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Infinitesimal Bishop–Gromov condition for Alexandrov spaces

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Abstract.

We prove the infinitesimal version of Bishop–Gromov volume comparison condition for Alexandrov spaces.

§1. Introduction

We first present the definition of the infinitesimal Bishop–Gromov volume comparison condition for Alexandrov spaces.

For a real number κ, we set

$$s_\kappa(r) := \begin{cases} \sin(\sqrt{\kappa}r)/\sqrt{\kappa} & \text{if } \kappa > 0, \\ r & \text{if } \kappa = 0, \\ \sinh(\sqrt{|\kappa|}r)/\sqrt{|\kappa|} & \text{if } \kappa < 0. \end{cases}$$

The function s_κ is the solution of the Jacobi equation $s_\kappa''(r) + \kappa s_\kappa(r) = 0$ with initial condition $s_\kappa(0) = 0$, $s_\kappa'(0) = 1$.

Let M be an Alexandrov space and set $r_p(x) := d(p, x)$ for $p, x \in M$, where d is the distance function. For $p \in M$ and $0 < t \leq 1$, we define a subset $W_{p,t} \subset M$ and a map $\Phi_{p,t} : W_{p,t} \to M$ as follows. We first set $\Phi_{p,t}(p) := p \in W_{p,t}$. A point x ($\neq p$) belongs to $W_{p,t}$ if and only if there exists $y \in M$ such that $x \in py$ and $r_p(x) : r_p(y) = t : 1$, where py is a minimal geodesic from p to y. Since a geodesic does not branch on an Alexandrov space, for a given point $x \in W_{p,t}$ such a point y is unique and we set $\Phi_{p,t}(x) := y$. The triangle comparison condition implies the

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local Lipschitz continuity of the map $\Phi_{p,t} : W_{p,t} \to M$. We call $\Phi_{p,t}$ the *radial expansion map.*

Let μ be a positive Radon measure with full support in M, and $n \geq 1$ a real number.

Infinitesimal Bishop–Gromov Condition $\mathrm{BG}(\kappa, n)$ **for** μ**:** For any $p \in M$ and $t \in (0,1]$, we have

$$d(\Phi_{p,t\,*}\mu)(x) \geq \frac{t\, s_\kappa(t\, r_p(x))^{n-1}}{s_\kappa(r_p(x))^{n-1}} d\mu(x)$$

for any $x \in M$ such that $r_p(x) < \pi/\sqrt{\kappa}$ if $\kappa > 0$, where $\Phi_{p,t\,*}\mu$ is the push-forward by $\Phi_{p,t}$ of μ.

For an n-dimensional complete Riemannian manifold, the Riemannian volume measure satisfies $\mathrm{BG}(\kappa, n)$ if and only if the Ricci curvature satisfies $\mathrm{Ric} \geq (n-1)\kappa$ (see Theorem 3.2 of [10] for the 'only if' part). We see some studies on similar (or same) conditions to $\mathrm{BG}(\kappa, n)$ in [2, 18, 6, 7, 15, 10, 21] etc. $\mathrm{BG}(\kappa, n)$ is sometimes called the Measure Contraction Property and is weaker than the curvature-dimension (or lower n-Ricci curvature) condition, $\mathrm{CD}((n-1)\kappa, n)$, introduced by Sturm [19, 20] and Lott–Villani [9] in terms of mass transportation. For a measure on an Alexandrov space, $\mathrm{BG}(\kappa, n)$ is equivalent to the $((n-1)\kappa, n)$-MCP introduced by Ohta [10]. In our paper [5, 8], we prove a splitting theorem under $\mathrm{BG}(0, N)$. For a survey of geometric analysis on Alexandrov spaces, we refer to [17]

The purpose of this paper is to prove the following

Theorem 1.1. *Let M be an n-dimensional Alexandrov space of curvature $\geq \kappa$. Then, the n-dimensional Hausdorff measure $\mathcal{H}^n$ on M satisfies the infinitesimal Bishop–Gromov condition* $\mathrm{BG}(\kappa, n)$.

