{A}_i = \begin{array}{c} \\ 0 \\ m \\ \bar{m} \\ \bar{0} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & n & \bar{n} & \bar{0} \end{array} \\ \begin{pmatrix} 0 & -\delta^n_i & 0 & 0 \\ 0 & 0 & C^{\bar{n}}_{im} & 0 \\ 0 & 0 & 0 & -g_{i\bar{m}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \bar{\mathcal{A}}_{\bar{i}} = \begin{array}{c} \\ 0 \\ m \\ \bar{m} \\ \bar{0} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & n & \bar{n} & \bar{0} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -g_{\bar{i}m} & 0 & 0 & 0 \\ 0 & \overline{C}^{n}_{\bar{i}\bar{m}} & 0 & 0 \\ 0 & 0 & -\delta^{\bar{n}}_{\bar{i}} & 0 \end{pmatrix} \end{array},$$

where we set $C^{\bar{n}}_{im} = \sqrt{-1}e^K C_{imn}g^{n\bar{n}}$ *and* $\overline{C}^{n}_{\bar{i}\bar{m}} = -\sqrt{-1}e^K \overline{C}_{\bar{i}\bar{m}\bar{n}}g^{n\bar{n}}$.

Proof. Let us consider, the three from $R(D_j\omega_0)$ in the Hodge decomposition (2.10). Then $D_iR(D_j\omega_0)$ is a three form and may be expressed in terms of the basis (2.10) as

$$D_iR(D_j\omega_0) = c_0R(\omega_0) + c_mR(D_m\omega_0) + d_m\overline{R(D_m\omega_0)} + d_0\overline{R(\omega_0)}\,.$$

Now, using $\langle R(D_n\omega_0), \overline{R(D_m\omega_0)}\rangle = -e^{-K}g_{n\bar{m}}$ and other orthogonal relations, it is easy to see $c_0 = c_m = d_0 = 0$ and

$$\begin{aligned} -d_m e^{-K} g_{k\bar{m}} &= \langle R(D_k\omega_0), D_iR(D_j\omega_0)\rangle \\ &= -\langle R(\omega_0), D_kD_iD_jR(\omega_0)\rangle = \sqrt{-1}C_{ijk}\ , \end{aligned}$$

where C_{ijk} is the Griffiths–Yukawa coupling (2.2) ($R(\omega_0) = \Omega_x$). One can continue similar arguments for other bases of the Hodge decomposition (2.10). Integrating three forms over the cycles $\{A^I, B_J\}$, we obtain the claimed linear relations for the row vectors of the period matrix. Q.E.D.

The connection matrix $\mathcal{A}_i, \mathcal{A}_{\bar{i}}$ was first determined by Strominger in [28]. The existence of the first order differential operator is due to the fact that the Hodge decomposition is 'flat' over $\mathcal{M}$, and we should have the compatibility relations for the first order system. Indeed one can see that

$$[D_i + \mathcal{A}_i, D_j + \mathcal{A}_j] = 0 = [\overline{D}_{\bar{i}} + \overline{\mathcal{A}}_{\bar{i}}, \overline{D}_{\bar{j}} + \overline{\mathcal{A}}_{\bar{j}}],$$

are ensured by the existence of the prepotential (2.2), and another mixed-type compatibility condition imposes a rather strong constraint on the Kähler geometry [28], which is called special Kähler geometry (see, e.g., [10] and references therein).

Theorem 2.2. *The compatibility condition*

$$[D_i + \mathcal{A}_i, \overline{D}_{\bar{j}} + \overline{\mathcal{A}}_{\bar{j}}] = 0$$

is equivalent to

$$-R_{i\bar{j}\;l}^{\;\;k} = g_{i\bar{j}}\delta_l^{\,k} + g_{\bar{j}l}\delta_i^{\,k} - e^{2K} C_{ilm} \overline{C}_{\bar{j}\bar{k}\bar{m}} g^{m\bar{m}} g^{k\bar{k}} \quad . \tag{2.12}$$

In Section (3-1), we will derive the relation (2.12) directly evaluating the metric connection. The both equations (2.11) and (2.12) are often referred to as special Kähler geometry relations, and will play central roles in solving BCOV anomaly equation.

(2-6) Notations. As we have summarized above, the special Kähler geometry relations, in the holomorphic local coordinate $x^i (i = 1, \cdots, r)$, on $\mathcal{M}$ arises from the flat property of the period matrix $\boldsymbol{\Omega}$ and its complex conjugate. Since the row vectors of $\boldsymbol{\Omega}$ correspond to the decomposition $H^{3,0}(Y_x) \oplus H^{2,1}(Y_x)$, it is convenient to introduce the (Greek letter) notation $\alpha = (0, i)$ and $\bar{\alpha} = (\bar{0}, \bar{i})$ for the indices of the row vectors and their complex conjugates, respectively. Then the period matrix may be written simply by

$$\boldsymbol{\Omega} = (D_\alpha X^I \, D_\alpha P_J) = (D_\alpha X^I) \, (\, E \;\; \tau_{IJ} \,) \ .$$

As remarked in subsection (2-4), $(r+1) \times (r+1)$ matrix $(D_\alpha X^I)$ is invertible. We set $\xi_\alpha^{\,I} := D_\alpha X^I$, $\bar{\xi}_{\bar{\alpha}}^{\,I} := \overline{D_\alpha X^I}$ and define $\xi_I^{\,\alpha}$ and $\bar{\xi}_I^{\,\bar{\alpha}}$, respectively, by

$$(\xi_I^{\,\alpha}) = (D_\alpha X^I)^{-1} \ , \ (\bar{\xi}_I^{\,\bar{\alpha}}) = (\overline{D_\alpha X^I})^{-1} \ .$$

Note that the i-th row of $\boldsymbol{\Omega}$ represents period integrals of the three form $R(D_i\omega_0) \in H^{2,1}(Y_x)$ for a symplectic basis $\{A^I, B_J\}$. Then we can see the Weil–Petersson metric in the following $(r+1)\times(r+1)$ hermitian matrix,

$$\begin{pmatrix} g_{0\bar{0}} & 0 \\ 0 & g_{i\bar{j}} \end{pmatrix} = -e^{K(x,\bar{x})}\sqrt{-1}\boldsymbol{\Omega} Q {}^t\overline{\boldsymbol{\Omega}} \quad (g_{0\bar{0}} \equiv -1).$$

Equivalently one can write this matrix relation by

$$g_{\alpha\bar{\beta}} = \sqrt{-1}e^{K(x,\bar{x})}\,\xi_\alpha^{\ I}\,(\tau-\bar{\tau})_{IJ}\,\bar{\xi}_{\bar{\beta}}^{\ J} \ .$$

For the tensor analysis in later sections, we introduce a "metric" by

$$(\mathcal{G}_{IJ}) = \sqrt{-1}e^{K(x,\bar{x})}\big(\tau-\bar{\tau}\big) \ , \ (\mathcal{G}^{IJ}) = -\sqrt{-1}e^{-K(x,\bar{x})}\big(\tau-\bar{\tau}\big)^{-1} .$$

Then we have $g_{\alpha\bar{\beta}} = \xi_\alpha^{\ I}\mathcal{G}_{IJ}\bar{\xi}_{\bar{\beta}}^{\ J}, g^{\alpha\bar{\beta}} = \xi_I^{\ \alpha}\mathcal{G}^{IJ}\bar{\xi}_J^{\ \bar{\beta}}$. With these metrics we will raise and lower the indices $I, J, \cdots$ as well as the Greek indices.

Now the special Kähler geometry relations in Theorem2.1 may be expressed by

$$(2.13)\quad \begin{cases} D_i\xi_0^{\ I} = \xi_i^{\ I} \\ D_i\xi_j^{\ I} = -\sqrt{-1}e^K C_{ijk}g^{k\bar{k}}\bar{\xi}_{\bar{k}}^{\ I} \\ D_i\bar{\xi}_{\bar{j}}^{\ I} = g_{i\bar{j}}\bar{\xi}_{\bar{0}}^{\ I} \\ D_i\bar{\xi}_{\bar{0}}^{\ I} = 0 \end{cases} , \begin{cases} \overline{D}_{\bar{i}}\xi_0^{\ I} = 0 \\ \overline{D}_{\bar{i}}\xi_j^{\ I} = g_{\bar{i}j}\xi_0^{\ I} \\ \overline{D}_{\bar{i}}\bar{\xi}_{\bar{j}}^{\ I} = \sqrt{-1}e^K\overline{C}_{\bar{i}\bar{j}\bar{k}}g^{k\bar{k}}\xi_k^{\ I} \\ \overline{D}_{\bar{i}}\bar{\xi}_{\bar{0}}^{\ I} = \bar{\xi}_{\bar{i}}^{\ I} \end{cases} .$$

It will be useful to note that the following relation holds by definition;

$$(2.14)\qquad D_i\xi_j^{\ J} = C_{ijm}S^{mn}\xi_n^{\ I} \ ,$$

where S^{mn} is the *propagator* that will be introduced in the next section.

