

Some questions on spectrum and arithmetic of locally symmetric spaces

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Abstract.

We consider the question that the spectrum and arithmetic of locally symmetric spaces defined by congruent arithmetical lattices should mutually determine each other. We frame these questions in the context of automorphic representations.

§1. Introduction

The classical spectral theories of light and sound can be mathematically interpreted and generalized in possibly two different ways: one, in the context of Riemannian manifolds and the spectrum of the Laplacian acting on a suitably defined space of smooth functions on the manifold. More generally, one can consider the spectrum of elliptic, self-adjoint, natural differential operators acting on subspaces defined by appropriate boundary conditions of the space of smooth sections of natural vector bundles on the manifold, for example, the deRham Laplacian acting on the space of smooth forms.

The concept of spectrum can also be generalized in the context of continuous representation theory of topological groups. In what follows, we restrict our attention to the class of locally symmetric Riemannian manifolds. These are spaces of the form $\Gamma \backslash G/K$, where G is a reductive Lie group, K is a maximal compact subgroup of G , and Γ is a lattice in G . The universal cover G/K carries a natural G -invariant metric, the Bergman metric, and we can relate the representation theory of G to the spectrum of Laplace type operators on $\Gamma \backslash G/K$.

It is well known that the spectra of Laplace type operators associated with locally symmetric Riemannian manifolds or the representation theory of G acting on the space of functions on $\Gamma \backslash G$, share many similarities with the arithmetic of such spaces. The beginnings of this

Received August 08, 2008.

Revised April 18, 2009.

analogy can be attributed to Maass, who introduced and studied the arithmetic properties of eigenfunctions of the Laplace operator on the upper half plane which are invariant under a congruence subgroup of $SL_2(\mathbb{Z})$ (called Maass forms) [8]. It was shown that Hecke operators can be defined on the space of such eigenfunctions of the Laplacian, and that the L -functions associated to Maass–Hecke eigenforms have many similar properties to the L -functions associated to holomorphic Hecke eigenforms.

The theory of Maass forms also provides a link between the representation theory of $G = PSL_2(\mathbb{R})$ acting on the space of square integrable functions $L^2(\Gamma \backslash G)$, and the spectrum of the hyperbolic Laplacian acting on $\Gamma \backslash \mathbb{H}$, where \mathbb{H} is hyperbolic upper half plane. For general groups, Langlands theory of automorphic representations provides the appropriate generalization to relate not only the spectrum of invariant differential operators to representation theory, but also to the arithmetic that is provided by this theory.

It was Selberg who brought the analogy between the spectrum and the arithmetic to the fore. Selberg considered primitive closed geodesics on a hyperbolic surface of finite area, as analogous to rational primes, and established an analogue of the prime number theorem for primitive closed geodesics [10]. The concept of the length spectrum can be introduced, and the Selberg trace formula establishes a relationship between the spectrum of the Laplacian acting on functions on a compact hyperbolic surface and the length spectrum of primitive closed geodesics on it. Although the relationship with the length spectrum is quite important, we will not pursue it out here.

Another analogy observed by Selberg, is the conjecture that

$$\lambda_1(\Gamma \backslash \mathbb{H}) \geq 1/4,$$

where $\lambda_1(\Gamma \backslash \mathbb{H})$ is the least non-zero eigenvalue of the Laplacian acting on the space of smooth functions on $\Gamma \backslash \mathbb{H}$. Here Γ is a congruent arithmetical lattice contained inside $SL_2(\mathbb{Z})$. Selberg's conjecture can be considered as the archimedean analogue of Ramanujan's conjecture on the size of the Hecke eigenvalues of holomorphic newforms [25].

In a similar vein, there are many common features between the representation theories of real and p -adic Lie groups. For example, the Harishchandra homomorphism identifies the center $Z(\mathfrak{g})$ of the universal enveloping algebra of a complex semisimple Lie algebra \mathfrak{g} with the ring of Weyl group invariants of the symmetric algebra of a maximal torus in \mathfrak{g} [12]. The Casimir elements belong to $Z(\mathfrak{g})$, and these project to invariant differential operators on the associated symmetric space. The non-archimedean counterpart of the Harishchandra isomorphism is the

Satake isomorphism: if G is a split reductive p -adic Lie group, K a hyperspecial maximal compact subgroup of G , T a maximal split torus of G , the Satake isomorphism identifies the Hecke algebra $H(G, K)$ of K -biinvariant functions on G , to the Weyl group invariants of the algebra of unramified characters of T (see Cartier's article in [5]). Thus the Hecke algebra is analogous to the algebra of invariant differential operators of a complex semisimple Lie algebra.

The inverse spectral problem is to know the properties of the Riemannian manifold that are determined by the spectrum. For instance, is the isometry class of the space determined by the spectrum? In the context of planar domains, this question was posed in a colloquial way by M. Kac as "Can one hear the shape of the drum?". Milnor and later Vigneras used arithmetical methods to construct pairs of isospectral but non-isometric manifolds. A fundamental construction of pairs of isospectral manifolds was given by Sunada. Sunada's method is completely analogous to Gassmann's method of exhibiting non-conjugate number fields having the same Dedekind zeta function. These examples indicate the existence of an arithmetical flavor to the inverse spectral problem.

Based on such examples, a well known heuristic is that the Laplacian can be considered as the Frobenius/Hecke operator at infinity, for the class of locally symmetric spaces defined by congruent arithmetical lattices. Such spaces have a natural Riemannian structure arising from the invariant metric on the universal cover. When the universal cover is hermitian symmetric, a natural arithmetical structure is provided by the theory of canonical models due to Shimura and Deligne. In many examples, these spaces are the moduli spaces of abelian varieties endowed with extra structure. In the general case, the Langlands theory of automorphic representations can be considered as an automorphic aspect of the arithmetic of such spaces.