Note that we claimed this theorem in Lemma 6.1 of [6], but the proof in [6] is insufficient. The theorem also completes the proof of Proposition 2.8 of [10].

For the proof of the theorem, we have the delicate problem that the topological boundary of the domain $W_{p,t}$ of the radial expansion $\Phi_{p,t}$ is not necessarily of $\mathcal{H}^n$-measure zero. In fact, we have an example of an Alexandrov space such that the cut-locus at a point is dense (see Remark 2.2), in which case the boundary of $W_{p,t}$ has positive $\mathcal{H}^n$-measure. This never happens for Riemannian manifolds. To solve this problem, we need some delicate discussion using the approximate differential of $\Phi_{p,t}$.

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§2. Preliminaries

2.1. Alexandrov spaces

In this paper, we mean by an Alexandrov space a complete locally compact geodesic space of curvature bounded below locally and of finite Hausdorff dimension. We refer to [1, 12, 4] for the basics for the geometry and analysis on Alexandrov spaces. Let M be an Alexandrov space of Hausdorff dimension n. Then, n coincides with the covering dimension of M which is a nonnegative integer. Take any point $p \in M$ and fix it. Denote by $\Sigma_p M$ the space of directions at p, and by $K_p M$ the tangent cone at p. $\Sigma_p M$ is an $(n-1)$-dimensional compact Alexandrov space of curvature ≥ 1 and $K_p M$ an n-dimensional Alexandrov space of curvature ≥ 0.

Definition 2.1 (Singular Point, δ-Singular Point)**.** A point $p \in M$ is called a *singular point of M* if $\Sigma_p M$ is not isometric to the unit sphere S^{n-1}. For $\delta > 0$, we say that a point $p \in M$ is *δ-singular* if $\mathcal{H}^{n-1}(\Sigma_p M) \leq \mathrm{vol}(S^{n-1}) - \delta$. Let us denote the set of singular points of M by S_M and the set of δ-singular points of M by S_δ.

We have $S_M = \bigcup_{\delta>0} S_\delta$. Since the map $M \ni p \mapsto \mathcal{H}^n(\Sigma_p M)$ is lower semi-continuous, the set S_δ of δ-singular points in M is a closed set.

Lemma 2.1 ([14])**.** *Let γ be a minimal geodesic joining two points p and q in M. Then, the space of directions, $\Sigma_x M$, at all interior points of γ, $x \in \gamma \setminus \{p, q\}$, are isometric to each other. In particular, any minimal geodesic joining two non-singular (resp. non-δ-singular) points is contained in the set of non-singular (resp. non-δ-singular) points (for any $\delta > 0$).*

The following shows the existence of differentiable and Riemannian structure on M.

Theorem 2.1. *For an n-dimensional Alexandrov space M, we have the following:*

(1) *There exists a number $\delta_n > 0$ depending only on n such that $M^* := M \setminus S_{\delta_n}$ is a manifold* ([1]) *and has a natural C^∞ differentiable structure* ([4])*.*
(2) *The Hausdorff dimension of S_M is $\leq n-1$* ([1, 12])*.*
(3) *We have a unique continuous Riemannian metric g on $M \setminus S_M \subset M^*$ such that the distance function induced from g coincides with the original one of M* ([12])*. The tangent space at each point in $M \setminus S_M$ is isometrically identified with the tangent cone* ([12])*. The volume measure on M^* induced from g coincides with the n-dimensional Hausdorff measure $\mathcal{H}^n$* ([12])*.*

Remark 2.1. In [4] we construct a C^∞ structure only on $M \setminus B(S_{\delta_n}, \epsilon)$, where $B(A, \epsilon)$ denotes the ϵ-neighborhood of A. However this is independent of ϵ and extends to M^*. The C^∞ structure is a refinement of the structures of [12, 11, 13] and is compatible with the DC structure of [13].

