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Couplings of the Brownian motion via discrete approximation under lower Ricci curvature bounds

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Abstract.

Along an idea of von Renesse, couplings of the Brownian motion on a Riemannian manifold and their extensions are studied. We construct couplings as a limit of coupled geodesic random walks whose components approximate the Brownian motion respectively. We recover Kendall and Cranston's result under lower Ricci curvature bounds instead of sectional curvature bounds imposed by von Renesse. Our method provides applications of coupling methods on spaces admitting a sort of singularity.

§1. Introduction

In stochastic analysis on Riemannian manifolds, coupling methods are effectively used for deriving several analytic estimates from geometric conditions of the space. Among them, a coupling of Brownian motions by reflection given by Kendall [8] was used by Cranston [6] to derive L^∞-gradient estimates for harmonic functions. von Renesse [19] attempted to extend their argument on more singular spaces. By a technical reason as explained below, his method needs a stronger assumption on the underlying space even when it is a Riemannian manifold. In this paper, we consider applications of coupling methods along the idea of von Renesse.

Let X be a complete Riemannian manifold and $(W(t), \mathbb{P}_x)$ the Brownian motion on X. Let $Z(t) = (Z_1(t), Z_2(t))$ be a coupling of the Brownian motion starting from $(x_1, x_2) \in X \times X$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. That is, the law of Z_i equals $\mathbb{P}_{x_i} \circ W^{-1}$ for $i = 1, 2$. Let us define a coupling time τ by $\tau := \inf\{t \geq 0\ ;\ Z_1(s) = Z_2(s) \text{ for all } s \geq t\}$.

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We are interested in constructing a coupling Z which provides a useful upper estimate of $\mathbb{P}[\tau > t]$. When Z is a coupling by reflection in [6], it enjoys a domination of the distance $d(Z(t))$ between $Z_1(t)$ and $Z_2(t)$ by a semimartingale. Since the bounded variation part of the semimartingale is dominated in terms of the Ricci curvature of X, a lower bound of the Ricci curvature yields an estimate of $\mathbb{P}[\tau > t]$ by the hitting probability to 0 of the semimartingale. In [6, 8], they considered an SDE on $M \times M$ to construct such a coupling. Unfortunately its coefficient becomes singular if $Z_2(t)$ is in the cut locus of $Z_1(t)$ and we somehow need to overcome this difficulty (see [20] for another approach than [6, 8]). von Renesse's approach is to consider a sequence of couplings $\{Z^\varepsilon(n)\}_{\varepsilon>0}$ of geodesic random walks which approximate the Brownian motion. He constructed a coupling by reflection by taking a scaling limit. As a result, he does not only succeed in avoiding the technical obstruction, but also in extending Cranston's coupling method on more singular spaces where we no longer use SDE theory directly. But, in his argument, a domination of $d(Z(t))$ comes from the corresponding estimate for $d(Z^\varepsilon(n))$. To obtain a suitable estimate, it seems to need a convergence of the dominant. For this purpose, he assumed a lower sectional curvature bound instead of the corresponding Ricci curvature bound (see remark 7 for details). As we will see in section 3, in fact, his stronger assumption is not necessary. We will show that the event of the suitable domination of $d(Z(t))$ occurs with an arbitrary high probability, instead of showing the convergence of dominants. Actually, we are interested in the probability of the event $\{\tau > t\}$, not in a domination of $d(Z(t))$ itself.

The organization of this paper is as follows. In the next section, we construct couplings via approximating geodesic random walks. Some basic estimates are also gathered there. The estimate of $\mathbb{P}[\tau > t]$ for a coupling by reflection is given in section 3 under a nonpositive Ricci curvature bound. In sections 4 and 5, we will derive two well-known applications of coupling methods from our approach. First we show the Lichnerowicz bound for the first nonzero eigenvalue of the Laplacian on a positively curved space. As the second application, we use a so-called synchronous coupling constructed in section 2 to derive a gradient estimate of the heat semigroup under a lower Ricci curvature bound. All these applications are extended in section 6 on spaces admitting a sort of singularities. Here we allow the underlying space to have mutually isolated singular points where the space of directions has a degenerate diameter. Recently, geometric analysis of metric spaces having a sort of lower Ricci curvature bounds are extensively studied (see [18] and the references therein). Our spaces are included in such a class.

Before closing this section, we mention that there are some related works in analysis on singular spaces. For the Lichnerowicz bound, Shioya [14] derived a stronger result when X is an orbifold. Ohta [12] derived the contraction estimate of the heat distribution with respect to the Wasserstein metric on a nonnegatively curved Alexandrov space. It is closely related to the gradient estimate of the heat semigroup (see [17]). It should be remarked that our methods are different from either of theirs. Though our spaces have a strong restriction, some of our results, for example the consequence of an estimate of the coupling time (Theorem 16 (i)), are not treated in their work.

§2. Preliminaries

2.1. Construction of coupling

As in section 1, let X be a complete Riemannian manifold and $(\{W(t)\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in X})$ the Brownian motion on X. Here, the Brownian motion stands for the diffusion process associated with $\Delta/2$, where Δ is the Laplacian on X. Set $m := \dim X$. We always assume $m \geq 2$ and that X has no boundary.

Let $D(X) := \{(x,x) \in X \times X \; ; \; x \in X\}$. For each $x, y \in X$ with $(x,y) \notin D(X)$, we choose a minimal geodesic $\tilde{\gamma}_{xy} : [0,1] \to X$ of constant speed with $\tilde{\gamma}_{xy}(0) = x$ and $\tilde{\gamma}_{xy}(1) = y$ in a symmetric way, i.e.$\tilde{\gamma}_{xy}(t) = \tilde{\gamma}_{yx}(1-t)$. For a smooth curve γ in X from x to y, we denote the parallel transport along γ with respect to the Levi–Civita connection by $/\!/_{\gamma} : T_xX \to T_yX$. Let us define $\tilde{m}_{xy} : T_yX \to T_yX$ by $\tilde{m}_{xy}v := v - 2\langle v, \dot{\tilde{\gamma}}_{xy}(1)\rangle \dot{\tilde{\gamma}}_{xy}(1)$. This is a reflection with respect to a hyperplane which is perpendicular to $\dot{\tilde{\gamma}}_{xy}(1)$. Set $m_{xy} := \tilde{m}_{xy} \circ /\!/_{\tilde{\gamma}_{xy}}$. Clearly m_{xy} is an isometry. Take a measurable section $\Phi : X \to \mathscr{O}(X)$ of the orthonormal frame bundle $\mathscr{O}(X)$ of X. Let us define maps $\Phi_i : X \times X \to \mathscr{O}(X)$ for $i = 1, 2$ by

$$\Phi_1(x,y) := \Phi(x),$$

$$\Phi_2(x,y) := \begin{cases} m_{xy}\Phi_1(x,y), & (x,y) \in X \times X \setminus D(X), \\ \Phi(x), & (x,y) \in D(X). \end{cases}$$

Here we have extended m_{xy} to a map from $\mathscr{O}_x(X)$ to $\mathscr{O}_y(X)$. Note that we can choose $\tilde{\gamma}_{xy}$ in a measurable way in an appropriate sense (see [19]). Thus we may assume Φ_2 to be measurable without loss of generality. Let us denote the unit-speed geodesic obtained from a re-parametrization of $\tilde{\gamma}_{xy}$ by $\gamma_{xy} : [0, d(x,y)] \to X$.

Take a sequence of independent, identically distributed random variables $\{\xi_n\}_{n\in\mathbb{N}}$ where ξ_1 is uniformly distributed on the unit disk in $\mathbb{R}^m$.