We expound on the theme that the Frobenius at infinity is the Laplacian, and our basic expectation is that the spectrum and arithmetic for such spaces should mutually determine each other.

Acknowledgements. My sincere thanks to Dipendra Prasad for many discussions on this topic. I thank T. Chinburg, L. Clozel, C. Deninger, S. Ganguly, G. Harder, S. Kudla, M. S. Narasimhan, G. Prasad, N. Ramachandran, A. W. Reid, P. Sarnak and A. Venkatesh for many useful discussions, remarks, etc.

During the course of this work, I availed of the hospitality of MPIM at Bonn during the year 2005–06 and April of 2009, the IAS at Princeton

for the first half of 2007, and the University of Münster during May–June of 2008. I thank all these Institutes for their support and wonderful hospitality.

§2. Archimedean analogue of Tate’s conjecture

Let M be a compact Riemann surface, uniformized by the upper half plane. The Jacobian $J(M)$ of M , the ‘motive’ associated to the Riemann surface M , can be considered as the ‘arithmetic’ associated to M .

We recall the theorem of Faltings proving Tate’s conjecture [7]: if the eigenvalues of the Frobenius elements acting on the l -adic cohomology groups of two smooth, projective curves defined over a number field coincide, then the Jacobians of the curves are isogenous. The analogy of the Laplacian with the Frobenius motivates us to ask a naive archimedean analogue of Tate’s conjecture:

Question 1. Suppose M_1, M_2 are compact Riemann surfaces of genus at least two, that are isospectral with respect to the hyperbolic Laplacian acting on the space of smooth functions. Are the Jacobians $J(M_1)$ and $J(M_2)$ isogenous?

This question turns out to have a false answer (see Corollary 2) in general, but can be verified for the examples constructed using the Gassmann–Sunada method ([21]).

Remark 1. A similar question was raised by M. S. Narasimhan (to T. Sunada) but requiring the stronger conclusion that the Jacobians be isomorphic. This seems unrealistic to expect even for the examples constructed using the Sunada method. From the arithmetic viewpoint it is natural to consider the isogeny class.

Narasimhan also raised the question of considering isospectrality with respect to other natural metrics defined on hyperbolic compact Riemann surfaces, for example the pullback of the Fubini–Study metric with respect to the canonical map to projective space, or the pullback of the translation invariant metric with respect to embedding the curve in its Jacobian, etc. But it is the hyperbolic metric that is natural in the context of automorphic forms (see Theorem 5), and thus seems natural to consider if we want to relate the spectrum and the arithmetic of such spaces.

2.1. Flat tori

By the theorem of Faltings, we see that two abelian varieties defined over a number field have isomorphic l -adic representations if and only if they are isogenous.

In the context of flat tori, define two flat tori to be *arithmetically equivalent* if they are isogenous. We define two flat tori to be *spectrally commensurable* if upto rescaling of metrics the spectra are mutually contained inside each other. The expectation that the spectrum determines the arithmetic translates to the following refinement of a conjecture of Kitaoka [11]:

Conjecture 1. Suppose L_1, L_2 are lattices in Euclidean n -space \mathbb{R}^n , such that the corresponding flat tori $\mathbb{R}^n/L_1, \mathbb{R}^n/L_2$ are spectrally commensurable. Then upto an isometry of \mathbb{R}^n , the lattices L_1 and L_2 are commensurable.

(To put it more colloquially, the thetas and the zetas mutually determine each other.)

The known examples (see [4] for some of them), for example Milnor's example of the isospectral pair of flat tori given by the lattices $(E_{16}, E_8 \oplus E_8)$, can be seen to be commensurable. Indeed, when the quadratic form associated to the lattices are integral valued, a brief hint of the proof of the above conjecture with the stronger assumption that the flat tori are isospectral, was given by Kitaoka [11], and a detailed proof of a more general result was given by Bayer and Nart [2].

§3. Arithmetic Archimedean analogue of Tate's conjecture

In the previous section, we had taken as our model for the arithmetic of a compact Riemann surface, the Jacobian of the surface. Here, we focus our attention on the example of projective Shimura curves: they have natural canonical models defined over a number field, and so we can consider the usual arithmetic of such spaces [28].

Let F be a number field and let D be an indefinite, quaternion division algebra over F . Let

$$G = SL_1(D), \quad \text{and} \quad \tilde{G} = GL_1(D).$$

Let K_∞ be a maximal compact subgroup of $G_\infty = G(F \otimes \mathbb{R})$ and let $M = G_\infty/K_\infty$ be the non-compact symmetric space associated to G . If D is unramified at r real places and $2r_2$ is the number of complex embeddings of F , then,

$$M \simeq \mathbb{H}_2^r \times \mathbb{H}_3^{r_2},$$

where \mathbb{H}_2 (resp. \mathbb{H}_3) is the hyperbolic 2-space (resp. 3-space). For a number field F , let \mathbf{A}_F denote the adèle ring of F , and $\mathbf{A}_{F,f}$ the subring of finite adèles. Let K be a compact open subgroup of $G(\mathbf{A}_{F,f})$. Let Γ_K be the co-compact lattice in G_∞ corresponding to K , given by the projection to G_∞ of the group $G_\infty K \cap G(F)$. If K is sufficiently small, then Γ_K will be a torsion-free lattice acting freely and properly discontinuously on M . The quotient space $M_K = \Gamma_K \backslash M$ will then be a compact, locally symmetric space.