Note that the metric g is defined only on $M^* \setminus S_M$ and does not continuously extend to any other point of M.

Definition 2.2 (Cut-locus). Let $p \in M$ be a point. We say that a point $x \in M$ is a *cut point of* p if no minimal geodesic from p contains x as an interior point. Here we agree that p is not a cut point of p. The set of cut points of p is called the *cut-locus of* p and denoted by Cut_p.

Note that Cut_p is not necessarily a closed set. For the $W_{p,t}$ defined in §1, it follows that $\bigcup_{0<t<1} W_{p,t} = X \setminus \mathrm{Cut}_p$. The cut-locus Cut_p is a Borel subset and satisfies $\mathcal{H}^n(\mathrm{Cut}_p) = 0$ (Proposition 3.1 of [12]).

Remark 2.2. There is an example of a 2-dimensional Alexandrov space M such that S_M is dense in M (see [12]). For such an example, Cut_p for any $p \in M$ is also dense in M.

2.2. Approximate differential

Definition 2.3 (Density; cf. 2.9.12 in [3]). Let X be a metric space with a Borel measure μ. A subset $A \subset X$ *has density zero at a point* $x \in X$ if

$$\lim_{r\to 0} \frac{\mu(B(x,r)\cap A)}{\mu(B(x,r))} = 0.$$

Definition 2.4 (Approximate Differential; cf. 3.1.2 in [3]). Let $A \subset \mathbb{R}^m$ be a subset and $f : A \to \mathbb{R}^n$ a map. A linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ is called the *approximate differential of f at a point $x \in A$* if the approximate limit of

$$\frac{|\, f(y) - f(x) - L(y-x)\,|}{|y-x|}$$

is equal to zero as $y \to x$, i.e., for any $\delta > 0$, the set

$$\left\{\, y \in A \setminus \{x\} \;\middle|\; \frac{|\, f(y) - f(x) - L(y-x)\,|}{|y-x|} \geq \delta \,\right\}$$

has density zero at x, where we consider the Lebesgue (or equivalently m-dimensional Hausdorff) measure on $\mathbb{R}^m$ to measure the density. We say that f *is approximately differentiable at a point $x \in A$* if the approximate differential of f at x exists. Denote by '$\mathrm{ap}\, df_x$' the approximate differential of f at x. It is unique at each approximate differentiable point.

Let M and N be two differentiable manifolds and let $A \subset M$. We give a map $f : A \to N$ and a point $x \in A$. Take two charts (U, φ) and (V, ψ) around x and $f(x)$ respectively. The map f is said to be *approximately differentiable at x* if $\psi \circ f \circ \varphi^{-1}$ is approximately differentiable at $\varphi(x)$. If f is approximately differentiable at x, then the *approximate differential* '$\operatorname{ap} df_x$' of f at x is defined by

$$\operatorname{ap} df_x := (d\psi_{f(x)})^{-1} \circ \operatorname{ap} d(\psi \circ f \circ \varphi^{-1})_{\varphi(x)} \circ d\varphi_x : T_x M \to T_{f(x)} N.$$

The approximate differentiability of f at x and $\operatorname{ap} df_x$ are both independent of (U, φ) and (V, ψ).

§3. Proof of Theorem 1.1

Let M be an Alexandrov space of curvature $\geq \kappa$. We first investigate the exponential map on M. Denote by o_p the vertex of the tangent cone $K_p M$ at a point $p \in M$. We denote by $U_p \subset K_p M$ the *inside of the tangential cut-locus of* p, i.e., $v \in U_p$ if and only if there is a minimal geodesic $\gamma : [0, a] \to M$ from p with $a > 1$ such that $\gamma'(0) = v$, where $\gamma'(t)$ denotes the element of $K_{\gamma(t)} M$ tangent to $\gamma|_{[t, t+\epsilon)}$, $\epsilon > 0$, and whose distance from $o_{\gamma(t)} \in K_{\gamma(t)} M$ is equal to the speed of parameter of γ. Note that U_p is not necessarily an open set. Since the exponential map $\exp_p|_{U_p} : U_p \to M \setminus \mathrm{Cut}_p$ is a homeomorphism and since $W_{p,t} \cap \bar{B}(p, r)$ is compact for any $0 < t \leq 1$ and $r > 0$, the set

$$U_p = \bigcup_{0 < t \leq 1,\ r > 0} (\exp_p|_{U_p})^{-1}(W_{p,t} \cap \bar{B}(p, r))$$

is a Borel subset of $K_p M$.