§3. BCOV rings

(3-1) BCOV ring $\mathcal{R}^0_{BCOV}$. Based on the special Kähler geometry relations summarized in the previous section, we introduce a differential ring over the meromorphic sections of a certain vector bundle over $\tilde{\mathcal{M}}$. Let us first introduce the so-called propagators;

Definition 3.1.

$$S_{\bar{\alpha}\bar{\beta}} = e^{2K}(\tau-\bar{\tau})_{IJ}\bar{\xi}_{\bar{\alpha}}^{\ I}\bar{\xi}_{\bar{\beta}}^{\ J} \ , \ S^{\alpha\beta} = g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}S_{\bar{\alpha}\bar{\beta}} \ .$$

Obviously $S_{\bar{\alpha}\bar{\beta}}$ and $S^{\alpha\beta}$ are symmetric with respect to the indices. Note that, since the factor e^{2K} has weight $(-2,-2)$, both $S^{\alpha\beta}$ and $S_{\bar{\alpha}\bar{\beta}}$ have weight $(-2,0)$ with respect to the line bundle $\mathcal{L}$ (see Section (2-1)).

Following [5], we will often use the notation S, S^i, S^{ij}, which are related to $S^{\alpha\beta}$ by

$$\begin{pmatrix} S^{00} & S^{0i} \\ S^{0i} & S^{ij} \end{pmatrix} = \begin{pmatrix} 2S & -S^i \\ -S^i & S^{ij} \end{pmatrix} .$$

If we use the property of τ and the prepotential introduced in (2-2), we have

$$S = \frac{1}{2}e^{2K}(\tau - \bar{\tau})_{IJ}\bar{X}^I\bar{X}^J = e^{2K}\left(\frac{1}{2}\tau_{IJ}\bar{X}^I\bar{X}^J - \overline{\mathcal{F}}(\bar{X})\right) .$$

Proposition 3.2. *The following relations hold,*

$$1)\quad \overline{D}_{\bar{i}}S = S_{\bar{0}\bar{i}} \qquad 2)\quad \overline{D}_{\bar{i}}\overline{D}_{\bar{j}}S = S_{\bar{i}\bar{j}} \qquad 3)\quad \overline{D}_{\bar{i}}\overline{D}_{\bar{j}}\overline{D}_{\bar{k}}S = e^{2K}\overline{C}_{\bar{i}\bar{j}\bar{k}} .$$

These are equivalent to 1) $\partial_{\bar{i}}S = g_{\bar{i}j}S^j$, 2) $\partial_{\bar{i}}S^j = g_{\bar{i}i}S^{ij}$ *and* 3) $\partial_{\bar{i}}S^{jk} = e^{2K}\overline{C}_{\bar{i}\bar{j}\bar{k}}g^{j\bar{j}}g^{k\bar{k}}$.

Proof. Acting $D_{\bar{i}}(=\partial_{\bar{i}})$ on $S = \frac{1}{2}e^{2K}(\tau - \bar{\tau})_{IJ}\bar{X}^I\bar{X}^J$, we obtain

$$\partial_{\bar{i}}S = e^{2K}(\tau - \bar{\tau})_{IJ}D_{\bar{i}}\bar{X}^I\bar{X}^J = S_{\bar{0}\bar{i}} ,$$

where we use $(\frac{\partial}{\partial\bar{X}^K}\bar{\tau}_{IJ})\bar{X}^J = \bar{X}^J(\frac{\partial}{\partial\bar{X}^J}\bar{\tau}_{IK}) = 0$. By similar calculations with the special Kähler geometry relation (2.13), the properties 2) and 3) above follow. Q.E.D.

S^{kl}, S^k, S are called propagators in physics literatures, and contain the anti-holomorphic prepotential $\bar{\mathcal{F}}(\bar{X})$ in their definitions. On the other hand, the holomorphic prepotential $\mathcal{F}(X)$ defines the so-called n-point functions on a Riemann sphere by

$$C_{i_1 i_2 \cdots i_n} = D_{i_1}D_{i_2}\cdots D_{i_n}\mathcal{F}(X) \quad (n \geq 3),$$

where the covariant derivative $D_i = \partial_i + 2\partial_i K(x,\bar{x}) - \Gamma^*_{i*}$ acts on $\mathcal{F}(X)$ by the standard contractions of the holomorphic indices (see (3.4) in general). By Kähler geometry, the holomorphic covariant derivatives commute with each other, so the n-point functions are symmetric tensor, and in particular C_{ijk} coincides with the Griffith–Yukawa coupling (2.2).

Definition 3.3. As an symmetric algebra, we define

$$\mathcal{R}^0_{BCOV} = \mathbf{Q}[S, S^i, S^{ij}, K_i, C_{i_1 i_2 i_3}, \cdots, C_{i_1 i_2 i_3 \cdots i_n}, \cdots] ,$$

where $K_i = \frac{\partial}{\partial x^i}K(x,\bar{x})$, We call this symmetric algebra *BCOV ring*.

In general this BCOV ring is infinitely generated. However we will see in the next section that if we consider the corresponding ring for an elliptic curve, the ring is finitely generated. In case of Calabi–Yau threefolds, as it turns out later that this problem of finiteness is related to the explicit form of the holomorphic function $h_{ijkl}(x)$ derived in (3.7) below. For convenience, we will often abbreviate the infinite series of the generators $C_{jkl}, D_{i_1}C_{jkl}, \cdots$ by $\{C_{ijk}\}$. With this convention, the BCOV ring may be written simply by

$$\mathcal{R}^0_{BCOV} = \mathbf{Q}[S, S^i, S^{ij}, K_i, \{C_{ijk}\}] \ . \tag{3.1}$$

The propagator $S^{\alpha\beta}$ and $C_{i_1 i_2 \cdots i_n}$ have respective weights $(-2,0)$ and $(2,0)$, and also K_i has weight $(0,0)$. Therefore the symmetric algebra is graded and defined in the set of global sections of

$$\bigoplus_{m,n\geq 0} \bigoplus_{k=-\infty}^{\infty} \left(\pi^*(T^*\mathcal{M})\right)^{\otimes m} \otimes \left(\pi^*(T\mathcal{M})\right)^{\otimes n} \otimes \mathcal{L}^k \ , \tag{3.2}$$

where $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ is the covering map and $\mathcal{L} \to \tilde{\mathcal{M}}$ is the line bundle introduced in (2-1). Precisely, K_i is a connection of the holomorphic line bundle $\mathcal{L}$ and therefore this is not a one form. However we regard K_i as a one form taking a global trivialization of $\mathcal{L}$ over $\tilde{\mathcal{M}}$. By the following lemma, we see how the BCOV ring $\mathcal{R}^0_{BCOV}$ depends on the sheets of the covering.

Lemma 3.4. *When we change the symplectic basis by* $\mathbf{\Omega} \to \mathbf{\Omega}\left(\begin{smallmatrix} D & B \\ C & A \end{smallmatrix}\right)$, *we have the corresponding change of the generators,*

$$S^{\alpha\beta} \to S^{\alpha\beta} + \xi_I^{\alpha} \left[C(D+\tau C)\right]^{IJ} \xi_J^{\beta} \ . \tag{3.3}$$

Proof. We keep our convention of the contraction by writing the indices of the symplectic matrix by $(X^I P_J) \to (X^I P_J) \begin{pmatrix} D_I^{\ J} & B_{IJ} \\ C^{IJ} & A^I_{\ J} \end{pmatrix}$. Then we have

$$\xi_\alpha^{\ I} \to \xi_\alpha^{\ J}(D+\tau C)_J^{\ I} \ , \ \ \tau_{IJ} \to \left[(D+\tau C)^{-1}(B+\tau A)\right]_{IJ} \ ,$$

and also, using $\left(\begin{smallmatrix} D & B \\ C & A \end{smallmatrix}\right) Q \, {}^t\!\left(\begin{smallmatrix} D & B \\ C & A \end{smallmatrix}\right) = Q$, we have

$$(\tau - \bar{\tau}) \to (D+\tau C)^{-1} (\tau - \bar{\tau}) \, {}^t\overline{(D+\tau C)}^{-1} \ .$$

After some algebra, the claimed transformation property follows directly from the definitions. Q.E.D.

We note that the metric connection Γ^k_{ij} and K_i define the covariant derivative D_i on the sections (3.2). Thus, for example, for the section $V_i^{\ jl}$ of weight $(k,0)$ we have

$$(3.4)\quad D_n V_i^{\ jl} = \frac{\partial}{\partial x^n} V_i^{\ jl} - \Gamma^m_{ni} V_m^{\ jl} + \Gamma^j_{nm} V_i^{\ ml} + \Gamma^l_{nm} V_i^{\ jm} + k K_n V_i^{\ jl} \ .$$

Theorem 3.5. *The BCOV ring $\mathcal{R}^0_{\mathrm{BCOV}}$ is a graded, differential, symmetric algebra with the (commuting) differentials D_i $(i = 1, \cdots, r)$.*

This theorem is a direct consequence of the following proposition.

Proposition 3.6. *The covariant derivative acts on the generators of $\mathcal{R}^0_{\mathrm{BCOV}}$ by*

$$(3.5)\quad \begin{aligned} D_i S^{kl} &= \delta^k_i S^l + \delta^l_i S^k - C_{imn} S^{mk} S^{nl} \ , \\ D_i S^k &= -C_{imn} S^m S^{nk} + 2\delta^k_i S \ , \\ D_i S &= -\frac{1}{2} C_{imn} S^m S^n \ , \\ D_i K_j &= -K_i K_j + C_{ijm} S^{mn} K_n - C_{ijm} S^m \ . \end{aligned}$$

Proof. The first three equations follow from the definitions and the special Kähler geometry relations (2.12), (2.13). There, it is useful to write $S^{\alpha\beta} = \frac{1}{\sqrt{-1}} e^K \mathcal{G}^{IJ} \xi_I^{\ \alpha} \xi_J^{\ \beta}$ and use the following relations,

$$D_i \mathcal{G}_{LM} = \sqrt{-1} e^K C_{ikl} \xi_L^{\ k} \xi_M^{\ l} \ , \quad D_i \xi_I^{\ \alpha} = -C_{imn} S^{n\alpha} \xi_I^{\ m} - \xi_I^{\ 0} \delta_i^{\ \alpha} \ .$$

For the fourth relation, we formulate the following two lemmas and use the relation (3.6) below. Q.E.D.