Suppose now F is totally real. We recall the theory of canonical models due to Shimura [27]. Let τ_1, \dots, τ_r be the real embeddings corresponding to archimedean places of F at which D splits. The reflex field F' of $(F, \tau_1, \dots, \tau_r)$ is the field generated by the sums

$$\sum_{i=1}^r \tau_i(x), \quad x \in F.$$

There exists a \mathbb{Q} -rational homomorphism

$$\lambda : R_{F'/\mathbb{Q}} \mathbf{G}_m \rightarrow R_{F/\mathbb{Q}} \mathbf{G}_m,$$

where for a number field K , $R_{K/\mathbb{Q}}$ denotes the Weil restriction of scalars. On the \mathbb{Q} -rational points, $\lambda : (F')^* \rightarrow F^*$ satisfies the following relation:

$$N_{F/\mathbb{Q}}(\lambda(x)) = N_{F'/\mathbb{Q}}(x)^r, \quad x \in (F')^*.$$

When $r = 1$, the reflex field can be taken to be F itself.

Let $\tilde{G}_{\infty+}$ (resp. $F_{\infty+}^*$) be the identity component of \tilde{G}_∞ (resp. F_∞^*). Let

$$\tilde{G}_+ = \{x \in \tilde{G}_{\infty+} \tilde{G}(\mathbf{A}_{F,f}) \mid \nu(x) \in \lambda(\mathbf{A}_{F'}^* F^* F_{\infty+}^*)\},$$

where $\nu : \tilde{G} \rightarrow \mathbf{G}_m$ is the reduced norm. For any compact open subgroup \tilde{K} of $\tilde{G}(\mathbf{A}_{F,f})$, define a subgroup $N(\tilde{K})$ of the group of ideles $\mathbf{A}_{F'}^*$ as,

$$N(\tilde{K}) = \{c \in \mathbf{A}_{F'}^* \mid \lambda(c) \in F^* \nu(\tilde{K})\}.$$

By class field theory, the subgroup $N(\tilde{K})$ defines an abelian extension $F'_{\tilde{K}}$ of F' . Given $x \in \tilde{G}_+$, choose an element $c \in \mathbf{A}_{F'}^*$ such that $\lambda(c)/\nu(x) \in F^* F_{\infty+}^*$. By the reciprocity morphism of class field,

$$\text{rec} : \mathbf{A}_{F'}^* / (F')^* \rightarrow \text{Gal}((F')^{ab} / F'),$$

we get an element $\sigma(x) \in \text{Gal}((F')^{ab} / F')$ by the prescription

$$\sigma(x) = \text{rec}(c^{-1}).$$

Associated to \tilde{K} , we get a lattice

$$\Gamma_{\tilde{K}} = \tilde{K}\tilde{G}_{\infty+} \cap \tilde{G}\mathbb{Q}.$$

Let $K = \tilde{K} \cap G(\mathbf{A}_{F,f})$, and assume the following:

Assumption (*). The natural inclusion of $\Gamma_K \subset \Gamma_{\tilde{K}}$ is an isomorphism modulo the centers.

This assumption ensures that the quotient of M by the actions of Γ_K and $\Gamma_{\tilde{K}}$ are isomorphic.

For sufficiently small \tilde{K} , the group $\Gamma_{\tilde{K}}$ modulo the centre acts freely and discontinuously on the associated symmetric space M . By the theory of canonical models, the complex varieties M_K acquire a canonical model defined over the abelian extension of $F'_{\tilde{K}}$, satisfying various compatibility properties. We will refer to such varieties as ‘Shimura varieties of quaternionic type’. One of the principal properties that we require is that there is an isomorphism between the spaces,

$$(3.1) \quad M_K \simeq M_{K^x}^{\sigma(x)}.$$

We now give an ‘arithmetical archimedean analogue of Tate’s conjecture’ given in [21]:

Conjecture 2. Suppose X and Y are Shimura varieties of quaternionic type of dimension s , which are isospectral with respect to the deRham Laplacian acting on the space of p -forms for $0 \leq p < r$. Then the Hasse–Weil zeta functions of X and Y are equal.

Remark 2. A more general conjecture for curves is given in [21], where we do not restrict ourselves to Shimura curves. But for such curves, it is not at all clear about the naturality of either the Riemannian metric nor of the arithmetic of such varieties. For Shimura curves and more generally for the class of spaces defined by quotients of symmetric spaces by congruent arithmetic lattices, both the arithmetic and the spectrum can be related to theory of automorphic forms, and hence it is reasonable to restrict our attention to such spaces.

For higher dimensional varieties, it is natural to add the spectrum of deRham Laplacian acting on the space of smooth differential forms of higher degrees, or even other natural elliptic self adjoint differential operators. But as we will see in Section 8, it is more natural to break up this conjecture into two parts: one, relating the the spectral side to representation equivalence of lattices; the second is to reformulate the above conjecture on the spectral side in terms of representation equivalence of lattices.

The first striking evidence for this conjecture that leads us to believe in the above conjecture, comes from the following theorem of A. Reid [24]:

Theorem 1 (Reid). *Suppose X and Y are Shimura curves of quaternionic type, associated respectively to quaternion division algebras D_X , D_Y defined respectively over totally real number fields F_X and F_Y . Assume that X and Y are isospectral with respect to the hyperbolic Laplacian acting on the space of smooth functions. Then*

$$F_X = F_Y \quad \text{and} \quad D_X = D_Y.$$

The proof of Reid's theorem starts with the equivalent hypothesis that X and Y are length isospectral, a consequence of the Selberg trace formula. The length spectrum is then related to the arithmetic of the quaternion division algebras. This latter step has been generalised by T. Chinburg, E. Hamilton, D. Long and A. Reid [3], and more generally by G. Prasad and A. S. Rapinchuk [22].