Denote by $\Theta(t|a, b, \dots)$ a function of $t, a, b, \dots$ such that $\Theta(t|a, b, \dots) \to 0$ as $t \to 0$ for any fixed $a, b, \dots$. We use $\Theta(t|a, b, \dots)$ as Landau symbols.

Lemma 3.1. *For any $p \in M$, $r > 0$, and for any $\mathcal{H}^n$-measurable subset $A \subset B(o_p, r) \subset K_p M$, we have*

$$(1) \qquad |\,\mathcal{H}^n(\exp_p(A \cap U_p)) - \mathcal{H}^n(A)\,| \leq \Theta(r|p, n)\, r^n,$$

$$(2) \qquad \mathcal{H}^n(B(o_p, r) \setminus U_p) \leq \Theta(r|p, n)\, r^n.$$

Note that $\Theta(r|p, n)$ here is independent of A.

Proof. Let $p \in M$ and $r > 0$. By the triangle comparison condition, $\exp_p : U_p \cap B(o_p, r) \to M$ is Lipschitz continuous with Lipschitz constant $1 + \Theta(r|p)$. Therefore, for any $\mathcal{H}^n$-measurable $A \subset B(o_p, r)$,

$$\mathcal{H}^n(A) \geq (1 - \Theta(r|p,n))\,\mathcal{H}^n(\exp_p(A \cap U_p)),$$
$$\mathcal{H}^n(B(o_p, r) \setminus A) \geq (1 - \Theta(r|p,n))\,\mathcal{H}^n(B(p,r) \setminus \exp_p(A \cap U_p)).$$

According to Lemma 3.2 of [16], we have

$$\lim_{\rho \to 0} \frac{\mathcal{H}^n(B(p,\rho))}{\rho^n} = \mathcal{H}^n(B(o_p, 1)) = \frac{\mathcal{H}^n(B(o_p, r))}{r^n}.$$

Combining those three formulas we have the lemma. Q.E.D.

Let $p \in M$ and $0 < t \leq 1$. We restrict the domain of the radial expansion map $\Phi_{p,t} : W_{p,t} \to M$ to the subset

$$W'_{p,t} := W_{p,t} \setminus (\Phi_{p,t}^{-1}(\mathrm{Cut}_p) \cup S_{\delta_n}),$$

where S_{δ_n} is as in Theorem 2.1.

Lemma 3.2. *We have $\Phi_{p,t}(W'_{p,t}) = M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$ and the map $\Phi_{p,t}|_{W'_{p,t}} : W'_{p,t} \to M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$ is bijective. In particular, the sets $W'_{p,t}$ and $\Phi_{p,t}(W'_{p,t})$ are both contained in the C^∞ manifold $M^* = M \setminus S_{\delta_n}$ without boundary.*

Proof. Let us first prove $\Phi_{p,t}(W'_{p,t}) \subset M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$. It is clear that $\Phi_{p,t}(W'_{p,t}) \subset M \setminus \mathrm{Cut}_p$. To prove $\Phi_{p,t}(W'_{p,t}) \subset M \setminus S_{\delta_n}$, we take any point $x \in W'_{p,t}$. Since $\Phi_{p,t}(x)$ is not a cut point of p and by Lemma 2.1, $\Phi_{p,t}(x)$ is not δ_n-singular. Therefore, $\Phi_{p,t}(W'_{p,t}) \subset M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$.