Let us define a continuously-interpolated coupled geodesic random walk $Z^\varepsilon(t) = (Z_1^\varepsilon(t), Z_2^\varepsilon(t))$ on $X \times X$ with a step size $\varepsilon > 0$ and a starting point $(x_1, x_2) \in X \times X$ inductively by $Z^\varepsilon(0) := (x_1, x_2)$ and for $t > 0$

$$\tag{2.1} \begin{cases} Z_1^\varepsilon(t) & := \exp_{Z_1^\varepsilon(\lfloor t \rfloor)} \left((t - \lfloor t \rfloor) \left(\varepsilon\sqrt{m+2}\Phi_1(Z^\varepsilon(\lfloor t \rfloor))\xi_{\lfloor t \rfloor + 1} \right) \right), \\ Z_2^\varepsilon(t) & := \exp_{Z_2^\varepsilon(\lfloor t \rfloor)} \left((t - \lfloor t \rfloor) \left(\varepsilon\sqrt{m+2}\Phi_2(Z^\varepsilon(\lfloor t \rfloor))\xi_{\lfloor t \rfloor + 1} \right) \right), \end{cases}$$

where $\lfloor t \rfloor := \sup \{ n \in \mathbb{N} \cup \{0\} \ ; \ n < t \}$. Note that our choice of ξ_n is a bit different from that in [19], where ξ_n is uniformly distributed on the unit sphere. However, the same argument still works. Set $\bar{Z}^\varepsilon(t) := Z^\varepsilon(\varepsilon^{-2}t)$. As in [19], we can show that $\{\bar{Z}^\varepsilon\}_{\varepsilon>0}$ is tight in the joint path space $C([0,\infty) \to X \times X)$ and each $\bar{Z}_i^\varepsilon$ weakly converges to the Brownian motion as $\varepsilon \to 0$. Let us denote a (subsequential) limit of $\{\bar{Z}^\varepsilon\}_{\varepsilon>0}$ by $\bar{Z}(t) = (\bar{Z}_1(t), \bar{Z}_2(t))$. Let τ be the first hitting time of $\bar{Z}$ to $D(X)$. We define $Z(t)$ by

$$\tag{2.2} Z(t) := \begin{cases} \bar{Z}(t) & \text{if } t < \tau, \\ (\bar{Z}_1(t), \bar{Z}_1(t)) & \text{if } t \geq \tau. \end{cases}$$

We call $Z(t)$ a coupling by reflection, or a Kendall–Cranston coupling. This is indeed a coupling of two Brownian motions starting at x_1 and x_2 respectively. Intuitively speaking, an infinitesimal motion $dZ_2(t)$ of $Z_2(t)$ is determined by a reflection $m_{Z_1(t)Z_2(t)}$ of $dZ_1(t)$ until they meet.

Even when we replace $\tilde{m}_{xy}$ with an identity map in the definition of m_{xy}, the same construction still works. In this case, we denote the coupling of geodesic random walks by $Y^\varepsilon(t) = (Y_1^\varepsilon(t), Y_2^\varepsilon(t))$. As above, we denote time-scaled random walks $Y^\varepsilon(\varepsilon^{-2}t)$ by $\bar{Y}^\varepsilon(t) = (\bar{Y}_1^\varepsilon(t), \bar{Y}_2^\varepsilon(t))$ and its subsequential limit by $\bar{Y} = (\bar{Y}_1, \bar{Y}_2)$. Finally we define a coupling $Y(t) = (Y_1(t), Y_2(t))$ from $\bar{Y}$ in the same way as (2.2). Intuitively speaking, an infinitesimal motion $dY_2(t)$ is parallel to $dY_1(t)$. We call it a coupling by parallel transport, or a synchronous coupling.

Throughout this paper, we use the term "$\varepsilon \to 0$" as a fixed subsequential limit so that $\bar{Z}^\varepsilon$ and $\bar{Y}^\varepsilon$ converge.

2.2. Variations of coupled geodesic random walks

From now to the end of section 5, we assume that the Ricci curvature $\mathrm{Ric}(\cdot,\cdot)$ of X is bounded from below by $(m-1)k$ for $k \in \mathbb{R}$. When $k > 0$, we assume in addition that the diameter of X is strictly less than $\pi/\sqrt{k}$.

Remark 1. Our assumption on the diameter of X in the case $k > 0$ is not so restrictive. In fact, by the Bonnet–Myers theorem (see [4] for example), the diameter is dominated by $\pi/\sqrt{k}$ and the equality holds only when X is the sphere of the constant sectional curvature k. When

X is a sphere, we can choose k strictly less than its sectional curvature to apply our result. By taking an increasing limit of k to the sectional curvature, the sharp estimate follows since we can take such a limit without changing the choice of couplings.

Fix a reference point $o \in X$. Let us define a functional $\bar{\sigma}_R$ on $C([0,\infty) \to X)$ by $\bar{\sigma}_R(w) := \inf\{t \geq 0 \,;\, d(o, w(t)) \geq R\}$. For $w_1, w_2 \in C([0,\infty) \to X)$, set $\sigma_R(w_1, w_2) := \bar{\sigma}_R(w_1) \wedge \bar{\sigma}_R(w_2)$. We begin with the following auxiliary lemma, which allows us to localize the state space X.

Lemma 2. *For any $T > 0$,*

$$\varlimsup_{R\to\infty} \varlimsup_{\varepsilon\to 0} \mathbb{P}[\sigma_R(\bar{Z}^\varepsilon) < T] = \varlimsup_{R\to\infty} \varlimsup_{\varepsilon\to 0} \mathbb{P}[\sigma_R(\bar{Y}^\varepsilon) < T] = 0.$$

Proof. We only show the assertion for $\bar{Z}^\varepsilon$ since the other can be shown in the same way. By the definition of σ_R,

$$\mathbb{P}\left[\sigma_R(\bar{Z}^\varepsilon) \leq T\right] \leq \mathbb{P}\left[\bar{\sigma}_R(\bar{Z}_1^\varepsilon) \leq T\right] + \mathbb{P}\left[\bar{\sigma}_R(\bar{Z}_2^\varepsilon) \leq T\right]. \tag{2.3}$$

Note that $\{w \,;\, \bar{\sigma}_R(w) \leq T\}$ is closed in $C([0,\infty) \to X)$. As $\varepsilon \to 0$, $\bar{Z}_i^\varepsilon$ converges in law to the Brownian motion W starting from x_i for $i = 1, 2$. Thus

$$\varlimsup_{\varepsilon\to 0} \mathbb{P}\left[\bar{\sigma}_R(\bar{Z}_i^\varepsilon) \leq T\right] \leq \mathbb{P}_{x_i}\left[\bar{\sigma}_R(W) \leq T\right].$$

Since W is conservative under a lower Ricci curvature bound (see [15] for example), $\lim_{R\to\infty} \mathbb{P}_{x_i}\left[\bar{\sigma}_R(W) \leq T\right] = 0$ holds. By combining it with (2.3), the conclusion follows. Q.E.D.

Next we review a basic estimate for geodesic variations of arclength introduced in [19] (Lemma 3 below). It makes a basis of estimates for variations of coupled geodesic random walks. We define two functions c_k and s_k by

$$c_k(s) := \begin{cases} \cos(\sqrt{k}s) & (k > 0), \\ 1 & (k = 0), \\ \cosh(\sqrt{-k}s) & (k < 0), \end{cases} \quad s_k(s) := \begin{cases} \sin(\sqrt{k}s) & (k > 0), \\ s & (k = 0), \\ \sinh(\sqrt{-k}s) & (k < 0). \end{cases}$$

Let ∇ be the Levi–Civita connection and $\mathscr{R}$ the curvature tensor associated with ∇. For a smooth curve γ and smooth vector fields U, V along γ, the index form $I_\gamma(U, V)$ is given by

$$I_\gamma(U, V) := \int_\gamma \left(\langle \nabla_{\dot\gamma} U, \nabla_{\dot\gamma} V\rangle - \langle \mathscr{R}(U, \dot\gamma)\dot\gamma, V\rangle\right) ds.$$

For simplicity, we write $I_\gamma(U, U) =: I_\gamma(U)$.

Lemma 3 (Lemma 5 in [19]). *Let $y_1, y_2 \in X$. Take $\zeta_i \in T_{y_i}X$ with $|\zeta_i| \le C\varepsilon$ for $i = 1, 2$ for some constant $C > 0$. We write $\gamma := \gamma_{y_1 y_2}$. Take an orthonormal frame $\{e_i\}_{i=1}^m$ along γ with $e_1(s) = \dot\gamma(s)$ and $\nabla_{\dot\gamma} e_i = 0$. Set $\zeta_1^i := \langle \zeta_1, e_i(0)\rangle$ and $\zeta_2^i := \langle /\!\!/_\gamma^{-1}\zeta_2, e_i(0)\rangle$. Let us define a vector field $V^\perp_{\zeta_1,\zeta_2}$ along γ by*

$$V^\perp_{\zeta_1,\zeta_2}(s) := \sum_{i=2}^m \left(\zeta_1^i c_k(s) + \frac{\zeta_2^i - \zeta_1^i c_k(d(y_1,y_2))}{s_k(d(y_1,y_2))} s_k(s) \right) e_i(s).$$

Then,

$$d(\exp_{y_1}(\zeta_1), \exp_{y_2}(\zeta_2)) \le d(y_1,y_2) + \zeta_2^1 - \zeta_1^1 + \frac{1}{2} I_\gamma(V^\perp_{\zeta_1,\zeta_2}) + o(\varepsilon^2).$$

Moreover, the following uniformity holds for the error term: Take constants $0 < \delta < R$. Then we can control $o(\varepsilon^2)$ uniformly in $y_1, y_2 \in X$, $\zeta_i \in T_{y_i}X$ $(i = 1, 2)$, γ and $\{e_i\}_{i=1}^m$ as long as $d(o, y_1) \vee d(o, y_2) \le R$ and $d(y_1, y_2) \ge \delta$ hold.