§4. Representation equivalence of lattices

In this section, we relate the representation theoretic approach to isospectral questions. We recall the Gassmann–Sunada method: A triple (G, H_1, H_2) consisting of a finite group G and two non-conjugate subgroups H_1, H_2 of G is said to form a *Gassmann–Sunada system* if it satisfies one of the following equivalent conditions:

- (1) The regular representations of G on $\mathbb{C}[H_1 \backslash G]$ and $\mathbb{C}[H_2 \backslash G]$ are equivalent.
- (2) Any G -conjugacy class intersects H_1 and H_2 in the same number of elements.
- (3) For any finite dimensional representation V of G , the dimension of the spaces of invariants with respect to the subgroups H_1 and H_2 are equal.

Example. There exists examples of Gassmann–Sunada systems. For example suppose H_1 and H_2 are two non-isomorphic finite groups such that for any natural number d , the number of elements of order d in H_1 and H_2 are equal (this is a necessary condition for H_1 and H_2 to be part of a Gassmann–Sunada system as above). In particular, the cardinalities of H_1 and H_2 are equal, say d . Then (S_d, H_1, H_2) forms a Gassmann–Sunada system, where S_d denotes the symmetric group on d symbols.

Gassmann proved the following theorem (see [21], [17]):

Theorem 2 (Gassmann). *Suppose K is a finite Galois extension of the rationals with Galois group isomorphic to G . Suppose G belongs to a Gassmann–Sunada system (G, H_1, H_2) . Then the field of invariants K^{H_1} and K^{H_2} are non-conjugate number fields having the same Dedekind zeta function.*

Sunada’s fundamental observation ([29]) is that an entirely analogous statement can be transplanted to the spectral side:

Theorem 3 (Sunada). *Suppose (G, H_1, H_2) is a Gassmann–Sunada system and assume that G acts freely and isometrically on a compact Riemannian manifold M . Then the quotient spaces of M by the action of the groups H_1 and H_2 are isospectral with respect to the metric induced from M .*

The proofs of the theorems of Gassmann and Sunada as well as the validity of Question 1 for the examples of pairs of isospectral compact Riemann surfaces constructed using the Sunada method, rest on the functorial nature of the Frobenius isomorphism i.e., given a G -space V , and H a subgroup of the finite group G , the Frobenius isomorphism,

$$V^H \simeq (V \otimes \mathbb{C}[H \backslash G])^G$$

is natural (see [21]) with respect to G -equivariant maps $\phi : V \rightarrow W$ of G -spaces. Sunada’s theorem follows by taking $V = C^\infty(M)$, and ϕ to be the Laplacian. Since G acts by isometries, the Laplacian is G -equivariant, and restricts to give the Laplacian of the space $V^{H_i} \simeq C^\infty(M/H_i)$, $i = 1, 2$.

Sunada’s condition for isospectrality can be extended to infinite groups, to the context of continuous Lie group actions and discrete subgroups of such Lie groups. Let G be a Lie group, and let Γ be a cocompact lattice in G . The existence of a cocompact lattice implies that G is unimodular. Let R_Γ denote the right regular representation of G on the space $L^2(\Gamma \backslash G)$ of square integrable functions with respect to the projection of the Haar measure on the space $\Gamma \backslash G$:

$$(R_\Gamma(g)f)(x) = f(xg) \quad f \in L^2(\Gamma \backslash G), \quad g, x \in G.$$

As a G -space, $L^2(\Gamma \backslash G)$ breaks up as a direct sum of irreducible unitary representations of G , with each irreducible representation occurring with finite multiplicity.

Definition 4.1. Let G be a Lie group and Γ_1 and Γ_2 be two cocompact lattices in G . The lattices Γ_1 and Γ_2 are said to be *representation equivalent in G* if the regular representations R_{Γ_1} and R_{Γ_2} of G are isomorphic.

We have the following generalization of Sunada's criterion for isospectrality proved by DeTurck and Gordon in [6]:

Proposition 4. *Let G be a Lie group acting on the left as isometries of a Riemannian manifold M . Suppose Γ_1 and Γ_2 are discrete, co-compact subgroups of G acting freely and properly discontinuously on M , such that the quotients $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are compact Riemannian manifolds.*

If the lattices Γ_1 and Γ_2 are representation equivalent in G , then $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are isospectral for the Laplacian acting on the space of smooth functions.

The conclusion in Proposition 4 can be strengthened to imply 'strong isospectrality', i.e., the spectrums coincide for natural self-adjoint differential operators besides the Laplacian acting on functions. In particular, this implies the isospectrality of the Laplacian acting on p -forms.

The natural hermitian vector bundles on spaces of the form $\Gamma \backslash G/K$ should be the class of automorphic vector bundles. On the universal cover G/K , the automorphic vector bundles are the G -invariant hermitian vector bundles that are associated to representations of K on some finite dimensional unitary vector space V . The centre $Z(\mathfrak{g})$ of the universal enveloping algebra acts on the space of smooth sections $C^\infty(\Gamma \backslash G, V)$ of these vector bundles. Since the image of $Z(\mathfrak{g})$ is generated by invariant self adjoint elliptic differential operators, it follows that if Γ is a uniform lattice in G , that for any character χ of $Z(\mathfrak{g})$ the χ -eigenspace,

$$\{\phi \in C^\infty(\Gamma \backslash G, V) \mid z \cdot \phi = \chi(z)\phi, \quad z \in Z(\mathfrak{g})\},$$

has finite dimension $m(\chi, \Gamma, V)$.