Let us next prove $\Phi_{p,t}(W'_{p,t}) \supset M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$. Take any point $y \in M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$ and join p to y by a minimal geodesic $\gamma : [0,1] \to M$. Then, $\Phi_{p,t}(\gamma(t)) = y$. Since $y \notin \mathrm{Cut}_p$, the geodesic γ is unique and so $\Phi_{p,t}|_{W'_{p,t}}$ is injective. By Lemma 2.1, $\gamma(t) = (\Phi_{p,t}|_{W'_{p,t}})^{-1}(y)$ is not δ_n-singular and belongs to $W'_{p,t}$. This completes the proof. Q.E.D.

By the local Lipschitz continuity of $\Phi_{p,t}$ and by 3.1.8 of [3], $\Phi_{p,t}|_{W'_{p,t}}$ is approximately differentiable $\mathcal{H}^n$-a.e. on $W'_{p,t}$. The following lemma is essential for the proof of Theorem 1.1.

Lemma 3.3. *Let $p \in M$ and $0 < t < 1$. Then, the approximate Jacobian determinant of $\Phi_{p,t}|_{W'_{p,t}}$ satisfies that*

$$|\det \mathrm{ap}\, d(\Phi_{p,t}|_{W'_{p,t}})_x| \leq \frac{s_\kappa(r_p(x)/t)^{n-1}}{t\, s_\kappa(r_p(x))^{n-1}}$$

for any approximately differentiable point $x \in W'_{p,t} \setminus S_M$ of $\Phi_{p,t}|_{W'_{p,t}}$.

Proof. Let $x \in W'_{p,t} \setminus S_M$ be an approximately differentiable point of $\Phi_{p,t}|_{W'_{p,t}}$ and let $\epsilon > 0$ be a small number. Note that K_xM and $K_{\Phi_{p,t}(x)}M$ are both isometric to $\mathbb{R}^n$ and identified with the tangent spaces. We take two charts (U, φ) and (V, ψ) of $M \setminus S_{\delta_n}$ around x and $\Phi_{p,t}(x)$ respectively such that $|\, |\varphi(y) - \varphi(z)|/d(y,z) - 1\,| < \epsilon$ for any different $y, z \in U$ and ψ satisfies the same inequality on V. In particular, every eigenvalue of the differentials $d\varphi_x : K_xM \to \mathbb{R}^n$ and $d\psi_{\Phi_{p,t}(x)} : K_{\Phi_{p,t}(x)}M \to \mathbb{R}^n$ is between $1 - \epsilon$ and $1 + \epsilon$. Put

$$\begin{aligned} \bar{\Phi} &:= \psi \circ \Phi_{p,t}|_{W'_{p,t}} \circ \varphi^{-1} : \varphi(W'_{p,t} \cap U) \to \psi(V), \\ \bar{x} &:= \varphi(x), \qquad L := \operatorname{ap} d\bar{\Phi}_{\bar{x}} : \mathbb{R}^n \to \mathbb{R}^n. \end{aligned}$$

For simplicity we set $D := \operatorname{ap} d(\Phi_{p,t}|_{W'_{p,t}})_x : K_xM \to K_{\Phi_{p,t}(x)}M$. Then,

$$D = (d\psi_{\Phi_{p,t}(x)})^{-1} \circ L \circ d\varphi_x.$$

By the definition of the approximate differential, for any $r > 0$ with $B(x,r) \subset U$, the set of $\bar{y} \in B(\bar{x}, r)$ satisfying

$$|\,\bar{\Phi}(\bar{y}) - \bar{\Phi}(\bar{x}) - L(\bar{y} - \bar{x})\,| \geq \epsilon\, |\,\bar{x} - \bar{y}\,|$$

has $\mathcal{H}^n$-measure $\leq \Theta(r|\bar{\Phi}, \bar{x})\, \mathcal{H}^n(B(\bar{x}, r))$, where $B(\bar{x}, r)$ is a Euclidean metric ball. Take any $u \in \Sigma_xM$ and fix it. Let $r > 0$ be any number. We set

$$C(u, r, \epsilon) := \{\, v \in B(o_x, r) \setminus \{o_x\} \subset K_xM \mid \angle(u, v) < \epsilon \,\}.$$