Lemma 3.7.

$$\partial_{\bar{k}} \xi_I^{\ i} = 0 \, , \quad \partial_{\bar{k}} \xi_I^{\ 0} = -\partial_{\bar{k}} (K_m \xi_I^{\ m}) \ .$$

Proof. The matrix $(\xi_I^{\ \alpha})$ is the inverse of $(\xi_\alpha^{\ I})$ by definition. From this, we have $\partial_{\bar{k}} \xi_I^{\ \alpha} = -\xi_I^{\ \beta} \partial_{\bar{k}} \xi_\beta^{\ J} \xi_J^\alpha$. Now using $\partial_{\bar{k}} \xi_\beta^{\ J} = \partial_{\bar{k}} D_\beta X^J = g_{\bar{k}\beta} \xi_0^{\ J}$, we obtain

$$\partial_{\bar{k}} \xi_I^{\ \alpha} = -\xi_I^{\ \beta} g_{\bar{k}\beta} \xi_0^{\ J} \xi_J^\alpha = -g_{\bar{k}\beta} \xi_I^{\ \beta} \delta_0^{\ \alpha}.$$

The first claimed relation is the case when $\alpha = i$. The second relation follows from the case $\alpha = 0$ together with the first relation. Q.E.D.

Lemma 3.8. *Define $f_{ij}^k = (\partial_i\partial_j X^I)\,\xi_I{}^k$, then f_{ij}^k is holomorphic and we have*

$$\partial_i K_j - K_i K_j = -C_{ijk}S^k + f_{ij}^m K_m + h_{ij} \;\; ,$$

where $h_{ij} = -(\partial_i\partial_j X^I)h_I$ with a holomorphic function $h_I = h_I(x)$.

Proof. When differentiating twice the defining relation of the Kähler potential $e^{-K} = \langle \Omega_x, \bar\Omega_x\rangle = \sqrt{-1}X^I(\bar\tau-\tau)_{IJ}\bar X^J$, we have

$$e^{-K}\big(-\partial_i K_j + K_i K_j\big) = \sqrt{-1}\partial_i\partial_j X^I(\bar\tau-\tau)_{IJ}\bar X^J - \sqrt{-1}\partial_i X^I \partial_j X^L \tau_{ILJ}\bar X^J,$$

where we set $\tau_{IJK} = \frac{\partial}{\partial X^K}\tau_{IJ}$. On the other hand, by definition of S^m, we have

$$\begin{aligned} C_{ijm}S^m &= -e^{2K}C_{ijm}(\tau-\bar\tau)_{IJ}\bar\xi_{\bar 0}^I\bar\xi_{\bar m}^J g^{m\bar m}g^{0\bar 0} \\ &= -\sqrt{-1}e^K C_{ijm}\mathcal{G}_{IJ}\bar X^I\bar\xi_{\bar m}^J g^{m\bar m} \\ &= -\sqrt{-1}e^K C_{ijm}\bar X^L \xi_L{}^m = -\sqrt{-1}e^K \tau_{IJL}\partial_i X^I \partial_j X^J \bar X^L \;, \end{aligned}$$

where we use $X^M\tau_{IJM} = 0$ which follows from the homogeneity property of $\mathcal{F}(X)$. Using $\mathcal{G}_{IJ} = \xi_I{}^\alpha g_{\alpha\bar\beta}\bar\xi_J^{\bar\beta}$ and $\bar\xi_{\bar 0}^J = \bar X^J$, we also have

$$\sqrt{-1}e^K\partial_i\partial_j X^I(\bar\tau-\tau)_{IJ}\bar X^J = -\partial_i\partial_j X^I\mathcal{G}_{IJ}\bar X^J = \partial_i\partial_j X^I\xi_I{}^0 \;\; ,$$

where, due to Lemma 3.7, we may use $\xi_I{}^0 = -\xi_I{}^m K_m + h_I$ with some holomorphic function h_I. Substituting all these relations into the first equation, we obtain the claimed formula. The holomorphicity of $f_{ij}^k = (\partial_i\partial_j X^I)\xi_I{}^k$ follows from the same Lemma 3.7. Q.E.D.

From the above lemma, and $\partial_{\bar m}S^k = g_{m\bar m}S^{mk}$, we obtain

$$\Gamma_{ij}^k = g_i^k K_j + g_j^k K_i - C_{ijm}S^{mk} + f_{ij}^k \;\; , \tag{3.6}$$

which derives the special Kähler relation (2.12) directly from the definitions.

The connection (3.6) contains non-geometric object $f_{ij}^k(x)$, however this does not appear in the formulas (3.5) since the first three equations follows directly from the special Kähler relations as we have already seen. For the the fourth equation, one observes that $f_{ij}^k(x)$ cancels in the evaluation of $D_i K_j$.

Remark 3.9. The BCOV ring $\mathcal{R}^0_{BCOV}$ is not finitely generated in general, however it is very 'close' to this property. This can be observed in the following formula

$$(3.7)\quad \begin{aligned} D_l C_{ijk} &= \sum_{\{a,b\}\cup\{c,d\}=\mathcal{I}} C_{abm}S^{mn}C_{ncd} + \tau_{IJKL}\xi_i{}^I\xi_j{}^J\xi_k{}^K\xi_l{}^L \\ &= \sum_{\{a,b\}\cup\{c,d\}=\mathcal{I}} C_{abm}S^{mn}C_{ncd} - \sum_{a\in\mathcal{I}} K_a C_{\mathcal{I}\setminus\{a\}} + h_{ijkl} \end{aligned}$$

where we set $\mathcal{I} = \{i,j,k,l\}$ and $h_{ijkl} := \tau_{IJKL}\partial_i X^I \partial_j X^J \partial_k X^K \partial_l X^L$ with $\tau_{IJKL} = \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}\frac{\partial}{\partial X^K}\frac{\partial}{\partial X^L}\mathcal{F}(X)$. Eq.(3.7) follows from $C_{ijk} = \tau_{IJK}\xi_i{}^I\xi_j{}^J\xi_k{}^K$ with the relations (2.14) and $D_k\tau_{IJK} = (\partial_k - K_k)\tau_{IJK} = \tau_{IJKL}\xi_k{}^L$. If the holomorphic term h_{ijkl} were zero, then the BCOV ring $\mathcal{R}^0_{BCOV}$ reduces to a finitely generated ring. This finite generating property can be realized in general by considering the following quotient: First, let us note that under the relation (3.7), the BCOV ring may be written as

$$\mathcal{R}^0_{BCOV} = \mathbf{Q}[S, S^i, S^{ij}, K_i, C_{ijk}, \{h_{ijkl}\}] \ ,$$

where $\{h_{ijkl}\}$ means the infinite sequence of the covariant derivations of h_{ijkl}. Considering a differential ideal $\mathbf{Q}[\{h_{ijkl}\}]$, we may reduce the ring $\mathcal{R}^0_{BCOV}$ to the quotient

$$(3.8)\qquad \mathcal{R}^{0,red}_{BCOV} = \mathcal{R}^0_{BCOV}/\mathbf{Q}[\{h_{ijkl}\}] \ .$$

We call this quotient ring *reduced BCOV ring.*

(3-2) BCOV ring $\mathcal{R}^\Gamma_{BCOV}$. As we have remarked in the previous section, the BCOV ring is defined in the algebra of global (meromorphic) sections of the bundle (3.2) over the covering space $\tilde{\mathcal{M}}$. Since there is a natural action of the covering group $\Gamma \subset Sp(2r+2,\mathbf{Z})$ on the sections, one may consider the invariants under this group action. Elements in $\mathcal{R}^0_{BCOV}$, in general, are not invariant under this group action, however they can be 'lifted' to define Γ-invariants by specifying a 'lift' for each propagator (see below). We then define $\mathcal{R}^\Gamma_{BCOV}$ the minimal differential ring of Γ-invariants which contains those Γ-invariants from $\mathcal{R}^0_{BCOV}$. We may call $\mathcal{R}^\Gamma_{BCOV}$ as a Γ-completion of the BCOV ring $\mathcal{R}^0_{BCOV}$.

Let us first note that the generators $K_i(x,\bar{x})$ and $C_{ijk}(x)$ are invariant under the group Γ by their definitions. Since the generators $S^{\alpha\beta}$ are transformed according to (3.3), we modify them to Γ-invariants $\tilde{S}^{\alpha\beta}$.

Let us assume $\tilde{S}^{kl} = S^{kl} + \Delta S^{kl}$ for a Γ-invariant lift. Then it may be determined simply by writing the equation (3.6) as

$$(3.9)\quad \Gamma^k_{ij} = \delta^k_i K_j + \delta^k_i K_j - C_{ijm}\tilde{S}^{mk} + \tilde{f}^k_{ij} \ , \quad (\tilde{f}^k_{ij} = f^k_{ij} + C_{ijm}\Delta S^{mk}) \ ,$$

and requiring Γ-invariance of $\tilde{f}_{ij}^k$. We will show, in the example of an elliptic curve, the simplest way to impose the invariance is to require $\tilde{f}_{ij}^k$ to be a rational function (section) on $\mathcal{M}$. In the original paper by BCOV, this process is referred to as 'fixing holomorphic (meromorphic) ambiguity'(Section 6.3 of [5]).