Here we give only a brief sketch of the proof of Lemma 3. If y_2 is not a conjugate point of y_1, the conclusion follows from the second variational formula and the index lemma. The error term is given as smooth functions and hence we can control it locally uniformly. The assumption $d(y_1, y_2) \ge \delta$ is imposed for realizing a regular geodesic variation uniformly in small enough $\varepsilon > 0$. Even if y_2 is conjugate to y_1, we can apply the second variational formula by dividing $\gamma_{y_1 y_2}$ into small geodesic segments. Such a division may grow the error term, but the number of necessary division is uniformly bounded above under the conditions on y_1, y_2 since the injectivity radius is locally uniformly away from 0.

For $(y_1, y_2) = Z^\varepsilon(n-1)$ and $\zeta_i = \varepsilon\sqrt{m+2}\Phi_i(Z^\varepsilon(n-1))\xi_n$, we will apply Lemma 3. To describe it, we define λ_n and $\Lambda_n^{(Z^\varepsilon)}$ by

$$\begin{aligned} \lambda_n &:= \sqrt{m+2}\langle \Phi_1(Z^\varepsilon(n-1))\xi_n, \dot\gamma_{Z_1^\varepsilon(n-1)Z_2^\varepsilon(n-1)}(0)\rangle, \\ \Lambda_n^{(Z^\varepsilon)} &:= (m+2) I_{\gamma_{Z_1^\varepsilon(n-1)Z_2^\varepsilon(n-1)}} \left(V^\perp_{\Phi_1(Z^\varepsilon(n-1))\xi_n, \Phi_2(Z^\varepsilon(n-1))\xi_n} \right) \end{aligned} \tag{2.4}$$

when $Z^\varepsilon(n-1) \notin D(X)$. For a technical reason, we also define λ_n and $\Lambda_n^{(Z^\varepsilon)}$ when $Z^\varepsilon(n-1) \in D(X)$ by $\lambda_n := \sqrt{m+2}\langle \xi_n, v\rangle$ and $\Lambda_n^{(Z^\varepsilon)} := 0$, where $v \in \mathbb{R}^m$ is a fixed vector with $|v| = 1$. Note that $\{\lambda_n\}_{n\in\mathbb{N}}$ are independent, identically distributed random variables with $\mathbb{E}\lambda_1 = 0$ and $\mathrm{Var}(\lambda_1) = 1$. By using λ_n and $\Lambda_n^{(Z^\varepsilon)}$, we have

$$d(Z^\varepsilon(n)) \le d(Z^\varepsilon(n-1)) - 2\varepsilon\lambda_n + \frac{\varepsilon^2}{2}\Lambda_n^{(Z^\varepsilon)} + o(\varepsilon^2) \tag{2.5}$$

if $d(Z^\varepsilon(n-1)) > 0$. Moreover, given $0 < \delta < R$, $o(\varepsilon^2)$ is uniformly small as long as $d(Z^\varepsilon(n-1)) > \delta$ and $d(Z_1^\varepsilon(n-1), o) \vee d(Z_2^\varepsilon(n-1), o) < R$ hold. To apply the same argument to $Y^\varepsilon(n)$, we define $\Lambda_n^{(Y^\varepsilon)}$ by replacing $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$ in (2.4) with $Y^\varepsilon = (Y_1^\varepsilon, Y_2^\varepsilon)$. Then we obtain

$$d(Y^\varepsilon(n)) \le d(Y^\varepsilon(n-1)) + \frac{\varepsilon^2}{2}\Lambda_n^{(Y^\varepsilon)} + o(\varepsilon^2). \tag{2.6}$$

For later use, we introduce two additional difference inequalities such as (2.5) and (2.6). Set $f_t(u) := \exp\left((m-1)kt/2\right) s_k(u/2)$.

Lemma 4. *Take $R > \delta > 0$. (i) The following inequality holds:*

$$\begin{aligned} &f_0(d(Z^\varepsilon(n))) - f_0(d(Z^\varepsilon(n-1))) \\ &\qquad \le -2\varepsilon f_0'(d(Z^\varepsilon(n-1)))\lambda_n + 2\varepsilon^2 f_0''(d(Z^\varepsilon(n-1)))\lambda_n^2 \\ &\qquad\qquad + \frac{\varepsilon^2}{2} f_0'(d(Z^\varepsilon(n-1)))\Lambda_n^{(Z^\varepsilon)} + o(\varepsilon^2). \end{aligned}$$

Here $o(\varepsilon^2)$ is controlled uniformly in n as long as $d(Z^\varepsilon(n-1)) > \delta$ and $d(o, Z_1^\varepsilon(n-1)) \vee d(o, Z_2^\varepsilon(n-1)) < R$ hold.
(ii) The following inequality holds:

$$\begin{aligned} &f_{\varepsilon^2 n}(d(Y^\varepsilon(n))) - f_{\varepsilon^2(n-1)}(d(Y^\varepsilon(n-1))) \\ &\le \frac{\varepsilon^2}{2}\left(f'_{\varepsilon^2(n-1)}(d(Y^\varepsilon(n-1)))\Lambda_n^{(Y^\varepsilon)} + (m-1)k f_{\varepsilon^2(n-1)}(d(Y^\varepsilon(n-1)))\right) \\ &\qquad + o(\varepsilon^2). \end{aligned}$$

Here $o(\varepsilon^2)$ is controlled uniformly in n as long as $d(Y^\varepsilon(n-1)) > \delta$ and $d(o, Y_1^\varepsilon(n-1)) \vee d(o, Y_2^\varepsilon(n-1)) < R$ hold.

Proof. (i) From the first variational formula for arclength, we have

$$d(Z^\varepsilon(n)) - d(Z^\varepsilon(n-1)) = -2\varepsilon\lambda_n + o(\varepsilon). \tag{2.7}$$

Thus the Taylor expansion of f_0 up to second order yields

$$\begin{aligned} &f_0(d(Z^\varepsilon(n))) - f_0(d(Z^\varepsilon(n-1))) \\ &\quad = f_0'(Z^\varepsilon(n-1))\left(d(Z^\varepsilon(n)) - d(Z^\varepsilon(n-1))\right) \\ &\qquad + \frac{f_0''(d(Z^\varepsilon(n-1)))}{2}\left(d(Z^\varepsilon(n)) - d(Z^\varepsilon(n-1))\right)^2 + o(\varepsilon^2). \end{aligned} \tag{2.8}$$

We can replace $(d(Z^\varepsilon(n)) - d(Z^\varepsilon(n-1)))^2$ in (2.8) with $4\varepsilon^2\lambda_n^2$ by using (2.7) because λ_n is bounded. Since $f_0'(d(Z^\varepsilon(n-1))) \ge 0$, (2.8) together

with (2.5) yields the desired inequality. The uniformity follows from the remark after (2.5) and the uniformity in (2.7).
(ii) The first variational formula implies $d(Y^\varepsilon(n)) - d(Y^\varepsilon(n-1)) = o(\varepsilon)$. Thus, as we obtained (2.8),

$$\begin{aligned} f_{\varepsilon^2 n}&(d(Y^\varepsilon(n))) - f_{\varepsilon^2(n-1)}(d(Y^\varepsilon(n-1))) \\ &= \frac{(m-1)k\varepsilon^2}{2} f_{\varepsilon^2(n-1)}(d(Y^\varepsilon(n-1))) \\ &\quad + f'_{\varepsilon^2(n-1)}(d(Y^\varepsilon(n-1)))\,(d(Y^\varepsilon(n)) - d(Y^\varepsilon(n-1))) + o(\varepsilon^2). \end{aligned}$$

Then the conclusion follows from (2.6) as we did above. Q.E.D.

In order to control our couplings by the lower bound of the Ricci curvature, we show some properties of $\Lambda_n^{(Q)}$ ($Q = Z^\varepsilon$ or Y^ε). For $Q = Z^\varepsilon$ or Y^ε, we define $J_{(Q),n} : [0, d(Q(n-1))] \to \mathbb{R}$ by

$$J_{(Q),n}(s) := c_k(s) + \frac{1 - c_k(d(Q(n-1)))}{s_k(d(Q(n-1)))} s_k(s).$$

Note that $J_{(Q),n}$ satisfies $J_{(Q),n}(0) = J_{(Q),n}(d(Q(n-1))) = 1$ and $J''_{(Q),n}(s) + kJ_{(Q),n}(s) = 0$.

Lemma 5. *For $R > 0$ and $Q = Z^\varepsilon$ or Y^ε, there is a constant $K = K(R)$ such that $|\Lambda_n^{(Q)}| < K$ if $n < \sigma_R(Q)$.*

Proof. Take $K > 0$ so that the absolute value of a sectional curvature at $x \in X$ is bounded by K whenever $d(x, o) \le 2R$. By the definition of $\Lambda_n^{(Q)}$, if $n < \sigma_R(Q)$, then

$$|\Lambda_n^{(Q)}| \le (m+2)|\xi_n|^2 \int_0^{d(Q(n-1))} \left(|J'_{(Q),n}(s)|^2 + K J^2_{(Q),n}(s) \right) ds.$$

Note that $J'_{(Q),n}$ and $J_{(Q),n}$ are bounded on $[0, d(Q(n-1))]$ in terms of R. Since $|\xi_n| \le 1$, the conclusion follows. Q.E.D.