Proposition 4 can be refined to say that the two spaces $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are *strongly compatibly isospectral*, in the sense that for any V , χ as above, we have an equality of multiplicities,

$$m(\chi, \Gamma_1, V) = m(\chi, \Gamma_2, V).$$

4.1. Converse to Gassmann–Sunada criterion

We see that representation equivalent lattices give rise to spectrally indistinguishable spaces. It is natural to ask whether the converse holds. When $G = PSL_2(\mathbb{R})$, basic facts from the representation theory of G , allow us to show that the converse holds. Given a hyperbolic compact Riemann surface X , let

$$\rho_X : \Gamma_X \rightarrow PSL(2, \mathbb{R}),$$

be an embedding of the fundamental group Γ_X arising from the uniformization of X by the upper half plane. This map is independent of the various choices made upto conjugation by an element of $PSL(2, \mathbb{R})$. A folklore observation is the following [19]:

Theorem 5. *Suppose X and Y are compact Riemann surfaces of genus at least two. Then X and Y are isospectral with respect to the hyperbolic Laplacian acting on the space of smooth functions if and only if the lattices $\rho_X(\Gamma_X)$ and $\rho_Y(\Gamma_Y)$ are representation equivalent in $PSL(2, \mathbb{R})$:*

$$L^2(PSL(2, \mathbb{R})/\Gamma_X) \simeq L^2(PSL(2, \mathbb{R})/\Gamma_Y),$$

as $PSL(2, \mathbb{R})$ -modules.

The proof that the spectrum determines the representation type of Γ_X in $PSL(2, \mathbb{R})$ follows from the classification of the irreducible unitary representations of $PSL(2, \mathbb{R})$.

Remark 3. Maass showed that a theory of Hecke operators can be defined on the spaces of eigenfunctions of the Laplacians invariant by the lattice, paving the way for the work of Selberg and Langlands.

Remark 4. One of the main reasons for considering the hyperbolic Laplacian for Riemann surfaces is that the spectrum of the hyperbolic Laplacian can be related to the representation theory of the isometry group $PSL(2, \mathbb{R})$ of the universal covering space. If we consider other natural metrics on a compact Riemann surface, then we can no longer expect any relationship with representation theory.

For groups apart from $PSL_2(\mathbb{R})$, the converse direction from the spectrum to the representation theory is not well understood. For compact quotients of hyperbolic spaces, H. Pesce [18] has shown that if two such spaces are strongly isospectral, then the corresponding lattices are representation equivalent in the group of isometries of the hyperbolic space. For more general groups, we formulate later a possible approach to a converse of the generalized Sunada criterion in the context of compact locally symmetric spaces (see Conjecture 3).

§5. Adelic conjugation of lattices

In this section, we provide some evidence for the conjectures raised in the previous sections. In the process, we construct new examples of isospectral but non-isometric spaces.

Let F be a number field and let D be an indefinite quaternion division algebra over F . We assume that there is at least one finite place

v_0 of F , at which D is ramified. Let K be a compact open subgroup of $G(\mathbf{A}_{F,f})$ having a factorisation of the form,

$$K = K_0 K',$$

where K_0 is a compact, open subgroup of $G(F_{v_0})$, and is an invariant subgroup of $D_{v_0}^*$. The group K' has no v_0 component, i.e., for any element $x \in K'$, the v_0 component $x_{v_0} = 1$. Let

$$\Gamma_K = G_\infty K \cap G(F),$$

be the co-compact lattice in G_∞ , and let $M_K = \Gamma_K \backslash M$ be the quotient space of M by Γ_K .

The following theorem is proved in [23], and can be considered as a geometric version of the concepts of L -indistinguishability due to Labesse, Langlands and Shelstad:

Theorem 6. *With the above notations and assumptions, for any element $x \in GL_1(D)(\mathbf{A}_{F,f})$, the lattices Γ_K and Γ_{K^x} are representation equivalent.*

The following corollary gives examples of isospectral but non-isometric compact Riemannian manifold, generalizing the class of examples constructed by Vigneras [30], reflecting essentially the failure of strong approximation in the adjoint group:

Corollary 1. *With the notation as in Theorem 6, assume further that K is small enough so that Γ_K is torsion-free. Let $\widehat{N(K)}$ denote the normalizer of the image of K in $PGL_1(D)(\mathbf{A}_{F,f})$. Choose an element $x \in \widehat{G}(\mathbf{A}_f)$ such that its projection to $PGL_1(D)(\mathbf{A}_{F,f})$ does not belong to the set $\widehat{N(K)}PGL_1(D)(F)$ (such elements exist by the failure of strong approximation in $PGL_1(D)$).*

Then X_K and X_{K^x} are strongly isospectral, but are not isometric.

Remark 5. Vigneras works with the length spectrum rather than the representation theoretic context in which the above theorem is placed. The relation between the Laplacian spectrum and the length spectrum holds only for hyperbolic surfaces and three folds. Further, she has to compute the length spectrum and show that indeed the lattices are length isospectral. For this reason, she has to restrict attention to K which come from maximal orders and use theorems of Eichler computing the number of elements having a given trace.

We also obtain the following corollary providing more evidence in support of the Conjecture 2:

Corollary 2. *Let F be a totally real number field and \tilde{K} be a compact open subgroup of $\tilde{G}(\mathbf{A}_{F,f})$. Assume that \tilde{K} satisfies the hypothesis of Theorem 6 and is such that the lattice Γ_K is torsion-free and satisfies Assumption (*).*

Then the spaces M_{K^x} for $x \in \tilde{G}(\mathbf{A}_{F,f})$ are isospectral and have the same Hasse–Weil zeta function for the canonical model defined by Shimura.