Once $\tilde{S}^{kl}$ is determined in this way, the form of other Γ-invariant propagators $\tilde{S}^k$, $\tilde{S}$ may be restricted, by requiring the relations $\partial_{\bar{k}}\tilde{S}^l = g_{k\bar{k}}\tilde{S}^{kl}$ and $g_{k\bar{k}}\tilde{S}^k = \partial_{\bar{k}}\tilde{S}$ given in Proposition 3.2, to

$$(3.10)\quad \tilde{S}^k = S^k + \Delta S^{kl}K_l + \Delta S^k \ , \quad \tilde{S} = S + \frac{1}{2}\Delta S^{kl}K_kK_l + \Delta S^k K_k + \Delta S \ ,$$

where ΔS^k and ΔS are suitable meromorphic sections. The form of ΔS^k can also be determined, in a similar way to (3.9), from

$$(3.11)\quad \partial_i K_j - K_iK_j = -C_{ijk}\tilde{S}^k + \tilde{f}_{ij}^m K_m + \tilde{h}_{ij} \ , \quad (\tilde{h}_{ij} = h_{ij} + C_{ijk}\Delta S^k),$$

by requiring that $\tilde{h}_{ij}$ is a rational function (section) on $\mathcal{M}$.

Proposition 3.10. *For the* Γ*-invariant propagators, we have*

$$(3.12)\quad \begin{aligned} D_i\tilde{S}^{kl} &= \delta_i^k\tilde{S}^l + \delta_i^l\tilde{S}^k - C_{imn}\tilde{S}^{mk}\tilde{S}^{nl} + \mathcal{E}_i^{kl} \ , \\ D_i\tilde{S}^k &= -C_{imn}\tilde{S}^m\tilde{S}^{nk} + 2\delta_i^k\tilde{S} + \mathcal{E}_i^{km}K_m + \mathcal{E}_i^k \ , \\ D_i\tilde{S} &= -\frac{1}{2}C_{imn}\tilde{S}^m\tilde{S}^n + \frac{1}{2}\mathcal{E}_i^{kl}K_kK_l + \mathcal{E}_i^m K_m + \mathcal{E}_i \ , \\ D_iK_j &= -K_iK_j + C_{ijm}\tilde{S}^{mn}K_n - C_{ijm}\tilde{S}^m + C_{ijm}\kappa^m \ , \end{aligned}$$

where we set $\kappa^m = \Delta S^m$ *and*

$$\mathcal{E}_i^{kl} = \mathcal{D}_i^f\Delta S^{kl} - \delta_i^k\Delta S^l - \delta_i^l\Delta S^k + C_{imn}\Delta S^{mk}\Delta S^{nl},$$

$$\mathcal{E}_i^k = \mathcal{D}_i^f\Delta S^k - 2\delta_i^k\Delta S + C_{imn}\Delta S^n\Delta S^{nl} \ , \quad \mathcal{E}_i = \mathcal{D}_i^f\Delta S + \frac{1}{2}C_{imn}\Delta S^m\Delta S^n \ .$$

We also define $\mathcal{D}_i^f := \partial_i + f_{i*}^*$ *a (covariant) derivative with* $f_{ij}^k(x)$ *in (3.6) being treated as a connection.*

Proof. Use the definitions $\tilde{S}^{kl}, \tilde{S}^k, \tilde{S}$, $\Gamma_{ij}^k = \delta_i^kK_j + \delta_j^kK_i - C_{ijm}S^{mk} + f_{ij}^k$ and Proposition 3.6 for the evaluations. After some algebra, the claimed formulas follow. Q.E.D.

If we define

$$(3.13)\quad \hat{\mathcal{E}}_i^{kl} = \mathcal{E}_i^{kl} \ , \quad \hat{\mathcal{E}}_i^k = \mathcal{E}_i^{km}K_m + \mathcal{E}_i^k \ , \quad \hat{\mathcal{E}}_i = \frac{1}{2}\mathcal{E}_i^{kl}K_kK_l + \mathcal{E}_i^m K_m + \mathcal{E}_i$$

then these are sections of weight $(-2,0)$ which are invariant under the action Γ. Considering all covariant derivatives of these tensors, and also κ^m, we have the minimal ring of Γ-invariants

$$\mathcal{R}^{\Gamma}_{BCOV} = \mathbf{Q}[\tilde{S}^{kl}, \tilde{S}^k, \tilde{S}, K_i, \{C_{ijk}\}, \{\hat{\mathcal{E}}^{kl}_i\}, \{\hat{\mathcal{E}}^k_i\}, \{\hat{\mathcal{E}}_i\}, \{\kappa^m\}] \ ,$$

where the bracket notation is used for the infinite set of the generators as before. We should note that the explicit forms of the new generators $\hat{\mathcal{E}}^{kl}_i, \hat{\mathcal{E}}^k_i, \hat{\mathcal{E}}_i$ and κ^m depend on the 'lifts' of the propagators $\tilde{S}^{ij}, \tilde{S}^k, \tilde{S}$. Hence the ring $\mathcal{R}^{\Gamma}_{BCOV}$ also depends on the lifts.

In case of elliptic curves, we will observe that the ring $\mathcal{R}^{\Gamma}_{BCOV}$ has a close similarity to the ring of almost holomorphic modular forms studied in Kaneko–Zagier [23].

(3-3) Example (elliptic curve). We have introduced the BCOV ring for Calabi–Yau threefolds, however if we replace the special Kähler geometry by the geometry of upper-half plane, it naturally reduces to the rather standard theory of (almost holomorphic) modular forms[23].

Let us consider a family of elliptic curves over $\mathcal{M}$ and its period integrals following (2-1). We consider a family of hypersurfaces $Y_{\vec{a}}$

$$\mathtt{W}(a) := a_0 + a_1 U + a_2 V + a_3 \frac{1}{U^3 V^2} = 0 \ \subset (\mathbf{C}^*)^2$$

in the torus $(\mathbf{C}^*)^2$. Compactifying $(\mathbf{C}^*)^2$ to a suitable toric variety $\mathbf{P}_\Sigma$, we obtain our family of elliptic curves. The moduli space $\mathcal{M}$ arises as the parameter space of the defining equation. Because of the natural torus actions on the parameters, it is easy to see that $\mathcal{M}$ is given by $\mathbf{P}^1$, and we have

$$\Omega_x = R\Big(\frac{a_0}{\mathtt{W}(a)} \frac{dU}{U} \frac{dV}{V}\Big) \quad \Big(x = \frac{a_1^3 a_2^2 a_3}{a_0^6} \in \mathbf{P}^1\Big).$$

Taking a symplectic basis A, B with $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we define the period integral $\vec{\omega} = (w_0(x), w_1(x))$. The period integrals satisfy the Picard–Fuchs differential equation of the form,

$$\{\theta_x^2 - 12x(6\theta_x + 1)(6\theta_x + 5)\}\omega_i(x) = 0 \ ,$$

where $\theta_x = x\frac{d}{dx}$. The 'Griffiths–Yukawa coupling' in this case is simply defined by

$$C_x := -\int_{Y_x} \Omega_x \cup \frac{d}{dx}\Omega_x = \frac{1}{(1-432x)x} \ .$$

Let us fix (uniquely) the A cycle by the condition that the corresponding period integral $w_0(x)$ is regular at $x = 0$ and normalized by $\omega_0(x) =$

$1+\cdots$. Then we take a dual cycle B to A. With this choice of the basis, the period matrix takes the form $\mathbf{\Omega} = (\omega_0(x)\,\omega_1(x)) = \omega_0(x)(\,1\;t\,)$ with $t \sim \frac{1}{2\pi i}\log x + \cdots$ near $x=0$. We invert the relation $t = \frac{\omega_1(x)}{\omega_0(x)}$ as $x = x(t)$. Then it is standard to obtain the following identities (see eg. [24]);

$$(3.14)\quad \frac{1}{2\pi i}\frac{1}{\omega_0(x)^2}C_x\frac{dx}{dt} = 1\;,\;\; \omega_0(x(t))^4 = E_4(t)\;,\;\; C_x(x(t)) = j(t)\;,$$

where $E_4(t)$ is the Eisenstein series, $j(t)$ is the normalized j-function with their Fourier expansion given by $q = e^{2\pi i t}$ (and $\frac{1}{2\pi i}\frac{d}{dt} = q\frac{d}{dq}$). We also have the following useful relation

$$(3.15)\qquad \frac{\omega_0(x(t))^{12}}{C_x(x(t))} = \eta(t)^{24}\;,$$

in terms of the Dedekind η-function. Note that the first identity of (3.14) simply represents the fact that there is no quantum correction to the Griffiths–Yukawa coupling.