Set $\bar{\Lambda}_n^{(Q)} := \mathbb{E}[\Lambda_n^{(Q)} | Q(n-1)]$, where $Q = Z^\varepsilon$ or Y^ε. Combining the computation of $\bar{\Lambda}_n^{(Q)}$ with the Ricci curvature bound yields the following estimate:

$$\begin{aligned} \bar{\Lambda}_n^{(Q)} &\le (m-1) \int_0^{d(Q(n-1))} \left(J'_{(Q),n}(s)^2 - kJ_{(Q),n}(s)^2 \right) ds \\ &= -\frac{2(m-1)k}{\sqrt{|k|}} \cdot \frac{s_k\,(d(Q(n-1))/2)}{c_k\,(d(Q(n-1))/2)}. \end{aligned} \tag{2.9}$$

Let $\{\mathscr{F}_n\}_{n \in \mathbb{N}}$ be a filtration given by $\mathscr{F}_n := \sigma(\xi_1, \ldots, \xi_n)$.

Lemma 6. *Let $Q = Z^\varepsilon$ or Y^ε and $\{a_n\}_{n\in\mathbb{N}}$ an $\{\mathscr{F}_n\}$-predictable process, which may depend on ε. Assume*

$$C_{T,R} := \overline{\lim_{\varepsilon\to 0}} \sup_{0\le n\le \lfloor \varepsilon^{-2}T\rfloor \wedge (\lfloor \sigma_R(Q)\rfloor +1)} |a_n| < \infty$$

for $R, T > 0$. Set $M_n := \sum_{j=1}^n a_j(\Lambda_j^{(Q)} - \bar{\Lambda}_j^{(Q)})$. Then (i) $\{M_n\}_{n\in\mathbb{N}}$ is an $\{\mathscr{F}_n\}$-local martingale. (ii) For any $\delta > 0$ and $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left[\sup_{0\le n\le \lfloor \varepsilon^{-2}T\rfloor} |M_n| > \frac{\delta}{\varepsilon^2}\right] = 0.$$

Proof. (i) Set $\hat\sigma_R := \lfloor \sigma_R(Q)\rfloor + 1$. Then $\hat\sigma_R$ is an $\{\mathscr{F}_n\}$-Markov time. Lemma 5 implies that $M_{n\wedge\hat\sigma_R}$ has a bounded increment and hence it is integrable. The martingale property of $M_{n\wedge\hat\sigma_R}$ directly follows from the definition of M_n. Since $\lim_{R\to\infty}\sigma_R(Q) = \lim_{R\to\infty}\hat\sigma_R = \infty$, the conclusion follows.
(ii) By the Doob inequality and Lemma 5, there is a constant $K_0 > 0$ depending on R, $C_{T,R}$ and $K(R)$ in Lemma 5 such that

(2.10)
$$\mathbb{P}\left[\sup_{0\le n\le \lfloor \varepsilon^{-2}T\rfloor} |M_{n\wedge\hat\sigma_R}| > \frac{\delta}{\varepsilon^2}\right] \le \frac{4\varepsilon^4}{\delta^2}\mathbb{E}\left[\left|M_{\lfloor \varepsilon^{-2}T\rfloor\wedge\hat\sigma_R}\right|^2\right] \le \frac{\varepsilon^2 K_0 T}{\delta^2}.$$

Note that we have

$$\begin{aligned}(2.11)\quad \mathbb{P}\left[\sup_{0\le n\le \lfloor \varepsilon^{-2}T\rfloor} |M_n| > \frac{\delta}{\varepsilon^2}\right] &\le \mathbb{P}\left[\hat\sigma_R \le \lfloor \varepsilon^{-2}T\rfloor\right] \\ &\quad + \mathbb{P}\left[\sup_{0\le n\le \lfloor \varepsilon^{-2}T\rfloor} |M_{n\wedge\hat\sigma_R}| > \frac{\delta}{\varepsilon^2}\right].\end{aligned}$$

Set $\bar{Q}(t) := Q(\varepsilon^{-2}t)$. Since $\mathbb{P}\left[\hat\sigma_R \le \varepsilon^{-2}T\right] \le \mathbb{P}\left[\sigma_R(\bar{Q}) \le T\right]$ holds, Lemma 2 and (2.10) imply the assertion by letting $R\to\infty$ after $\varepsilon\to 0$ in (2.11). Q.E.D.

Remark 7. From (2.5), a domination of $d(\bar{Z}^\varepsilon(t))$ can be obtained as a sum of $\varepsilon\lambda_n$ and that of $\varepsilon^2\Lambda_n^{(Z^\varepsilon)}$ (see (3.3)). As $\varepsilon\to 0$, the former one enjoys the invariance principle. For a domination of the second, we can apply Lemma 6 for replacing $\Lambda_n^{(Z^\varepsilon)}$ with $\bar\Lambda_n^{(Z^\varepsilon)}$, which is estimated in (2.9). Thus, in order to derive a domination of $d(Z(t))$ as $\varepsilon\to 0$, it seems necessary to consider these two different limit theorems at the same time though λ_n and $\Lambda_n^{(Z^\varepsilon)}$ are *not independent.* To avoid this difficulty, von

Renesse [19] assumed lower sectional curvature bounds instead of Ricci curvature bounds. In such a case, $\Lambda_n^{(Z^\varepsilon)}$ itself can be dominated in terms of the sectional curvature bound.

§3. Non-successful probability

The goal in this section is to show the following theorem.

Theorem 8. *Assume $k \le 0$. Then we have*

$$\mathbb{P}[\tau > T] \le \left(\frac{1}{4\sqrt{2\pi T}} + \frac{(m-1)\sqrt{-k}}{2} \right) d(x_1, x_2). \tag{3.1}$$

The following is a well-known consequence of Theorem 8:

Corollary 9. *For any bounded measurable function ψ on X,*

$$\begin{aligned} \mathbb{E}_{x_1}[\psi(W(t))] - \mathbb{E}_{x_2}[\psi(W(t))] \\ \le \|\psi\|_\infty \left(\frac{1}{2\sqrt{2\pi t}} + (m-1)\sqrt{-k} \right) d(x_1, x_2). \end{aligned} \tag{3.2}$$

In particular, if ψ is harmonic, then $\|\nabla\psi\|_\infty \le (m-1)\sqrt{-k}\|\psi\|_\infty$.

Proof of Corollary 9. By the definition of $Z = (Z_1, Z_2)$ and τ,

$$\begin{aligned} \mathbb{E}_{x_1}[\psi(W(t))] - \mathbb{E}_{x_2}[\psi(W(t))] &= \mathbb{E}[\psi(Z_1(t)) - \psi(Z_2(t))] \\ &= \mathbb{E}[\psi(Z_1(t)) - \psi(Z_2(t)) \; ; \; \tau > t] \le 2\|\psi\|_\infty \mathbb{P}[\tau > t]. \end{aligned}$$

Thus Theorem 8 implies the conclusion. When ψ is harmonic, we have $\mathbb{E}_x[\psi(W_{t\wedge\sigma_R(W)}))] = \psi(x)$ for any $x \in X$ and $R > 0$ by the Itô formula. Since W is conservative and ψ is bounded, the dominated convergence theorem implies $\mathbb{E}_x[\psi(W_t)] = \lim_{R\to\infty} \mathbb{E}_x[\psi(W_{t\wedge\sigma_R(W)})] = \psi(x)$. Hence the desired result follows by letting $t \to \infty$ and $x_2 \to x_1$. Q.E.D.

Remark 10. When X is compact and ψ is an eigenfunction of Δ, the inequality (3.2) gives a quantitative uniform upper bound of its Lipschitz constant. This fact is used in section 6.