It follows from Lemma 3.1(1) that

$$\begin{aligned} &\mathcal{H}^n(\varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x))) \\ &\geq (1 - \epsilon)^n\, \mathcal{H}^n(\exp_x(C(u, r/2, \epsilon) \cap U_x)) \\ &\geq (1 - \epsilon)^n\, (\mathcal{H}^n(C(u, 1/2, \epsilon)) - \Theta(r|x, n))\, r^n. \end{aligned}$$

Since $\mathcal{H}^n(C(u, 1/2, \epsilon))$ is positive, we have

$$\lim_{r \to 0} \frac{\mathcal{H}^n(\varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x)))}{\mathcal{H}^n(B(\bar{x}, r))} > 0.$$

Note that $\varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x))$ is contained in $B(\bar{x}, r)$ because ϵ is small enough. Therefore, supposing $r \ll \epsilon$, there is a point $\bar{y} \in B(\bar{x}, r)$ such that

$$\begin{aligned} &\bar{y} \in \varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x)), \\ &|\,\bar{\Phi}(\bar{y}) - \bar{\Phi}(\bar{x}) - L(\bar{y} - \bar{x})\,| < \epsilon\, d(\bar{x}, \bar{y}). \end{aligned}$$

Setting $y := \varphi^{-1}(\bar{y})$ and $v_{xy} := (\exp_x|_{U_x})^{-1}(y)$, we have $\angle(u, v_{xy}) < \epsilon$. For simplicity we write $a \le (1+\Theta(\epsilon|p,t,x))\, b + \Theta(\epsilon|p,t,x)$ by $a \lesssim b$. Note that since $r \ll \epsilon$, all $\Theta(r|\cdots)$ become $\Theta(\epsilon|\cdots)$. Since $|v_{xy}| = d(x,y)$ and $|d\varphi_x(v_{xy}) - (\bar{y} - \bar{x})| \le \Theta(\epsilon|x)\, d(x,y)$ (cf. Lemma 3.6(2) of [12]), we have

$$\begin{aligned}|D(u)| \lesssim |D(v_{xy}/|v_{xy}|)| &\lesssim \frac{|L(\bar{y}-\bar{x})|}{d(x,y)} \\ &\lesssim \frac{|\bar{\Phi}(\bar{y}) - \bar{\Phi}(\bar{x})|}{d(x,y)} \lesssim \frac{d(\Phi_{p,t}(x), \Phi_{p,t}(y))}{d(x,y)}.\end{aligned}$$

We are going to estimate the last formula. Denote by $M^2(\kappa)$ a complete simply connected 2-dimensional space form of curvature κ. We take three points $\tilde{p}, \tilde{x}, \tilde{y} \in M^2(\kappa)$ such that $d(\tilde{p}, \tilde{x}) = d(p,x)$, $d(\tilde{p}, \tilde{y}) = d(p,y)$, and $d(\tilde{x}, \tilde{y}) = d(x,y)$. The triangle comparison condition tells that $d(\Phi_{p,t}(x), \Phi_{p,t}(y)) \le d(\Phi_{\tilde{p},t}(\tilde{x}), \Phi_{\tilde{p},t}(\tilde{y}))$, where $\Phi_{\tilde{p},t}$ is the radial expansion on $M^2(\kappa)$. Since $d(\tilde{x}, \tilde{y}) = d(x,y) < r \ll \epsilon$, we have

$$\frac{d(\Phi_{\tilde{p},t}(\tilde{x}), \Phi_{\tilde{p},t}(\tilde{y}))}{d(\tilde{x}, \tilde{y})} \lesssim |d(\Phi_{\tilde{p},t})_{\tilde{x}}(v_{\tilde{x}\tilde{y}}/|v_{\tilde{x}\tilde{y}}|)|.$$