(3-3.a) The BCOV ring $\mathcal{R}^0_{BCOV} = \mathbf{Q}[S, K_x, C_x]$. For an elliptic curve, Definition 3.1 of $S = S^{00}$ should be read as

$$S = \frac{1}{2\pi i}g^{0\bar{0}}g^{0\bar{0}}e^{2K}(t-\bar{t})\bar{\xi}_{\bar{0}}\bar{\xi}_{\bar{0}} = \frac{1}{2\pi i}e^{2K}(t-\bar{t})\bar{\omega}_0\bar{\omega}_0\;.$$

with the period matrix $\mathbf{\Omega} = \omega_0(\,1\,t\,)$. Here, for elliptic curves, we introduce the factor $\frac{1}{2\pi i}$ in the definition of S. For the Kähler potential, we have $e^{-K} = i\,\mathbf{\Omega}Q\,{}^t\overline{\mathbf{\Omega}}$. Then it is straightforward to obtain

$$S = \frac{1}{2\pi i}\frac{1}{\omega_0^2}\frac{1}{\bar{t}-t}\;,\quad K_x = \frac{dt}{dx}\frac{1}{\bar{t}-t} - \frac{d}{dx}\log(\omega_0(x))\quad.$$

Proposition 3.11. *The BCOV ring $\mathcal{R}^0_{BCOV}$ is finitely generated by S, K_x and C_x. The covariant differential D_x acts on the generators by*

$$(3.16)\qquad D_xS = -C_xSS\,,\;\; D_xK_x = -K_xK_x - 60\,C_x\,,\;\; D_xC_x = 0\;.$$

Proof. It is sufficient to derive the differentials of generators (3.16). The metric connection Γ^x_{xx} may be determined from the relation;

$$(3.17)\qquad \begin{aligned}&(-\partial_xK_x + K_xK_x)e^{-K}\\ &= i\frac{d^2}{dx^2}\mathbf{\Omega}\,Q\,{}^t\overline{\mathbf{\Omega}} = \Big(-\frac{C'_x}{C_x}K_x + 60\,C_x\Big)e^{-K}\,,\end{aligned}$$

where we use the Picard–Fuchs equation

$$\frac{d^2}{dx^2}\vec{\omega} - \frac{C_x'}{C_x}\frac{d}{dx}\vec{\omega} - 60C_x\vec{\omega} = \vec{0} \ ,$$

to rewrite $i\frac{d^2}{dx^2}\,\mathbf{\Omega}\,Q\,{}^t\overline{\mathbf{\Omega}} = i\frac{d^2}{dx^2}\vec{\omega}Q^t\overline{\vec{\omega}}$ as above. After differentiating (3.17) by $\partial_{\bar{x}}$, we have $\Gamma^x_{xx} = 2K_x + \frac{C_x'}{C_x}$. Now using (3.17) again, we have

$$D_xK_x = (\partial_x + \Gamma^x_{xx})K_x = -K_xK_x - 60C_x \ .$$

Similarly, noting the generators S, C_x have their weights $(-2,0)$ and $(2,0)$, respectively, and using the relations (3.14), it is straightforward to obtain the claimed relations. Q.E.D.

It will be useful to have the following expression for the connection $\Gamma^x_{xx} = 2K_x + \frac{C_x'}{C_x}$;

$$\text{(3.18)} \quad \Gamma^x_{xx} = 2\frac{dt}{dx}\frac{1}{\bar{t}-t} + \frac{d}{dx}\log\Big(\frac{C_x}{\omega_0(x)^2}\Big) = 2\frac{dt}{dx}\frac{1}{\bar{t}-t} + \frac{dx}{dt}\frac{d}{dx}\frac{dt}{dx} \ ,$$

where we use the identify $\frac{dt}{dx} = \frac{1}{2\pi i}\frac{C_x}{\omega_0^2}$ in (3.14). In particular, writing the first identity of (3.18) as

$$\Gamma^x_{xx} = 2C_xS + \frac{d}{dx}\log\Big(\frac{C_x}{\omega_0(x)^2}\Big) = 2C_xS + f^x_{xx} \ ,$$

one may regard this as the corresponding relation to (3.6).

(3-3.b) The BCOV ring $\mathcal{R}^{\Gamma}_{BCOV}$. In our example, the covering group Γ is given by the (genus one) modular subgroup $\langle\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle \subset SL(2,\mathbf{Z})$. S is not invariant under the Γ action, however it is clear from the form $\frac{1}{\bar{t}-t}$ that S can be lifted to a Γ-invariant by

$$S \mapsto \tilde{S} = \frac{1}{\omega_0^2(x)}\Big\{\frac{1}{2\pi i}\frac{1}{\bar{t}-t} - \frac{1}{12}E_2(t)\Big\}$$

in terms of the Eisenstein series $E_2(t)$. $\tilde{S}$ is invariant since $E_2(t) - \frac{1}{2\pi i}\frac{12}{\bar{t}-t} = E_2^*(t)$ is the almost holomorphic (elliptic) modular form of weight 2 and $\omega_0(x)^2 = \sqrt{E_4(t)}$ for the denominator.

In our general formulation based on (3.9), the invariance arises in a rather weak form as follows: We first start with the 'shift';

$$\Gamma^x_{xx} = 2C_xS + f^x_{xx} = 2C_x\tilde{S} + \tilde{f}^x_{xx} \quad (\tilde{f}^x_{xx} := f^x_{xx} - 2C_x\Delta S) \ ,$$

where $\tilde{S} = S + \Delta S$. Accordingly, the formula $D_x S$ changes to

$$D_x \tilde{S} = -C_x \tilde{S}\tilde{S} + \mathcal{E}_x \quad \left(\mathcal{E}_x := \partial_x \Delta S + C_x \Delta S \Delta S + 2\frac{\omega_0'}{\omega_0}\Delta S\right) .$$

Proposition 3.12. *$\tilde{f}_{xx}^x$ is a rational function of x if and only if we set*

$$\Delta S = -\frac{1}{C_x}\frac{\omega_0'}{\omega_0} + r(x) \ ,$$

with some rational function $r(x)$. In terms of $r(x)$, $\mathcal{E}_x$ is given by $r'(x) + C_x r^2(x) - 60$. When $\mathcal{E}_x = \lambda C_x$ with some constant $\lambda \in \mathbf{Q}$, then the BCOV ring $\mathcal{R}_{BCOV}^{\Gamma}$ is finitely generated by $\tilde{S}, K_x, C_x$ with the following differentials,

$$D_x \tilde{S} = -C_x \tilde{S}\tilde{S} + \lambda C_x \ , \quad D_x K_x = -K_x K_x - 60\, C_x \ , \quad D_x C_x = 0 \ .$$

Proof. We evaluate $\tilde{f}_{xx}^x$ as

$$\tilde{f}_{xx}^x = \partial_x \log\left(\frac{C_x}{\omega_0^2}\right) - 2C_x \Delta S = -2\frac{\omega_0'}{\omega_0} + \frac{C_x'}{C_x} - 2C_x \Delta S \ ,$$

from which the first claim is clear. For the evaluation of $\mathcal{E}_x$, we use the Picard–Fuchs equation satisfied the period integral $\omega_0(x)$. The third claim is clear since the differentials closes among the generators. Q.E.D.

The differential equation $\mathcal{E}_x = \lambda C_x$ for $r(x)$ may be solved by hypergeometric series. From the solution, one may observe that there are infinitely many λ for which $r(x)$ becomes rational. The simplest result is given by

$$\lambda = \frac{1}{144} \ , \quad r(x) = \frac{1}{12}\frac{C_x'}{C_x^2} \ , \quad \tilde{S} = -\frac{1}{12}\frac{E_2^*}{\omega_0^2} \ ,$$

where we evaluate $S + \Delta S = S + \frac{1}{12}\frac{1}{C_x}\partial_x \log \frac{C_x}{\omega_0^{12}} = S - \frac{1}{12}\frac{1}{C_x}\partial_x \log \eta(t)^{12}$ for $\tilde{S}$. Similarly, for $\lambda = \frac{25}{144}, \frac{49}{144}, \frac{121}{144}$, for example, we obtain $r(x) = \frac{5}{12}\frac{C_x'}{C_x^2}, \frac{1}{12}\frac{C_x'}{C_x^2} - \frac{1}{2}\frac{1}{1-864x}, \frac{5}{12}\frac{C_x'}{C_x^2} - \frac{1}{2}\frac{1}{1-864x}$ and

$$\tilde{S} = \frac{-1}{12\omega_0^2}\left\{E_2^* + 4\frac{E_6}{E_4}\right\}, \ \frac{-1}{12\omega_0^2}\left\{E_2^* + 6\frac{E_4^2}{E_6}\right\}, \ \frac{-1}{12\omega_0^2}\left\{E_2^* + 4\frac{E_6}{E_4} + 6\frac{E_4^2}{E_6}\right\},$$

respectively.

We note that, when the ring is finitely generated, the BCOV ring is very close to the ring of almost holomorphic modular forms studied in Kaneko–Zagier [23]. For comparison, it might be useful to write our generators (for the case $\lambda = \frac{1}{144}$) in terms of the elliptic modular forms;

$$(3.19)\quad \tilde{S} = \frac{-1}{12}\frac{E_2^*(t)}{\omega_0(x)^2}\,,\; C_x = j(t)\,,\; K_x = \frac{-1}{12}\frac{j(t)}{\omega_0(x)^2}\Big\{E_2^*(t) - \frac{E_6(t)}{E_4(t)}\Big\}.$$

One should note, however, that the weight assignment in the BCOV ring is different from that of almost holomorphic modular forms. Also, in the BCOV ring, we have additional indices of the cotangents $\big(\pi^*(T^*\mathcal{M})\big)^{\otimes m}$.