For the proof of Theorem 8, we define a functional τ_δ on a joint path space $C([0,\infty) \to X \times X)$ by

$$\tau_\delta(w_1, w_2) := \inf\{t \ge 0 \; ; \; d(w_1(t), w_2(t)) \le \delta\}.$$

Proof of Theorem 8. Take $R > \delta > 0$. By applying (2.5) iteratively,

$$(3.3)\quad d(\bar{Z}^\varepsilon(t)) \le d(x_1, x_2) - 2\varepsilon \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \lambda_i - 2\varepsilon \left(\varepsilon^{-2} t - \lfloor \varepsilon^{-2} t \rfloor\right) \lambda_{\lfloor \varepsilon^{-2} t \rfloor + 1}$$
$$+ \frac{\varepsilon^2}{2} \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \Lambda_i^{(Z^\varepsilon)} + \frac{\varepsilon^2}{2} \left(\varepsilon^{-2} t - \lfloor \varepsilon^{-2} t \rfloor\right) \Lambda_{\lfloor \varepsilon^{-2} t \rfloor + 1}^{(Z^\varepsilon)} + o(1)$$
$$=: d(x_1, x_2) - 2\varepsilon \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \lambda_i + \frac{\varepsilon^2}{2} \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \Lambda_i^{(Z^\varepsilon)} + \chi_t(\varepsilon)$$

for sufficiently small $\varepsilon > 0$ if $t < \sigma_R(\bar{Z}^\varepsilon) \wedge \tau_\delta(\bar{Z}^\varepsilon)$. The remark after (2.5) together with Lemma 5 and the boundedness of λ_n yields

$$(3.4)\qquad \lim_{\varepsilon \to 0} \mathbb{P}\left[|\chi_t(\varepsilon)| > \frac{\delta}{4} \text{ for some } t \le T \wedge \sigma_R(\bar{Z}^\varepsilon) \wedge \tau_\delta(\bar{Z}^\varepsilon) \right] = 0$$

for any $T > 0$. Let E_ε be an event that

$$d(\bar{Z}^\varepsilon(t)) \le d(x_1, x_2) - 2\varepsilon \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \lambda_i + \frac{\varepsilon^2}{2} \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \bar{\Lambda}_i^{(Z^\varepsilon)} + \frac{\delta}{2}$$

occurs for all $0 \le t \le T \wedge \sigma_R(\bar{Z}^\varepsilon) \wedge \tau_\delta(\bar{Z}^\varepsilon)$. Lemma 6 for $a_n \equiv 1$ and (3.4) imply that $\lim_{\varepsilon \to 0} \mathbb{P}[E_\varepsilon^c] = 0$. Note that $\bar{\Lambda}_n^{(Z^\varepsilon)} \le 2(m-1)\sqrt{-k}$ follows from (2.9). Hence, on E_ε,

$$(3.5)\qquad d(\bar{Z}^\varepsilon(t)) \le d(x_1, x_2) - 2\varepsilon \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} \lambda_i + \left((m-1)\sqrt{-k}\right) t + \frac{\delta}{2}$$

holds for each $0 \le t \le T \wedge \sigma_R(\bar{Z}^\varepsilon) \wedge \tau_\delta(\bar{Z}^\varepsilon)$. Let us denote the right hand side of (3.5) by $r^\varepsilon(t) + \delta/2$. Let us define $A_t \subset C([0,\infty) \to \mathbb{R})$ by $A_t := \{u \,;\, \inf_{0 \le s \le t} u(s) \ge \delta/2\}$. Then

$$(3.6)\qquad \begin{aligned} \varlimsup_{\varepsilon \to 0} \mathbb{P}\left[\tau_\delta(\bar{Z}^\varepsilon) > T\right] &= \varlimsup_{\varepsilon \to 0} \mathbb{P}\left[\left\{\tau_\delta(\bar{Z}^\varepsilon) > T\right\} \cap E_\varepsilon\right] \\ &\le \varlimsup_{\varepsilon \to 0} \mathbb{P}\left[\left\{r^\varepsilon \in A_{T \wedge \sigma_R(Z^\varepsilon)}\right\} \cap E_\varepsilon\right] \\ &= \varlimsup_{\varepsilon \to 0} \mathbb{P}\left[r^\varepsilon \in A_{T \wedge \sigma_R(Z^\varepsilon)}\right]. \end{aligned}$$

Note that $\mathbb{P}[r^\varepsilon \in A_{T \wedge \sigma_R(Z^\varepsilon)}] \le \mathbb{P}[\sigma_R(\bar{Z}^\varepsilon) \le T] + \mathbb{P}[r^\varepsilon \in A_T]$. Let β be the standard 1-dimensional Brownian motion and $r^0(t) := d(x_1, x_2) + 2\beta(t) + (m-1)\sqrt{-k}\, t$. The invariance principle asserts that r^ε converges in law to r^0 as $\varepsilon \to 0$. Thus $\overline{\lim}_{\varepsilon\to 0} \mathbb{P}[r^\varepsilon \in A_T] \le \mathbb{P}[r^0 \in A_T]$ holds since A_T is closed. By combining these observations with Lemma 2 and the fact that $\{w\ ;\ \tau_\delta(w) > T\}$ is open, $\mathbb{P}\left[\tau_\delta(\bar{Z}) > T\right] \le \mathbb{P}\left[r^0 \in A_T\right]$ follows as $R \to \infty$ in (3.6). By taking a limit $\delta \to 0$, we obtain

$$\mathbb{P}\left[\tau_0(\bar{Z}) > T\right] \le \overline{\lim_{\delta\to 0}} \mathbb{P}\left[\tau_\delta(\bar{Z}) > T\right] \le \mathbb{P}\left[\inf_{0\le t\le T} r^0(t) > 0\right].$$

Since $\tau_0(\bar{Z}) = \tau$, the conclusion follows from a computation of the right hand side of the above inequality (see [1] for example). Q.E.D.

§4. Eigenvalue estimate

Our goal in this section is to show the Lichnerowicz bound (Theorem 12 below) from our coupling method. We refer to [4] for the Lichenerowicz bound; see [5] also for a proof based on coupling methods. Note that (2.9) implies

$$f_t'(d(Q(n-1)))\bar{\Lambda}_n^{(Q)} \le -(m-1)kf_t(d(Q(n-1))), \tag{4.1}$$

where $Q = Z^\varepsilon$ or Y^ε.

Lemma 11. *Assume $k > 0$. Then we have*

$$\mathbb{E}\left[f_0\left(d(Z(t))\right)\right] \le f_0\left(d(x_1, x_2)\right) - \frac{mk}{2}\int_0^t \mathbb{E}\left[f_0\left(d(Z(s))\right)\right] ds.$$

Proof. Note that X is compact by the Bonnet–Myers theorem. Hence $\sigma_R(\bar{Z}^\varepsilon) = \infty$ holds for sufficiently large R. Fix such an $R > 0$. Set $\hat{\tau}_\delta := \lfloor \tau_\delta(Z^\varepsilon)\rfloor + 1$. Then $\hat{\tau}_\delta$ becomes an $\{\mathscr{F}_n\}$-Markov time. As we did in (3.3), an iteration of Lemma 4 (i) yields

$$\begin{aligned} f_0(d(\bar{Z}^\varepsilon(t \wedge (\varepsilon^2\hat{\tau}_\delta)))) &\le f_0(d(x_1, x_2)) \\ &+ \sum_{n=1}^{\lfloor \varepsilon^{-2}t\rfloor \wedge \hat{\tau}_\delta} -2\varepsilon f_0'(d(Z^\varepsilon(n-1)))\lambda_n + 2\varepsilon^2 f_0''(d(Z^\varepsilon(n-1)))\lambda_n^2 \\ &\qquad + \frac{\varepsilon^2}{2}\sum_{n=1}^{\lfloor \varepsilon^{-2}t\rfloor \wedge \hat{\tau}_\delta} f_0'(d(Z^\varepsilon(n-1)))\Lambda_n^{(Z^\varepsilon)} + o(1). \end{aligned} \tag{4.2}$$

By using the martingale property, for any $N \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\left[\sum_{n=1}^{N\wedge\hat{\tau}_\delta} f_0'(d(Z^\varepsilon(n-1)))\lambda_n\right] &= \mathbb{E}\left[\sum_{n=1}^{N\wedge\hat{\tau}_\delta} f_0''(d(Z^\varepsilon(n-1)))(\lambda_n^2 - 1)\right] \\ &= 0.\end{aligned}$$

Thus, taking an expectation in (4.2), applying Lemma 6 (i) for $a_n := f_0'(d(\bar{Z}^\varepsilon(n-1)))$ and using (4.1), we obtain

$$\mathbb{E}\left[f_0\left(d(\bar{Z}^\varepsilon(t \wedge (\varepsilon^2\hat{\tau}_\delta)))\right)\right] \le f_0\left(d(x_1, x_2)\right) - \frac{mk\varepsilon^2}{2}\mathbb{E}\left[\sum_{n=1}^{\lfloor \varepsilon^{-2}t \rfloor \wedge \hat{\tau}_\delta} f_0(d(\bar{Z}^\varepsilon(\varepsilon^2(n-1))))\right] + o(1).$$

Set $\tau_\delta^\varepsilon := \tau_\delta(\bar{Z}^\varepsilon)$ and $\tau_\delta^0 := \tau_\delta(\bar{Z})$. Note that $|\varepsilon^{-2}\tau_\delta^\varepsilon - \hat{\tau}_\delta| \le 1$. Then

$$\varepsilon^2 \sum_{n=1}^{\lfloor \varepsilon^{-2}t \rfloor \wedge \hat{\tau}_\delta} f_0(d(\bar{Z}^\varepsilon(\varepsilon^2(n-1)))) = \int_0^{t \wedge \tau_\delta^\varepsilon} f_0(d(\bar{Z}^\varepsilon(s)))ds + o(\varepsilon).$$