If D is ramified at all real places except one, then the Jacobians of M_K and M_{K^x} are not isogenous but are conjugate by an automorphism of \mathbb{Q} .

This gives us the counterexamples to Question 1. It is tempting to conjecture that the last statement of the foregoing corollary will be the exception to Question 1: if two compact Riemann surfaces are isospectral, then the Jacobian of one is isogenous to a conjugate of the Jacobian of the other by an automorphism $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$, where σ preserves the spectrum of the Riemann surface. If moreover σ is not identity or of order two, then the pair of Riemann surfaces arise from an arithmetical context, i.e., are Shimura curves as considered in this paper.

§6. L -equivalence

The proof of Theorem 6 given in [23], uses the theory of L -indistinguishable automorphic representations initiated by Langlands, in particular the computation of the multiplicity by Labesse and Langlands with which a representation of $SL(1, D)(\mathbf{A}_F)$ occurs in the space of automorphic representations of $SL(1, D)(\mathbf{A}_F)$ [5], [14]. An examination of the proof of Theorem 6 indicates that it yields a more general formulation, which will form the basis of reformulating the above conjectures in a more general framework.

From the viewpoint of Langlands theory, it is more natural to classify the L -packets of irreducible representations ([1, 5, 13, 15, 26]). We will assume that there exists a suitable notion of L -packets. We do not attempt out here the definition of either the Langlands or Arthur L -packets, nor what is the correct notion of L -packets that we require for the relationship between the spectrum and arithmetic to hold. A coarse expectation that two representations are L -equivalent (or L -indistinguishable or equivalently belong to the same L -packet) is that the L -functions and ϵ -factors attached to them are equal. It is expected that L -packets have finite cardinality. For representations of real Lie groups, a property that members of the same L -packet share is that they all have the same infinitesimal character, i.e., they cannot be distinguished by means of spectral data alone.

Example. If F is a local field, the L -packets of representations of $SL(2, F)$ (or any of its inner forms $SL(1, D)(F)$) consists of those representations that are conjugate to each other by an element of $GL(2, F)$ (resp. by $GL(1, D)(F)$). Thus the discrete series $\{\pi_k^+, \pi_k^-, k \geq 2\}$ constitute a single L -packet for $SL(2, \mathbb{R})$.

Let F be a global field. Two representations of $SL(2, \mathbf{A}_F)$ (resp. of $SL(1, D)(\mathbf{A}_F)$) are said to be L -equivalent if their local components at each place of F are L -indistinguishable as representations of the local group. Equivalently, they are conjugate by an element of $GL(2, \mathbf{A}_F)$ (resp. $GL(1, D)(\mathbf{A}_F)$).

Given an irreducible unitary representation π of a semisimple Lie group G (real or p -adic or adelic), we denote by $[\pi]$, the L -packet of G containing π . For an automorphic π of $G(\mathbf{A})$, we denote by π^K the space of K -fixed vectors of the space underlying the representation π and by $m_0(\pi)$ the (finite) multiplicity with which π occurs in the space of cusp forms. We define a notion of automorphic equivalence of lattices:

Definition 6.1. Let G be a reductive algebraic group over F and K be a compact open subgroup of $G(\mathbf{A}_{F,f})$. The *cuspidal automorphic spectrum* associated to $(G(\mathbf{A}_F), K)$, is the collection of cuspidal automorphic representations π of $G(\mathbf{A}_F)$ counted with multiplicity $m(\pi, K)$, where

$$m(\pi, K) = m_0(\pi) \dim(\pi^K).$$

The *L -cuspidal spectrum* associated to $(G(\mathbf{A}_F), K)$ is the collection of cuspidal L -packets of $[\pi]$ of $G(\mathbf{A}_F)$ counted with multiplicity $m([\pi], K)$ defined as,

$$m([\pi], K) = \sum_{\eta \in [\pi]} m(\eta, K).$$

Definition 6.2. Let G_1 and G_2 be semisimple groups over F . Suppose K_1 and K_2 are compact open subgroups of $G_1(\mathbf{A}_{F,f})$ and $G_2(\mathbf{A}_{F,f})$ respectively. Then the pair $(G_1(\mathbf{A}_{F,f}), K_1)$ and $(G_2(\mathbf{A}_F), K_2)$ are said to be *cuspidally equivalent* (resp. *L -cuspidally equivalent*) if there exists an isomorphism

$$\phi: G_1(\mathbf{A}_F) \rightarrow G_2(\mathbf{A}_F)$$

such that for any cuspidal representation π of $G_2(\mathbf{A}_F)$, the multiplicities $m(\pi, K_2)$ and $m(\pi \circ \phi, K_1)$ (resp. $m([\pi], K_2)$ and $m([\pi \circ \phi], K_1)$) are equal.

The reasons for the above definitions stem from the following strengthening of Theorem 6:

Theorem 7. *With the same hypothesis as in Theorem 6, the lattices K and K^x are L -equivalent in $SL_1(D)(\mathbf{A}_F)$.*

The proof of Theorem 6 given in [23] carries over to prove this more general theorem.

§7. From arithmetic to spectrum

Suppose X is a projective Shimura curve. We take for a tentative definition of the arithmetic of X the data given by Hasse–Weil zeta functions attached to ‘natural’ local systems on X , i.e., those local systems which arise by restriction to the lattice from algebraic representations of the group $\tilde{G} = GL_1(D)$. The irreducible representations are essentially given by the symmetric powers of the standard representation of $GL(2)$. We are then led to ask the converse to the above questions that the spectrum determines the arithmetic:

Question 2. Suppose X and Y are Shimura curves, such that the Hasse–Weil zeta functions associated to the local systems coming from the n -th symmetric power of the standard representation of $GL(2)$ are equal for all $n \geq 0$.