(3-4) BCOV ring $\mathcal{R}^{hol}_{BCOV}$. For the applications to Gromov–Witten theory of Calabi–Yau manifolds, the most relevant form of the BCOV ring is the holomorphic limits of the invariants $\mathcal{R}^{\Gamma}_{BCOV}$, which is often referred to as "$\bar{t} \to \infty$" limit in physics literatures. For the above example of an elliptic curve, the meaning "$\bar{t} \to \infty$" should be clear as the 'limit' taking the constant term of $\sum_{n\geq 0} a_m(t)\big(\frac{1}{\bar{t}-t}\big)^n$. We need to formulate a precise meaning for Calabi–Yau threefolds. However the idea of the limit should be clear from the structure $\mathcal{R}^{\Gamma}_{BCOV}$ with the differentials (3.12). Namely, all the differentials are with respect to holomorphic coordinate, and therefore "throwing away" the anti-holomorphic dependence, at the cost of Γ-invariance, should be compatible with the differentiations.

To describe the holomorphic limit in more detail, let us introduce the so-called flat coordinate. We first fix a symplectic basis $\{A^I, B_J\}$ and denote the corresponding period integrals $(X^I(x), P_J(x))$. By the property 1) in Section (2-2), the (half) period maps $x \in B(\subset \mathcal{M}) \mapsto [X^I(x)] \in \mathbf{P}^r$ provides a local isomorphism. Due to this property we may introduce the so-called *flat* coordinate $(t^a)_{a=1,\cdots r}$ by the relation

$$(X^0(x), X^1(x), \cdots, X^r(x)) = X^0(x)(1, t^1, \cdots, t^r)\ ,$$

near $X^0(x) \neq 0$. In this flat coordinate we have for the Kähler potential $e^{-K(x,\bar{x})} = iX^0(x)\overline{X^0(x)}e^{-\mathcal{K}(t,\bar{t})}$ with

$$e^{-\mathcal{K}(t,\bar{t})} = 2\overline{F(t)} - 2F(t) + (t^a - \bar{t}^a)\Big(\frac{\partial F}{\partial t^a} + \overline{\frac{\partial F}{\partial t^a}}\Big),$$

and $F(t) = \frac{1}{(X^0)^2}\mathcal{F}(X) = \mathcal{F}(\frac{X^a}{X^0})$. Connections of the bundles in these two local coordinates (x^i) and (t^a) are related by

$$K_i = -\partial_i \log X^0(x) + \frac{\partial t^a}{\partial x^i}\mathcal{K}_{t^a}\,,\; \Gamma^k_{ij} = \frac{\partial x^k}{\partial t^c}\Gamma^{t^c}_{t^a t^b}\frac{\partial t^a}{\partial x^i}\frac{\partial t^b}{\partial x^j} + \frac{\partial x^k}{\partial t^a}\frac{\partial}{\partial x^i}\frac{\partial t^b}{\partial x^j}\ .$$

As we see in the formula $e^{-\mathcal{K}(t,\bar{t})}$, holomorphic and anti-holomorphic dependences are not separated by a factor like $f(t)g(\bar{t})$ (or $f(t) + g(\bar{t})$

in $\log\mathcal{K}$). We assume that the 'constant terms' against to the anti-holomorphic dependences are selected simply by setting to zero those expressions written by $\mathcal{K}_{t^a}(t,\bar{t})$ and $\Gamma^{t^c}_{t^a t^b}$ (and also their holomorphic derivatives).

Definition 3.13. Choose a symplectic basis $\mathcal{B} := \{A^I, B_J\}$. Then we define the holomorphic limit of the elements in $\mathcal{R}^{\Gamma}_{BCOV}$, with respect to $\mathcal{B}$, by the following replacements of the connections:

$$K_i \to \mathtt{K}_i := -\partial_i \log X^0(x) \ , \ \Gamma^k_{ij} \to \Gamma^k_{ij} := \frac{\partial x^k}{\partial t^a}\frac{\partial}{\partial x^i}\frac{\partial t^a}{\partial x^j} \ .$$

As remarked above, the holomorphic limit commutes with the holomorphic differentials D_i, and hence we have the same differentials as (3.12). We denote the holomorphic limit of the generators $\tilde{S}^{ij}, \tilde{S}^k, \tilde{S}$, respectively by $\mathtt{S}^{ij}, \mathtt{S}^k, \mathtt{S}$. Also by $\mathtt{D}_i (= \partial_i \pm k\mathtt{K}_i \pm \Gamma^*_{i*})$, we represent the holomorphic limit of the differential D_i. Accordingly, $\mathtt{K}_i$ should be assumed in the definitions (3.13) of $\hat{\mathcal{E}}^k_i, \hat{\mathcal{E}}_i$, although we use the same notation for these. Thus, taking the holomorphic limit of the Γ-invariant BCOV ring, $\mathcal{R}^{\Gamma}_{BCOV}$, we will have *holomorphic* BCOV ring,

$$\mathcal{R}^{hol}_{BCOV} = \mathbf{Q}[\mathtt{S}^{ij}, \mathtt{S}^k, \mathtt{S}, \mathtt{K}_i, \{C_{ijk}\}, \{\hat{\mathcal{E}}^{kl}_i\}, \{\hat{\mathcal{E}}^k_i\}, \{\hat{\mathcal{E}}_i\}, \{\kappa^m\}] \ .$$

The concrete form of the generators $\mathtt{S}^{ij}$ may be determined from the holomorphic limit of the relation (3.9);

$$\Gamma^k_{ij} = \delta^k_i \mathtt{K}_j + \delta^k_j \mathtt{K}_i - C_{ijm}\mathtt{S}^{mk} + \tilde{f}^k_{ij} \ . \tag{3.20}$$

Similarly, for $\mathtt{S}^k$, we can use (3.11),

$$\partial_i \mathtt{K}_j - \mathtt{K}_i \mathtt{K}_j = -C_{ijk}\mathtt{S}^k + \tilde{f}^m_{ij}\mathtt{K}_m + \tilde{h}_{ij} \ . \tag{3.21}$$

These equations are used to determine the propagators in [5]. As noted there, when $r \geq 2$, the first relation (3.20) provides an overdetermined system for $\mathtt{S}^{ij}$, and the form of $\tilde{f}^k_{ij}$ should be restricted so that there exist solutions $\mathtt{S}^{ij}$. Similarly, $\tilde{h}_{ij}$ should be restricted so that the relation (3.21) has a solution $\mathtt{S}^k$. If we find a set of solutions $\mathtt{S}^{ij}, \mathtt{S}^k$, the first equation of (3.12) determines $\mathcal{E}^{kl}_i$, and the second relation of (3.12) determines $\mathtt{S}$ up to $\mathcal{E}^k_i$. As argued in [5], the possible forms of rational functions (sections) $\tilde{f}^k_{ij}, \tilde{h}_{ij}$ may be restricted, to some extent, by imposing regularity (or singularity) of $\mathtt{K}_i$ at certain degeneration loci of the family, see [5].

Example 1 (elliptic curve): When we take the symplectic basis $\mathcal{B} = \{A, B\}$ as in the previous section, the holomorphic limit of $\mathcal{R}^{\Gamma}_{BCOV}$ is exactly the map taking the constant term of $\sum c_m\left(\frac{1}{t-t}\right)^m$. From the example in the previous section, it is immediate to obtain (for $\lambda = \frac{1}{144}$) that

$$\mathtt{S} = -\frac{1}{12}\frac{E_2(t)}{\omega_0(x)^2} \ , \ \mathtt{K}_x = -\partial_x \log(\omega_0(x)) \ , \ \Gamma^x_{xx} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\frac{\partial x}{\partial t} \ .$$

As for the differentials, we have the same form as those in $\mathcal{R}^{\Gamma}_{BCOV}$, i.e.,

$$\mathtt{D}_x\mathtt{S} = -C_x\mathtt{S}\mathtt{S} + \frac{C_x}{144} \ , \ \mathtt{D}_x\mathtt{K}_x = -\mathtt{K}_x\mathtt{K}_x - 60\, C_x \ , \ \mathtt{D}_x C_x = 0 \ .$$

These relations define the BCOV ring $\mathcal{R}^{hol}_{BCOV} = \mathbf{Q}[\mathtt{S}, \mathtt{K}_x, C_x]$. The form of the generators are given simply by $E_2^*(t) \to E_2(t)$ in (3.19).

Example 2 (mirror quintic Calabi–Yau threefold): The construction of a symplectic basis $\mathcal{B}$ about the so-called large complex structure limit has been done in [8] (see also [17] and references therein for its combinatorial construction). We consider the holomorphic limit with respect to this basis. To fix the propagators $\mathtt{S}^{ij}, \mathtt{S}^k$, we have to solve the equations (3.20) and (3.21) finding suitable choices for the rational functions $\tilde{f}^x_{xx}, \tilde{h}_{xx}$. In [5], it has been found that these unknowns are uniquely fixed by requiring expected properties for the higher genus Gromov–Witten potential, $\mathcal{F}_g$, which comes from the anomaly equation. Here we simply translate their results into our conventions. First, the propagators $\mathtt{S}^{xx}, \mathtt{S}^x$ are determined by the choice $\tilde{f}^x_{xx} = -\frac{8}{5}\frac{1}{x}, \tilde{h}_{ij} = \frac{2}{25}\frac{1}{x^2}$ in

$$\Gamma^x_{xx} = 2\mathtt{K}_x - C_{xxx}\mathtt{S}^{xx} - \frac{8}{5}\frac{1}{x} \, , \ \partial_x\mathtt{K}_x - \mathtt{K}_x\mathtt{K}_x = -C_{xxx}\mathtt{S}^x - \frac{8}{5}\frac{1}{x}\mathtt{K}_x + \frac{2}{25}\frac{1}{x^2} ,$$

where $C_{xxx} = \frac{5}{x^3(1-5^5x)}$ and x is related to ψ in [4], [5] by $x = \frac{1}{\psi^5}$. Then the differentials are evaluated to be