Let $\tilde{\gamma}$ be the minimal geodesic from $\tilde{p}$ passing through $\tilde{x}$. We denote by $\tilde{\theta}$ the angle between $v_{\tilde{x}\tilde{y}}$ and $\tilde{\gamma}'(t_{\tilde{x}})$, where $t_{\tilde{x}}$ is taken in such a way that $\tilde{\gamma}(t_{\tilde{x}}) = \tilde{x}$. Set

$$\lambda(\xi) := \sqrt{\frac{1}{t^2}\cos^2\xi + \frac{s_\kappa(r_p(x)/t)^2}{s_\kappa(r_p(x))^2}\sin^2\xi}, \qquad \xi \in \mathbb{R}.$$

A calculation using Jacobi fields yields $|d(\Phi_{\tilde{p},t})_{\tilde{x}}(v_{\tilde{x}\tilde{y}}/|v_{\tilde{x}\tilde{y}}|)| = \lambda(\tilde{\theta})$. Combining the above estimates, we have

$$|D(u)| \lesssim \lambda(\tilde{\theta}).$$

Let γ be the minimal geodesic from p passing through x and let t_x be a number such that $\gamma(t_x) = x$. Denote by θ the angle between v_{xy} and $\gamma'(t_x)$ and by θ_u the angle between u and $\gamma'(t_x)$. It follows from $\angle(u, v_{xy}) < \epsilon$ that $|\theta - \theta_u| < \epsilon$. By 5.6 of [1] we have $|\,\theta - \tilde{\theta}\,| \le \Theta(r|p,t,x) \le \Theta(\epsilon|p,t,x)$. Therefore we have $|D(u)| \lesssim \lambda(\theta_u)$. Taking the limit as $\epsilon \to 0$ yields that

$$|D(u)| \le \lambda(\theta_u)$$

for any $u \in \Sigma_x M$, which together with Hadamard's inequality implies

$$|\det D| \le \lambda(0)\, \lambda(\pi/2)^{n-1} = \frac{s_\kappa(r_p(x)/t)^{n-1}}{t\, s_\kappa(r_p(x))^{n-1}}.$$

This completes the proof of Lemma 3.3. Q.E.D.

Proof of Theorem 1.1. For the proof, it suffices to prove that

$$(3.1) \qquad \int_{W_{p,t}} f \circ \Phi_{p,t}(x)\, d\mathcal{H}^n(x) \geq \int_M f(y)\, \frac{t\, s_\kappa(t\, r_p(y))^{n-1}}{s_\kappa(r_p(y))^{n-1}}\, d\mathcal{H}^n(y)$$

for any $\mathcal{H}^n$-measurable function $f : M \to [\,0, +\infty\,)$ with compact support. Since $\Phi_{p,t}|_{W'_{p,t}} : W'_{p,t} \to M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})$ is bijective, the area formula (cf. 3.2.20 of [3]) implies that

$$(3.2) \qquad \begin{aligned} &\int_{W'_{p,t}} F \circ \Phi_{p,t}(x)\, |\det \mathrm{ap}\, d(\Phi_{p,t}|_{W'_{p,t}})_x|\, d\mathcal{H}^n(x) \\ &= \int_{M \setminus (\mathrm{Cut}_p \cup S_{\delta_n})} F(y)\, d\mathcal{H}^n(y) \end{aligned}$$

for any $\mathcal{H}^n$-measurable function $F : M \to [\,0, +\infty\,)$ with compact support. We set

$$F(y) := f(y)\, \frac{t\, s_\kappa(t\, r_p(y))^{n-1}}{s_\kappa(r_p(y))^{n-1}}, \quad y \in M \setminus \mathrm{Cut}_p,$$

in (3.2). Then, since $\mathcal{H}^n(\mathrm{Cut}_p) = \mathcal{H}^n(S_{\delta_n}) = 0$ and by Lemma 3.3, we obtain (3.1). This completes the proof of the theorem. Q.E.D.

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