Are X and Y isospectral with respect to the hyperbolic Laplacian acting on the space of functions?

There is one context however where the arithmetic associated to a pair of varieties is expected to be the same. For this, we assume that X is associated to congruence lattices $K_X \subset G(\mathbf{A}_{F,f})$ as described in Section 3. In this case, the canonical models of the varieties are defined over an abelian extension of the reflex field F' . Now it follows from the theory of canonical models for algebraic local systems as above, that if $\sigma \in \text{Gal}((F')^{ab}/F')$, then the conjugate $Y = X^\sigma$ is first of all again a Shimura curve associated to D ; secondly, it has the same arithmetic as X in the sense described above.

In this case, we have the following theorem that Galois conjugation preserves the spectrum:

Theorem 8. *With notation as in Section 3, assume that \tilde{K} satisfies the hypothesis of Theorem 6 and is such that the lattices Γ_{K^x} are torsion-free and satisfies Assumption (*).*

For any $\sigma \in \text{Gal}((F')^{ab}/F')$, the spaces X and X^σ are isospectral for the hyperbolic Laplacian acting on functions.

The proof of this theorem is based on the observation that by the properties of canonical models, any σ as above can be written as $\sigma(x)$ for some $x \in GL_1(D)(\mathbf{A}_{F,f})$, and then appealing to Theorem 6. In some

sense, the ‘correct interpretation’ of Theorem 6 is given by the above theorem: *Galois conjugation over the reflex field preserves the spectrum.*

§8. General conjectures

We try to generalize the discussion so far to the general case of quotients of symmetric spaces defined by congruence arithmetic lattices. The relationship between the spectrum and arithmetic can be broken into two steps: one, the relationship between the spectrum and L -equivalence of uniform lattices in real Lie groups. The other aspect is to relate L -equivalence of lattices in the real points of a reductive or semisimple algebraic group to suitable notions of arithmetic, i.e., to relate to more global aspects.

8.1. Spectrum to arithmetic

Suppose Γ is a cocompact lattice in a real Lie group G . We have a direct sum decomposition,

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \pi,$$

where \hat{G} denotes the set of equivalence classes of irreducible unitary representations of G , and $m(\pi, \Gamma)$ the (finite) multiplicity with which π occurs in $L^2(\Gamma \backslash G)$. Define the multiplicity $m([\pi], \Gamma)$ of an L -packet $[\pi]$ by,

$$m([\pi], \Gamma) := \sum_{\eta \in [\pi]} m(\eta, \Gamma).$$

Define two uniform lattices Γ_1, Γ_2 to be L -equivalent or L -indistinguishable in G , if for any $\pi \in \hat{G}$,

$$m([\pi], \Gamma_1) = m([\pi], \Gamma_2).$$

With this definition, we can expect the following conjecture to hold:

Conjecture 3. Let G be a semisimple Lie group, K a maximal compact subgroup of G and Γ_1, Γ_2 two torsion-free uniform lattices in G .

Then Γ_1 and Γ_2 are L -equivalent in G if and only if the associated compact locally symmetric spaces $\Gamma_1 \backslash G/K$ and $\Gamma_2 \backslash G/K$ are compatibly strongly isospectral as defined in Section 4.

One can replace the isospectrality on forms by weaker assumptions: for instance, isospectrality for the deRham Laplacian on p -forms for all $0 \leq p \leq \dim(G/K)$. Or for the compact quotients of symmetric spaces of noncompact type, it is even tempting to ask whether isospectrality on functions is sufficient to guarantee the rest of the implication.

Remark 6. When the lattices are no longer cocompact, one can define a similar notion of two lattices being *cuspidally L -equivalent* by working with the cuspidal spectrum of $L^2(\Gamma \backslash G)$.

The relationship between L -equivalence and representation equivalence of lattices is not clear. For instance, the following question can be raised:

Question 3. Does there exist L -equivalent lattices which are not representation equivalent?

Remark 7. For $SL(2, \mathbb{R})$, it seems that Hodge duality will ensure that L -equivalent lattices are representation equivalent.

Remark 8. To start the discussion rolling from the spectral side, a first question to ask is whether any space isospectral to a compact locally symmetric space is itself locally symmetric? If we assume strong isospectrality, then this is proved by Gilkey [9]. A suitable assumption will be to impose isospectrality on forms, but for the compact quotients of non-compact symmetric spaces it would be interesting to know whether just the spectrum on functions will suffice.

Assuming the truth of the above question, the natural sequel to it is to say that the spectrum determines the symmetric space, in other words the isometry group of the universal cover.

Such results can be considered as spectral analogues of Langlands conjectures on conjugation of Shimura varieties, that the Galois conjugate of a Shimura variety is again a Shimura variety.

We now generalize the conjecture that the spectrum determines the arithmetic:

Conjecture 4. Let G_1, G_2 be semisimple algebraic groups over number fields F_1 and F_2 respectively. Let $K_i \subset G_i(\mathbf{A}_{F,f})$, $i = 1, 2$ be compact open subgroups, and Γ_{K_i} be the corresponding arithmetic lattices in $G_{i,\infty}$. Suppose $G_{1,\infty} \simeq G_{2,\infty} = G_0$, and that Γ_{K_1} and Γ_{K_2} are L -cuspidal, equivalent lattices in G_0 .

Then the adèle groups $G_1(\mathbf{A})$ and $G_2(\mathbf{A})$ are isomorphic, and the L -cuspidal spectrums of $(G_1(\mathbf{A}_F), K_1)$ and $(G_2(\mathbf{A}_F), K_2)$ are equal.