$$\begin{aligned}
\mathtt{D}_x\mathtt{S}^{xx} &= 2\,\mathtt{S}^x - C_{xxx}\mathtt{S}^{xx}\mathtt{S}^{xx} + \frac{x}{25} \, ,\\
\mathtt{D}_x\mathtt{S}^x &= 2\,\mathtt{S} - C_{xxx}\mathtt{S}^x\mathtt{S}^{xx} + \frac{x}{25}\mathtt{K}_x - \frac{1}{125} \, ,\\
\mathtt{D}_x\mathtt{S} &= -\frac{1}{2}C_{xxx}\mathtt{S}^x\mathtt{S}^x + \frac{x}{50}\mathtt{K}_x\mathtt{K}_x - \frac{1}{125}\mathtt{K}_x + \frac{2}{3125}\frac{1}{x} \, ,\\
\mathtt{D}_x\mathtt{K}_x &= -\mathtt{K}_x\mathtt{K}_x + C_{xxx}\mathtt{S}^{xx}\mathtt{K}_x - C_{xxx}\mathtt{S}^x + \frac{2}{25}\frac{1}{x^2} \, ,
\end{aligned}$$

from which we read $\mathcal{E}^{xx}_x = \frac{x}{25}, \mathcal{E}^x_x = -\frac{1}{125}, \mathcal{E}_x = \frac{2}{3125}\frac{1}{x}$ and $\kappa^x = \frac{2}{25}\frac{1}{x^2}\frac{1}{C_{xxx}}$. With these, the BCOV ring is determined by

$$\mathcal{R}^{hol}_{BCOV} = \mathbf{Q}[\mathtt{S}^{xx}, \mathtt{S}^x, \mathtt{S}, \mathtt{K}_x, \{C_{xxx}\}, \{\hat{\mathcal{E}}^{xx}_x\}, \{\hat{\mathcal{E}}^x_x\}, \{\hat{\mathcal{E}}_x\}, \{\kappa^x\}] \ .$$

Example 3: The Calabi–Yau manifolds whose higher genus Gromov–Witten invariants are studied in [19] are not complete intersections in toric varieties, but has an interesting property: there exist two different large complex structure limits (cusps) in the deformation space [27]. By mirror symmetry, this phenomenon is related to non-birational Calabi–Yau manifolds whose derived categories of coherent sheaves are equivalent [27], [3], [22]. The cusps of the example in [27], [19] are located at $x=0$ and $z=\frac{1}{x}=0$. The BCOV ring $\mathcal{R}^{hol}_{BCOV}$ with respect to a symplectic basis $\mathcal{B}_0$ at $x=0$ has a similar form as in Example 2 with

$$\mathcal{E}^{xx}_x=\frac{-1}{14}\frac{xp(x)}{(x-3)^2}\,,\;\mathcal{E}^x_x=\frac{1}{14}\frac{p(x)}{(x-3)^2}\,,\;\mathcal{E}_x=\frac{-1}{28}\frac{p(x)(x+14)+q(x)}{(x-3)^3}\,,$$

and $\kappa^x=\frac{2}{x^2}\frac{1}{C_{xxx}}$, where $p(x)=x^4-716x^3+422x^2+452x-15$, $q(x)=12374x^3-7166x^2-7630x+246$. The BCOV ring $\mathcal{R}^{hol}_{BCOV}$ at the other cusp ($z=0$) is defined with respect to a different symplectic basis $\mathcal{B}_\infty$. However we verify from the results in [19] that the ring is determined with $\mathcal{E}^{zz}_z=\mathcal{E}^{xx}_x\left(\frac{dz}{dx}\right)$, $\mathcal{E}^z_z=\mathcal{E}^x_x$, $\mathcal{E}_z=\mathcal{E}_x\left(\frac{dx}{dz}\right)$ and $\kappa^z=\kappa^x\left(\frac{dz}{dx}\right)$. From this, we observe that the BCOV ring $\mathcal{R}^\Gamma_{BCOV}$ is invariant under the symplectic transformation which connects $\mathcal{B}_0$ and $\mathcal{B}_\infty$.

§4. BCOV holomorphic anomaly equation in $\mathcal{R}^{0,red}_{BCOV}$

(4-1) BCOV holomorphic anomaly equation. The original form of the BCOV anomaly equation has been formulated based on the special Kähler geometry over the moduli space $\mathcal{M}$. Although mathematical ground of this anomaly equation has not yet been established, up to now, this equation provides the only way to a systematic calculation of higher genus Gromov–Witten potential $\mathtt{F}_g(t)$ for Calabi–Yau complete intersections and some cases beyond them. About this equation, recently, several important progress has been made in physics literatures [32], [1], [16], [19], [2]. In particular, the polynomial property found by Yamaguchi and Yau [32] and also in [2] is the one which we followed for our definition of the BCOV ring $\mathcal{R}^\Gamma_{BCOV}$.

To summarize the recursive procedure given in [5], let us write the anomaly equation in the following form, which appeared in [32] and [2],

$$\text{(4.1)}\quad \begin{aligned}\frac{\partial\mathcal{F}^{(g)}}{\partial\tilde{S}^{ij}}&=\frac{1}{2}D_iD_j\mathcal{F}^{(g-1)}+\frac{1}{2}\sum_{h=1}^{g-1}D_i\mathcal{F}^{(g-h)}D_j\mathcal{F}^{(h)}\;,\\ 0&=\tilde{S}^{jk}\frac{\partial\mathcal{F}^{(g)}}{\partial\tilde{S}^k}+\tilde{S}^j\frac{\partial\mathcal{F}^{(g)}}{\partial\tilde{S}}+\frac{\partial\mathcal{F}^{(g)}}{\partial\tilde{K}_j}\qquad(g\geq 2).\end{aligned}$$

Then the recursion proceeds as follows, with $\tilde{f}_{ij}^k, \tilde{h}_{ij}, \mathcal{E}_i^k$ in (3.9), (3.11), (3.12), respectively, being unknown:

Step 1. We start with the fact that there exists a polynomial $f_0(x)$ and a choice $\tilde{f}_{ij}^k(x)$ such that

$$D_i\mathcal{F}^{(1)} = \frac{1}{2}C_{imn}\tilde{S}^{mn} - \big(\frac{\chi}{24} - 1\big)K_i + f_{1,i}(x)\ , \quad (f_{1,i} := \partial_i \log f_0)$$

gives the genus one Gromov–Witten potential $\mathtt{F}_1(t)$ of the mirror Calabi–Yau manifold X (with its Euler number χ) when we take the holomorphic limit. We refer [4], [5] for details of $\mathtt{F}_1(t)$. The polynomial $f_0(x)$ is essentially given by the discriminant of the family. We define, using the bracket notation,

$$\mathcal{R}_{BCOV}^{\Gamma,1} = \mathcal{R}_{BCOV}^{\Gamma}[\{f_{1,i}(x)\}]\ ,$$

and regard $D_i\mathcal{F}^{(1)}$ (and $f_{1,i}$) as an element of weight zero in $\mathcal{R}_{BCOV}^{\Gamma,1}$.

Step 2. Suppose we have $D_i\mathcal{F}^{(1)}$ and $\mathcal{R}_{BCOV}^{\Gamma,1}$ as above. Consider the anomaly equation (4.1) for $g = 2$ in the ring $\mathcal{R}_{BCOV}^{\Gamma,1}$ to find a (unique) solution $\mathcal{F}_0^{(2)}$ of weight $(-2,0)$ under the condition $\mathcal{F}_0^{(2)}|_{\tilde{S}^{ij}=\tilde{S}^k=\tilde{S}=0} = 0$. Then the observation made in [5] is that there exist a rational section $f_2(x)$ of $(\mathcal{L}^{-1})^{\otimes 2}$ and suitable choices $\tilde{h}_{ij}(x)$ and $\mathcal{E}_i^k(x)$ such that

$$\mathcal{F}^{(2)} = \mathcal{F}_0^{(2)} + f_2(x)\ ,$$

gives the Gromov–Witten potential $\mathtt{F}_2(t)$ under the holomorphic limit. ($\tilde{f}_{ij}^k(x)$ in step 1 and $\tilde{h}_{ij}(x), \mathcal{E}_i^k(x)$ in step 2 fix the lifting $S^{\alpha\beta}$ to $\tilde{S}^{\alpha\beta}$, and thus $\mathcal{R}_{BCOV}^{\Gamma}$.) We extend our BCOV ring to $\mathcal{R}_{BCOV}^{\Gamma,2} = \mathcal{R}_{BCOV}^{\Gamma,1}[\{f_2(x)\}]$.

For $g \geq 3$, this procedure continues genus by genus enlarging the BCOV ring by some rational section $f_g(x)$ of the line bundle $\mathcal{L}^{2-2g}$ to $\mathcal{R}_{BCOV}^{\Gamma,g+1} = \mathcal{R}_{BCOV}^{\Gamma,g}[\{f_g\}]$. When we define a notation

$$\mathcal{R}_{BCOV}^{\Gamma,\infty} = \lim_{\to} \mathcal{R}_{BCOV}^{\Gamma,g}\ ,$$

the solutions $\mathcal{F}_g$ are (scalar) elements in this ring of weight $(2-2g, 0)$.