Here the remainder term is uniform in $\bar{Z}^\varepsilon$. Thus we obtain

$$\text{(4.3)} \quad \mathbb{E}\left[f_0(d(\bar{Z}^\varepsilon(t \wedge (\varepsilon^2\hat{\tau}_\delta))))\right] + \frac{mk}{2}\int_0^t \mathbb{E}\left[f_0(d(\bar{Z}^\varepsilon(s)))\ ;\ \tau_\delta^\varepsilon > s\right] ds \le f_0(d(x_1, x_2)) + o(1).$$

Since $\{w\ ;\ \tau_\delta(w) > s\}$ is open and $f_0 \ge 0$, we have

$$\begin{aligned}
\mathbb{E}\left[f_0(d(\bar{Z}(s \wedge \tau_\delta^0)))\right] &= \mathbb{E}\left[f_0(d(\bar{Z}(s)))\ ;\ \tau_\delta^0 > s\right] + f_0(\delta)\mathbb{P}[\tau_\delta^0 \le s] \\
&\le \varliminf_{\varepsilon \to 0} \mathbb{E}\left[f_0(d(\bar{Z}^\varepsilon(s)))\ ;\ \tau_\delta^\varepsilon > s\right] + f_0(\delta) \\
&\le \varliminf_{\varepsilon \to 0} \mathbb{E}\left[f_0(d(\bar{Z}^\varepsilon(s \wedge \tau_\delta^\varepsilon)))\right] + f_0(\delta) \\
&= \varliminf_{\varepsilon \to 0} \mathbb{E}\left[f_0(d(\bar{Z}^\varepsilon(s \wedge (\varepsilon^2\hat{\tau}_\delta))))\right] + f_0(\delta)
\end{aligned}$$

for small $\delta > 0$. Thus, by applying the Fatou lemma in (4.3) as $\varepsilon \to 0$,

$$\text{(4.4)} \quad \mathbb{E}\left[f_0(d(\bar{Z}(t \wedge \tau_\delta^0)))\right] + \frac{mk}{2}\int_0^t \mathbb{E}\left[f_0(d(\bar{Z}(s \wedge \tau_\delta^0)))\right] ds \le f_0(d(x_1, x_2)) + \left(1 + \frac{mkt}{2}\right) f_0(\delta).$$

Note that we have $\mathbb{E}\left[f_0(d(\bar{Z}(s \wedge \tau_\delta^0)))\right] = \mathbb{E}\left[f_0(d(Z(s \wedge \tau_\delta(Z))))\right]$. Since $f_0(0) = 0$, we obtain

$$\lim_{\delta \to 0} \mathbb{E}\left[f_0(d(\bar{Z}(s \wedge \tau_\delta^0)))\right] = \mathbb{E}\left[f_0(d(Z(s \wedge \tau)))\right] = \mathbb{E}\left[f_0(d(Z(s)))\right].$$

Hence the conclusion follows by letting $\delta \to 0$ in (4.4). Q.E.D.

Theorem 12. *Assume $k > 0$. Let λ be the first nonzero eigenvalue of $-\Delta$. Then $\lambda \geq mk$.*

Proof. Multiplying $\exp(mkt/2)$ on both sides of the inequality in Lemma 11 and taking an integration, we obtain

$$\int_0^t \mathbb{E}\left[f_0(d(Z(s)))\right] ds \leq \frac{2(1-\mathrm{e}^{-mkt/2})}{mk} f_0(d(x_1,x_2)). \tag{4.5}$$

Let φ be an eigenfunction of $-\Delta$ corresponding to λ. Then $\mathbb{E}[\varphi(Z_i(t))] = \mathrm{e}^{-\lambda t/2}\varphi(x_i)$ holds for $i = 1, 2$ since $Z_1(t)$ and $Z_2(t)$ are both diffusion processes generated by $\Delta/2$. Let us define a constant $C_L > 0$ by

$$C_L := \sup_{y_1, y_2 \in X} \frac{\varphi(y_1) - \varphi(y_2)}{f_0(d(y_1, y_2))}.$$

Note that $C_L < \infty$ holds because φ is continuously differentiable and $f_0'(0+) \neq 0$. Thus (4.5) implies

$$\begin{aligned} \frac{2(1-\mathrm{e}^{-\lambda t/2})}{\lambda}(\varphi(x_1) - \varphi(x_2)) &= \int_0^t \mathbb{E}\left[\varphi(Z_1(s)) - \varphi(Z_2(s))\right] ds \\ &\leq C_L \int_0^t \mathbb{E}\left[f_0(d(Z(s)))\right] ds \leq C_L f_0(d(x_1,x_2)) \frac{2(1-e^{-mkt/2})}{mk}. \end{aligned}$$

Hence, dividing both sides by $f_0(d(x_1,x_2))$ and taking supremum with respect to x_1, x_2, we obtain the desired result. Q.E.D.

§5. Gradient estimate for heat semigroups

For a bounded measurable function $\psi : X \to \mathbb{R}$, we define $T_t\psi$ by $T_t\psi(x) = \mathbb{E}_x\left[\psi(W_t)\right]$. We show the following gradient estimate (see [21] for example, as well as for possible applications).

Theorem 13. *For $\psi \in C^2(X)$ with $\sup_{x\in X} |\psi(x)| \vee |\nabla\psi(x)| < \infty$ and $z \in X$, $|\nabla(T_t\psi)|(z) \leq \mathrm{e}^{-(m-1)kt/2} T_t(|\nabla\psi|)(z)$ holds.*

Proof. Take $R > \delta > 0$. As in the proof of Lemma 11, we write $\tau_\delta(\bar{Y}^\varepsilon) =: \tau_\delta^\varepsilon$ and $\tau_\delta(\bar{Y}) =: \tau_\delta^0$. An iteration of Lemma 4 (ii) from $n = 1$ to $n = \lfloor \varepsilon^{-2} t \rfloor$ together with (4.1) yields

$$\begin{aligned} f_{\varepsilon^2\lfloor\varepsilon^{-2}t\rfloor}(d(\bar{Y}^\varepsilon(t))) &\leq f_0(d(x_1,x_2)) \\ &+ \frac{\varepsilon^2}{2} \sum_{n=1}^{\lfloor\varepsilon^{-2}t\rfloor} f'_{\varepsilon^2(n-1)}(d(\bar{Y}^\varepsilon(n-1)))(\Lambda_n^{(Y^\varepsilon)} - \bar{\Lambda}_n^{(Y^\varepsilon)}) + o(1). \end{aligned} \tag{5.1}$$

In the same way as we remarked in section 3, the remainder term is controlled uniformly in $\bar{Y}^\varepsilon$ as long as $t < \sigma_R(\bar{Y}^\varepsilon) \wedge \tau_\delta^\varepsilon$. Let $E_\varepsilon \subset \Omega$ be an event defined by

$$E_\varepsilon := \left\{ f_{\varepsilon^2 \lfloor \varepsilon^{-2} t \rfloor}(d(\bar{Y}^\varepsilon(t))) \le f_0(d(x_1, x_2)) + \delta \right\} \cap \{ t < \sigma_R(\bar{Y}^\varepsilon) \}.$$

By virtue of Lemma 6 (ii) for $a_n := f'_{\varepsilon^2(n-1)}(d(\bar{Y}^\varepsilon(n-1)))$, (5.1) and the remark after that imply

$$\lim_{\varepsilon \to 0} \mathbb{P}\left[f_{\varepsilon^2 \lfloor \varepsilon^{-2} t \rfloor}(d(\bar{Y}^\varepsilon(t))) > f_0(d(x_1, x_2)) + \delta,\ t < \sigma_R(\bar{Y}^\varepsilon) \wedge \tau_\delta^\varepsilon \right] = 0.$$