Remark 9. The results of Prasad–Rapinchuk [22] show that if the spaces corresponding to the two lattices satisfy a ‘weak commensurability’ property, then the corresponding adèle groups are isomorphic.

The following GL_1 -aspect of the above conjecture can be made:

Conjecture 5. Suppose F_1, F_2 are two totally real number fields of degree r over \mathbb{Q} are such that the unit groups U_{F_1} and U_{F_2} are spectrally

commensurable as lattices in the natural embedding into \mathbb{R}^{r-1} . Then are F_1 and F_2 arithmetically equivalent, i.e., do they have the same Dedekind zeta function?

It is known (see [17]) that if two number fields are arithmetically equivalent then the unit groups have the same rank. The methods of [21] will also directly show that the lattices are commensurable with respect to the natural embeddings, since it is known that arithmetically equivalent number fields arise from the Gassmann construction. But for the converse direction, we need to restrict our attention to totally real number fields. It is for this reason that we have restricted our attention mostly to semisimple groups, although it is quite natural to frame the conjectures in the wider context of reductive algebraic groups.

8.2. From arithmetic to spectrum

We now consider the possible definitions of arithmetic not only in the context of higher dimensional Shimura varieties but also for more general locally symmetric spaces arising from congruent arithmetic lattices.

Let G be a semisimple algebraic group over a number field F , and K be a compact open subgroup of $G(\mathbf{A}_{F,f})$. One possible definition for the arithmetic associated to $(G(\mathbf{A}_F), K)$ is to consider the cuspidal L -spectrum attached to it. A natural question is the following:

Conjecture 6. Let G_1, G_2 be semisimple algebraic groups over number field F_1 and F_2 respectively. Let $K_i \subset G_i(\mathbf{A}_{F_i,f})$, $i = 1, 2$ be compact open subgroups, and Γ_{K_i} be the corresponding arithmetic lattices in $G_{i,\infty}$. Suppose that the adèle groups $G_1(\mathbf{A})$ and $G_2(\mathbf{A})$ are isomorphic, and that the L -cuspidal spectrums of $(G_1(\mathbf{A}_F), K_1)$ and $(G_2(\mathbf{A}_F), K_2)$ are equal.

Then Γ_{K_1} and Γ_{K_2} are L -equivalent lattices in $G_{1,\infty} \simeq G_{2,\infty}$.

This conjecture seems obvious when $F_1 = F_2$, and G_1 and G_2 are isomorphic. However Lubotzky, Vishne and Samuels [16] have constructed non-trivial examples satisfying the hypothesis of the above conjecture, concluding thereby that there exists non-commensurable arithmetic lattices in $PGL_d(\mathbb{R})$ which are representation equivalent. The above conjecture should yield to suitable generalizations of Jacquet–Langlands type comparison between automorphic representations on inner forms of algebraic groups.

We now try to define a suitable notion of arithmetic for general groups not giving raise to hermitian symmetric spaces, which closely reflects the customary meaning of arithmetic of Shimura varieties as discussed in Section 7.

We recall that an automorphic representation π of $G(\mathbf{A}_F)$ is cohomological if its infinity type π_∞ is a cohomological representation of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$. For example, the discrete series representations π_k^\pm , $k \geq 2$ are representations of cohomological type for $SL(2, \mathbb{R})$ (together with the trivial representation and the Steinberg representation they constitute all the representations of $SL(2, \mathbb{R})$ with cohomology).

It follows from the congruence relation of Eichler–Shimura and the work of Ihara, Langlands and others ([20], [5]), that in case of Shimura varieties the automorphic representations that describe the Hasse–Weil zeta functions of natural local systems on such spaces, are cohomological.

Definition 8.1. The *cohomological L -cuspidal spectrum* associated to $(G(\mathbf{A}_F), K)$ is the collection of L -packets of cohomological automorphic representations of $G(\mathbf{A}_F)$ counted with multiplicity $m([\pi], K)$.

It's this definition, that we would like to consider as the suitable candidate for the notion of arithmetic of the space $\Gamma_K \backslash G_\infty / K_\infty$. However there are a couple of problems with this definition: one, is that computations done for general groups seem to indicate a sparsity of cohomological representations; the other is that we will have to consider non-unitary cohomological representations also in the above definition.

We now generalize the conjecture that the arithmetic of Shimura curves determines the spectrum to arbitrary locally symmetric spaces:

Conjecture 7. Let G_1, G_2 be semisimple algebraic groups defined over number fields F_1 and F_2 respectively. Let $K_i \subset G_i(\mathbf{A}_{F_i, f})$, $i = 1, 2$ be compact open subgroups, and Γ_{K_i} be the corresponding arithmetic lattices in $G_{i, \infty}$. Suppose that the adèle groups $G_1(\mathbf{A})$ and $G_2(\mathbf{A})$ are isomorphic, and that cohomological L -cuspidal spectrums of $(G_1(\mathbf{A}_F), K_1)$ and $(G_2(\mathbf{A}_F), K_2)$ are equal.

Then Γ_{K_1} and Γ_{K_2} are L -equivalent lattices in $G_{1, \infty} \simeq G_{2, \infty}$.

A particular consequence of Conjectures 4 and 7 is that the cuspidal cohomological L -spectrum should determine the full cuspidal L -spectrum.

Remark 10. If we consider the example of $SL(1, D)$, where D is an indefinite quaternion division algebra over a totally real number field, the arithmetic is concentrated on those automorphic representations whose archimedean component is a discrete series. The spectrum on the other hand gives information about those automorphic representations with archimedean component an unramified principal series representation.

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