Remark 4.1. In general, the ring $\mathcal{R}_{BCOV}^{\Gamma,\infty}$ is a large ring. However, we note that, since $\mathcal{R}_{BCOV}^{\Gamma,\infty}$ consists of Γ-invariants, working in an affine coordinate $U \subset \mathcal{M}$ of the toric variety $\mathcal{M} = \mathbf{P}_{Sec(\Sigma)}$ makes sense. Let us consider an affine coordinate of U and consider the field of rational functions $\mathbf{Q}(x)$ on U. We assume that the unknowns $\mathcal{E}_i^{kl}, \mathcal{E}_i^k, \mathcal{E}_i, \kappa^m$ are rational functions on U (as seen in Example 2 and 3 of (3-4)). Then, due to the rationality of C_{ijk} and $f_g(x)$, we observe

$$\mathcal{R}^{\Gamma,\infty}_{BCOV}\big|_U \subset \mathbf{Q}(x)[\tilde{S}^{ij}, \tilde{S}^k, \tilde{S}, K_i] \ . \tag{4.2}$$

The ring in the r.h.s. of the inclusion is the local form which we see the BCOV ring $\mathcal{R}^{\Gamma,\infty}_{BCOV}$ in physics literatures, for example [32], [16], [19]. In reference [16], in particular, following the idea of [29], [13], an efficient way to impose certain boundary conditions to determine $f_g(x)$ has been found.

(4-2) BCOV anomaly equation in $\mathcal{R}^{0,red}_{BCOV}$. As briefly sketched above, solving BCOV anomaly equation (4.1) contains a process finding a suitable $f_g(x)$ at each genus. This is the main problem to determine the Gromov–Witten potential $\mathbf{F_g}$ from the anomaly equation. Apart from this important problem, we can extract some algebraic (combinatorial) structure of the equation by considering the same BCOV anomaly equation (4.1) in the reduced ring $\mathcal{R}^{0,red}_{BCOV}$ defined in (3.8).

Let us first note that $\mathcal{R}^{0,red}_{BCOV}$ is generated by S^{ij}, S^k, S, K_i and C_{ijk} over $\mathbf{Q}$ in the quotient ring, with the 'reduced' differential D_i (see Remark 3.9). Hereafter all manipulations should be understood in this quotient ring, although we abuse the same notations. The BCOV anomaly equation has the same form as (4.1) with obvious replacements of the generators, e.g. $\tilde{S}^{ij}$ by S^{ij}. Then the following property is due to [2]:

Proposition 4.2. *Define new generators by*

$$\hat{S}^{ij} = S^{ij} \ , \ \hat{S}^k = S^k - S^{km}K_m \ , \ \hat{S} = S - S^m K_m + \frac{1}{2}S^{mn}K_m K_n \ ,$$

then the second equation of (4.1) implies simply $\frac{\partial \mathcal{F}^{(g)}}{\partial K_m} = 0$, *namely*

$$\mathcal{F}^{(g)} \in \mathbf{Q}[\hat{S}^{ij}, \hat{S}^k, \hat{S}, C_{ijk}] \subset \mathbf{Q}[S^{ij}, S^k, S, K_m, C_{ijk}] \ (= \mathcal{R}^{0,red}_{BCOV}) \ .$$

Using the new generators above, the l.h.s of the first equation of (4.1) may be written as

$$\frac{\partial \mathcal{F}^{(g)}}{\partial \hat{S}^{ij}} - K_j \frac{\partial \mathcal{F}^{(g)}}{\partial \hat{S}^i} + \frac{1}{2} K_i K_j \frac{\partial \mathcal{F}^{(g)}}{\partial \hat{S}} \ ,$$

while the r.h.s. of that equation has the following expansion,

$$\begin{aligned}(4.3)\qquad &\frac{1}{2}D_iD_j\mathcal{F}^{(g-1)}+\frac{1}{2}\sum_{h=1}^{g-1}D_i\mathcal{F}^{(g-h)}D_j\mathcal{F}^h\\ &=Q_{ij}^{(g-1)}+Q_iK_j^{(g-1)}+Q_j^{(g-1)}K_i+\frac{1}{2}Q^{(g-1)}K_iK_j\ .\end{aligned}$$

Here one should note that by the above Proposition, the dependence on K_i comes only from the covariant derivatives. Now comparing each coefficient of $1, K_i$, K_iK_j, we have,

Proposition 4.3. *The BCOV anomaly equation in $\mathcal{R}^{0,red}_{BCOV}$ is equivalent to the following first order system of linear differential equations;*

$$(4.4)\quad \frac{\partial\mathcal{F}^{(g)}}{\partial\hat{S}^{ij}}=Q_{ij}^{(g-1)}\ ,\ \frac{\partial\mathcal{F}^{(g)}}{\partial\hat{S}^{i}}=-Q_{i}^{(g-1)}\ ,\ \frac{\partial\mathcal{F}^{(g)}}{\partial\hat{S}}=Q^{(g-1)}\ \ (g\geq 2).$$

With the initial data $Q_{ij}^{(1)}, Q_i^{(1)}, Q^{(1)}$ which follow from (4.3) with $D_i\mathcal{F}^{(1)} = \frac{1}{2}C_{ijk}\hat{S}^{jk} - (\frac{\chi}{24}-1)K_i$, this equation has a unique solution $\mathcal{F}^{(g)}\in\mathbf{Q}[\hat{S}^{ij},\hat{S}^k,\hat{S},C_{ijk}]$ of weight $(2-2g,0)$.

The uniqueness of the solution above follows from the weight consideration for the possible 'constants of integration' in the ring $\mathcal{R}^{0,red}_{BCOV}$.

For the application to Gromov–Witten potential $\mathtt{F}_g(t)$, the BCOV anomaly equation should be considered in the ring $\mathcal{R}^{\Gamma,\infty}_{BCOV}$, as we have summarized briefly in the previous section. However, the simple structure (4.4) extracted above in $\mathcal{R}^{0,red}_{BCOV}$ is still valid for the BCOV anomaly equation in the ring $\mathcal{R}^{\Gamma,\infty}_{BCOV}$. In fact, the form (4.4) of the BCOV equation has appeared first in [[19], Section (3-4)] to make the solutions $\mathcal{F}^{(g)}$ $(\mathtt{F}_g(t))$.

§5. Conclusions and discussions

After a self-contained introduction to the special Kähler geometry, we have introduced the differential ring $\mathcal{R}^0_{BCOV}$, which is geometric in nature. Combined with the modular property, we considered the Γ-invariant 'lifts' $\tilde{S}^{ij}, \tilde{S}^k, \tilde{S}$ of the propagators. The 'lifting' process has been identified with that of fixing 'meromorphic ambiguities' in [5]. With a choice of Γ-invariant lifts of the propagators, we defined the ring $\mathcal{R}^{\Gamma}_{BCOV}$, which depends on the choice of the lifts. After taking a symplectic basis $\mathcal{B}$, we defined the holomorphic limit $\mathcal{R}^{hol}_{BCOV}$ following [5].

In case of an elliptic curve, we have shown a close relation of our BCOV rings to the theory of quasi-modular forms due to Kaneko–Zagier [23].

Considering a suitable quotient, we have reduced the ring $\mathcal{R}^0_{BCOV}$ to a finitely generated differential ring $\mathcal{R}^{0,red}_{BCOV}$. In this reduced ring, we have extracted a simple algebraic structure of the BCOV holomorphic anomaly equation which still exists before the reduction.

As briefly summarized in Section 4, our construction of the BCOV ring is an abstraction of the important progress made in [32] and [2] for the solutions of BCOV holomorphic anomaly equation. In 1999, in case of a rational elliptic surface $\frac{1}{2}K3$, M.-H. Saito, A. Takahashi and the present author [21] found a similar recursion formula for Gromov–Witten potentials,

$$\frac{\partial Z_{g;n}}{\partial E_2} = \frac{1}{24} \sum_{\substack{g'+g''=g \\ g',g''\geq 0}} \sum_{s=1}^{n-1} s(n-s) Z_{g';s} Z_{g'';n-s} + \frac{n(n+1)}{24} Z_{g-1;n} \ , \tag{5.1}$$

in terms of quasi-modular forms $Z_{g;n}=P_{g;n}\frac{q^{\frac{n}{2}}}{\eta(\tau)^{12n}}$ $(P_{g;n} \in \mathbf{Q}[E_2, E_4, E_6])$ with the initial data $Z_{0;1} = \frac{q^{\frac{1}{2}}E_4}{\eta(\tau)^{12}}$ (this generalized a previous result in [25], [26] for $g = 0$ case). Later it has been conjectured that the above recursion relation (5.1) is equivalent to the BCOV holomorphic anomaly equation evaluating $\mathcal{F}^{(g)}$ for $g \leq 3$ [18]. Due to recent progress made in [32] and [2], we have now the BCOV anomaly equation of the form,

$$\frac{\partial \mathcal{F}^{(g)}}{\partial S^{yy}} = \frac{1}{2}\Big(\sum_{h=1}^{g-1} D_y \mathcal{F}^{(g-h)} D_y \mathcal{F}^{(h)} + D_y D_y \mathcal{F}^{(g-1)} \Big) \ ,$$

which is defined over a suitable BCOV ring $\mathcal{R}^{\Gamma,\infty}_{BCOV}$. In this form, we can prove the equivalence of the 'modular anomaly equation' (5.1) to the BCOV holomorphic anomaly equation. It is almost clear that the close relationship between the ring of the quasi-modular forms and our BCOV ring presented in Section 3 plays a central role for the equivalence. The detailed results will be reported elsewhere [20].

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Graduate School of Mathematical Sciences
University of Tokyo
Meguro-ku, Tokyo 153-8914
Japan
E-mail address: hosono@ms.u-tokyo.ac.jp