Thus Lemma 2 yields $\overline{\lim}_{R \to \infty} \overline{\lim}_{\varepsilon \to 0} \mathbb{P}[E_\varepsilon^c \cap \{ t < \tau_\delta^\varepsilon \}] = 0$. Take $\eta > 0$ arbitrary. By the Taylor expansion, there exists $\tilde{\delta} > 0$ such that, for $q, q' \in X$ with $d(o, q) < R$ and $d(q, q') < \tilde{\delta}$, we have

$$|\psi(q) - \psi(q')| \le (|\nabla \psi|(q) + \eta)\, d(q, q').$$

Choose x_2 and δ so that $f^{-1}_{\varepsilon^2 \lfloor \varepsilon^{-2} t \rfloor}(f_0(d(x_1, x_2)) + \delta) < \tilde{\delta}$ holds for sufficiently small $\varepsilon > 0$. Then we have

$$\begin{aligned}
(5.2) \qquad & \mathbb{E}\left[\left| \psi(\bar{Y}_1^\varepsilon(t \wedge \tau_\delta^\varepsilon)) - \psi(\bar{Y}_2^\varepsilon(t \wedge \tau_\delta^\varepsilon)) \right| \right] \\
& \le \mathbb{E}\left[\left| \psi(\bar{Y}_1^\varepsilon(t)) - \psi(\bar{Y}_2^\varepsilon(t)) \right| \ ;\ t < \tau_\delta^\varepsilon \right] + \|\nabla \psi\|_\infty \delta \\
& \le \mathbb{E}\left[\left| \psi(\bar{Y}_1^\varepsilon(t)) - \psi(\bar{Y}_2^\varepsilon(t)) \right| \ ;\ E_\varepsilon \cap \{ t < \tau_\delta^\varepsilon \} \right] \\
& \qquad + 2 \|\psi\|_\infty \mathbb{P}[E_\varepsilon^c \cap \{ t < \tau_\delta^\varepsilon \}] + \|\nabla \psi\|_\infty \delta \\
& \le \left(\mathbb{E}\left[|\nabla \psi| (\bar{Y}_1^\varepsilon(t)) \right] + \eta \right) f^{-1}_{\varepsilon^2 \lfloor \varepsilon^{-2} t \rfloor}(f_0(d(x_1, x_2)) + \delta) \\
& \qquad + 2 \|\psi\|_\infty \mathbb{P}[E_\varepsilon^c \cap \{ t < \tau_\delta^\varepsilon \}] + \|\nabla \psi\|_\infty \delta.
\end{aligned}$$

Since $\{ w \ ;\ \tau_\delta(w) > t \}$ is open,

$$\begin{aligned}
& \left| \mathbb{E}\left[\psi(\bar{Y}_1(t \wedge \tau_\delta^0)) - \psi(\bar{Y}_2(t \wedge \tau_\delta^0)) \right] \right| \\
& \le \mathbb{E}\left[\left| \psi(\bar{Y}_1(t)) - \psi(\bar{Y}_2(t)) \right| \ ;\ \tau_\delta^0 > t \right] + \|\nabla \psi\|_\infty \delta \\
& \le \underline{\lim}_{\varepsilon \to 0} \mathbb{E}\left[\left| \psi(\bar{Y}_1^\varepsilon(t)) - \psi(\bar{Y}_2^\varepsilon(t)) \right| \ ;\ \tau_\delta^\varepsilon > t \right] + \|\nabla \psi\|_\infty \delta \\
& \le \underline{\lim}_{\varepsilon \to 0} \mathbb{E}\left[\left| \psi(\bar{Y}_1^\varepsilon(t \wedge \tau_\delta^\varepsilon)) - \psi(\bar{Y}_2^\varepsilon(t \wedge \tau_\delta^\varepsilon)) \right| \right] + \|\nabla \psi\|_\infty \delta.
\end{aligned}$$

By combining this inequality with the fact $\bar{Y}(t \wedge \tau_\delta^0) = Y(t \wedge \tau_\delta(Y))$,

$$\begin{aligned}
(5.3) \qquad & \left| \mathbb{E}\left[\psi(Y_1(t)) - \psi(Y_2(t)) \right] \right| \\
& = \left| \mathbb{E}\left[\psi(Y_1(t \wedge \tau)) - \psi(Y_2(t \wedge \tau)) \right] \right| \\
& = \lim_{\delta \to 0} \left| \mathbb{E}\left[\psi(Y_1(t \wedge \tau_\delta(Y))) - \psi(Y_2(t \wedge \tau_\delta(Y))) \right] \right| \\
& \le \underline{\lim}_{\delta \to 0} \underline{\lim}_{\varepsilon \to 0} \mathbb{E}\left[\left| \psi(\bar{Y}_1^\varepsilon(t \wedge \tau_\delta^\varepsilon)) - \psi(\bar{Y}_2^\varepsilon(t \wedge \tau_\delta^\varepsilon)) \right| \right].
\end{aligned}$$

Thus, by letting $\varepsilon \to 0$, $R \to \infty$ and $\delta \to 0$ in (5.2) and by applying (5.3),

$$|T_t\psi(x_1) - T_t\psi(x_2)| \le f_t^{-1} \circ f_0(d(x_1,x_2))(T_t|\nabla\psi|(x_1) + \eta).$$

Dividing both sides by $d(x_1,x_2)$ and letting $x_2 \to x_1$, we obtain

$$|\nabla T_t\psi(x_1)| \le \mathrm{e}^{-(m-1)kt/2}(T_t|\nabla\psi|(x_1) + \eta).$$

Since η is arbitrary, the conclusion follows. Q.E.D.

§6. Spaces admitting singularities

We extend our argument when the underlying space X has some singularity. In what follows, we state some notions of metric geometry including the definition and some basic properties of Alexandrov spaces. For details, refer to [2, 3] for example.

Let us suppose that X is a complete metric space. We also suppose that X is a geodesic space, that is, for any $x, y \in X$ there is a length-minimizing curve whose length realizes the distance between x and y. We call the curve a minimal geodesic and denote it by xy. Note that a minimal geodesic for $x, y \in X$ is not unique in general. A triangle $\triangle xyz$ in X means a set of three points $x, y, z \in X$ with a set of three minimal geodesics xy, yz and zx. For $\kappa \in \mathbb{R}$ and a triangle $\triangle xyz$, a κ-comparison triangle means a triangle $\triangle\tilde{x}\tilde{y}\tilde{z}$ in a complete simply connected space form of sectional curvature κ satisfying $d(\tilde{x},\tilde{y}) = d(x,y)$, $d(\tilde{y},\tilde{z}) = d(y,z)$ and $d(\tilde{z},\tilde{x}) = d(z,x)$. We call X an Alexandrov space if for each relatively compact open set $\Omega \subset X$ there exists $\kappa = \kappa(\Omega) \in \mathbb{R}$ which satisfies the following property: For any triangle $\triangle xyz \subset \Omega$, there exists a κ-comparison triangle $\triangle\tilde{x}\tilde{y}\tilde{z}$ such that, for any $p \in yz$ and $\tilde{p} \in \tilde{y}\tilde{z}$ with $d(\tilde{p},\tilde{y}) = d(p,y)$, we have $d(x,p) \ge d(\tilde{x},\tilde{p})$. Note that any complete Riemannian manifold is an Alexandrov space. Let $x \in X$ and $\gamma_1(s)$ and $\gamma_2(s)$ minimal geodesics satisfying $\gamma_1(0) = \gamma_2(0) = x$. We denote the angle of the comparison triangle $\triangle\widetilde{\gamma_1(s)}\tilde{x}\widetilde{\gamma_2(t)}$ at x by $\tilde{\angle}\gamma_1(s)x\gamma_2(t)$. When X is an Alexandrov space, $\lim_{t,s\downarrow 0}\tilde{\angle}\gamma_1(s)x\gamma_2(t)$ exists. We call it the angle between γ_1 and γ_2 at x. For a triangle $\triangle xyz$, we denote the angle at x between xy and xz by $\angle yxz \in [0,\pi]$. Note that $\tilde{\angle}yxz \le \angle yxz$ always holds.

Let us state the assumption on X. Let X be an Alexandrov space with a Hausdorff dimension $m \in [2,\infty)$. It is known that $m < \infty$ implies $m \in \mathbb{N}$ and local compactness of X. Assume that there exists an at most countable subset $\{z_j\}_{j\in J} \subset X$ such that $X_0 := X \setminus \{z_j\}_{j\in J}$ becomes a (non-complete) m-dimensional Riemannian manifold without boundary.

Its metric is assumed to be compatible with the original distance and the Ricci curvature of X_0 is bounded from below by $(m-1)k$ for some $k \in \mathbb{R}$. We assume two additional conditions on $\{z_j\}_{j \in J}$. First we assume that there exist constants $\theta_j < \pi$ such that each angle between geodesics emanating from z_j is smaller than θ_j. Second we assume that there exists $\delta_0 > 0$ such that $d(z_i, z_j) > \delta_0$ for any $i \neq j \in J$. The existence of such θ_j means that the diameter of the space of directions at z_j is strictly less than π. We can easily show that this singularity prevents z_j from being on a minimal geodesic xy if $z_j \neq x, y$. It also follows from more general result in [13]. Under these assumptions, X is included in a larger class of spaces studied in [10]. In particular, there is a canonical strongly local regular Dirichlet form on X (see [9, 10]). Thus we have the corresponding diffusion process on X (see [7]). We call it the Brownian motion and denote by $W_\cdot$.

Let us turn to construct coupled geodesic random walks on X. We need a little modification on the construction in (2.1) because the exponential map cannot be defined beyond singular points $\{z_j\}_{j \in J}$. We can choose Φ_1 and Φ_2 on $X_0 \times X_0$ in the same way as in section 2.1. To define a coupling by reflection Z^ε of geodesic random walks starting from $(x_1, x_2) \in X_0 \times X_0$, we use (2.1) when $d(Z_i^\varepsilon(n), z_j) > \sqrt{m+2}\varepsilon$ holds for $i = 1, 2$ and $j \in J$. When $d(Z_i^\varepsilon(n), z_j) \leq \sqrt{m+2}\varepsilon$ holds for $i = 1$ or $i = 2$ and some $j \in J$, we choose $Z_i^\varepsilon(n+1)$ $(i = 1, 2)$ as being uniformly distributed on the ball of radius $\sqrt{m+2}\varepsilon$ centered at $Z_i^\varepsilon(n)$ respectively. In this case, we take $Z_1^\varepsilon(n+1)$ and $Z_2^\varepsilon(n+1)$ independently and we define $Z_i^\varepsilon(t)$ for $t \in (n, n+1)$ by a geodesic interpolation of $Z_i^\varepsilon(n)$ and $Z_i^\varepsilon(n+1)$. Such a geodesic interpolation can be chosen in a measurable way as we choose Φ_i. The geodesic random walk in each component converges and its limit is identified with the Brownian motion on X_0. It follows from a similar argument as in the proof of Proposition 1 in [19] since $\{z_j\}_{j \in J}$ is polar (see [9]) and isolated from each other. Hence we obtain a coupling by reflection $Z = (Z_1, Z_2)$ of the Brownian motion on X. The same argument also yields a coupling by parallel transport $Y = (Y_1, Y_2)$.

In what follows, we will show two auxiliary lemmas for proving the main theorem (Theorem 16 below). Take a reference point $o \in X$. We denote an open metric ball of radius r centered at $x \in X$ by $B(x, r)$.

Lemma 14. *Fix $j \in J$. Let $R > 0$ and $z_j \in B(o, 2R)$. Then, for any $\eta > 0$, there is a constant $C_j(\eta, R) > 0$ such that it satisfies the following: If $B(z_j, \eta) \cap xy \neq \emptyset$ for a minimal geodesic xy joining $x, y \in B(o, R)$, then $x \in B(z_j, C_j(\eta, R))$ or $y \in B(z_j, C_j(\eta, R))$ holds.*

In addition, we can choose $C_j(\eta, R)$ *so that* $\lim_{\eta\downarrow 0} C_j(\eta, R) = 0$ *holds for each fixed* $R > 0$.

Proof. We write $\kappa := \kappa(B(o, 2R))$, which appeared in the definition of Alexandrov space. We may assume $\kappa < 0$ without loss of generality. Take $p \in xy$ so that $d(z_j, p) < \eta$. Take a κ-comparison triangle $\triangle\tilde{x}\tilde{z}_j\tilde{y}$ of $\triangle xz_jy$. Take $\tilde{p} \in \tilde{x}\tilde{y}$ satisfying $d(\tilde{x}, \tilde{p}) = d(x, p)$. Then the definition of Alexandrov space yields $d(\tilde{z}_j, \tilde{p}) \le d(z_j, p) < \eta$. Set $l_1 = d(x, z_j)$ and $l_2 = d(y, z_j)$. By the triangular inequality,

$$d(\tilde{x}, \tilde{y}) = d(\tilde{x}, \tilde{p}) + d(\tilde{p}, \tilde{y}) \ge l_1 + l_2 - 2d(\tilde{p}, \tilde{z}_j) \ge l_1 + l_2 - 2\eta.$$

Suppose $l_1 \wedge l_2 \ge \eta$. Then $l_1 + l_2 - 2\eta \ge 0$ and thus the cosine formula on κ-space form implies

$$c_\kappa(l_1)c_\kappa(l_2) - s_\kappa(l_1)s_\kappa(l_2)\cos\angle\tilde{x}\tilde{z}_j\tilde{y} \ge c_\kappa(l_1 + l_2 - 2\eta).$$

Note that $\cos\angle\tilde{x}\tilde{z}_j\tilde{y} \ge \cos\theta_j > -1$ holds. From the above inequality, we obtain

$$\begin{aligned} 0 < \cos\theta_j + 1 &\le \frac{c_\kappa(l_1 + l_2) - c_\kappa(l_1 + l_2 - 2\eta)}{s_\kappa(l_1)s_\kappa(l_2)} \\ &\le 2\sqrt{-\kappa}\eta\frac{s_\kappa(l_1 + l_2)}{s_\kappa(l_1)s_\kappa(l_2)} = 2\sqrt{-\kappa}\eta\left(\frac{c_\kappa(l_2)}{s_\kappa(l_2)} + \frac{c_\kappa(l_1)}{s_\kappa(l_1)}\right). \end{aligned}$$

Note that $t_\kappa(t) := s_\kappa(t)/c_\kappa(t)$ is increasing. Hence the conclusion follows by taking $C_j(\eta, R) := \eta \vee t_\kappa^{-1}(5\sqrt{-\kappa}\eta/(1 + \cos\theta_j))$. Q.E.D.

Let us define $\bar{S}_{\eta,R} : C([0,\infty) \to X) \to [0,\infty]$ by

$$\bar{S}_{\eta,R}(w) := \bar{\sigma}_R(w) \wedge \inf_{\substack{j\in J \\ d(z_j,o)<2R}} \inf\{t \ge 0 \,;\, d(w(t), z_j) \le C_j(\eta, R)\}.$$

For $w_1, w_2 \in C([0,\infty) \to X)$, set $S_{\eta,R}(w_1, w_2) := \bar{S}_{\eta,R}(w_1) \wedge \bar{S}_{\eta,R}(w_2)$.

Lemma 15. $\overline{\lim}_{R\to\infty} \overline{\lim}_{\eta\to 0} \overline{\lim}_{\varepsilon\to 0} \mathbb{P}[S_{\eta,R}(\bar{Z}^\varepsilon) < T] = 0$ *holds for* $T > 0$. *The same is also true for* $\bar{Y}^\varepsilon$.

Proof. Note that $\lim_{\eta\to 0} \mathbb{P}_x\left[\bar{S}_{\eta,R}(W) < T\right] = \mathbb{P}_x[\bar{\sigma}_R(W) < T]$ holds for all $x \in X_0$ since $\{z_j\}_{j\in J}$ is polar. Note that X satisfies the (m, k)-measure contraction property (see [11, 16]). Thus we obtain the Bishop–Gromov theorem which controls volume of metric balls. Since our framework satisfies all assumptions in [15] (see [9]), the Brownian motion on X is conservative by the Bishop–Gromov theorem. Hence the conclusion follows from a similar argument as in the proof of Lemma 2. Q.E.D.

Theorem 16.

(i) *Suppose $k \leq 0$. Then, for a coupling by reflection Z starting at $(x_1, x_2) \in X_0 \times X_0$, we have* (3.1) *and therefore* (3.2).

(ii) *Suppose $k > 0$. Then the first nonzero eigenvalue λ of $-\Delta$ satisfies $\lambda \geq mk$.*

(iii) *Let T_t be the semigroup corresponding to the Brownian motion on X. For $\psi \in C^2(X_0)$ with $\sup_{x \in X_0} |\psi(x)| \vee |\nabla \psi(x)| < \infty$ and $z \in X_0$, $|\nabla(T_t\psi)|(z) \leq \mathrm{e}^{-(m-1)kt/2} T_t(|\nabla\psi|)(z)$ holds.*

Proof. (i) Take $y_1, y_2 \in X_0$. Since $\gamma_{y_1 y_2} \cap (X \backslash X_0) = \emptyset$, we can apply the second variational formula. Given $R, \delta, \eta > 0$, suppose $d(y_1, o) \vee d(y_2, o) < R$, $d(y_1, y_2) > \delta$ and $d(y_1, z_j) \wedge d(y_2, z_j) > C_j(\eta, R)$ for every $j \in J$ with $d(z_j, o) < 2R$. Then Lemma 14 yields that the distance from z_j to $\gamma_{y_1 y_2}$ is bounded below by η. In this case, the injectivity radius is uniformly bounded below on $B(o, 2R) \setminus \bigcup_{j \in J} B(z_j, \eta)$. Thus, the proof of Lemma 3 also works with a uniform control (depending on R, δ, η) of the error term under the above-mentioned conditions. Once we obtain it, we can verify that the same argument as in subsection 2.2 and section 3 works by replacing σ_R with $S_{\eta,R}$ by virtue of Lemma 15.

(ii) Note that the measure contraction property implies the Bonnet–Myers theorem and hence X is compact. We obtain the same estimate as Lemma 11 by using $\tau_\delta \wedge S_{\eta,R}$ instead of τ_δ. The compactness of X implies that Δ has a discrete spectrum and its eigenfunctions are locally Hölder continuous (see [9]). Take an eigenfunction φ corresponding to the first nonzero eigenvalue λ. By applying the first assertion with $k = 0$, Remark 10 yields that φ is globally Lipschitz continuous. Hence we obtain the desired result by extending the argument in Theorem 12.

(iii) The conclusion follows in the same way as (i) and (ii). Q.E.D.

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