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Fixed-point theorems for random groups

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Abstract.

This article is an exposition of fixed-point theorems for random groups of the triangular model and of the graph model obtained in joint works with Izeki and Nayatani [11, 12].

§1. Introduction

The study of random groups has been a subject in geometric group theory ever since Gromov referred to the "genericity" of hyperbolic groups via a definite statistical meaning in [6]. That paper contains no proof, but later, Olshanskii [16] gave a confirmation of Gromov's claim.

Furthermore, Gromov proved in [7] that in the density model, random groups with density smaller than $1/2$ are infinite hyperbolic. Here, the density of a subset A of some finite set X is $0 < d < 1$ when the number of elements of A is the number of elements of the whole set X to the power d. We use the expression "random groups with density d have a certain property P" to mean that the probability for a group defined by a randomly chosen (with respect to the uniform measure) density d subset of the set of words of length l to have property P goes to 1 when l goes to infinity. For the detailed proof of this theorem in [7] and some generalizations, see [14].

In general, random groups are a probability distribution of finitely generated groups as above. The main motivations for the study of random groups are to investigate what a typical property of finitely generated groups is and to construct groups with new properties. For a survey of the study of random groups, see [5, 15].

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Żuk showed in [19] that random groups have Kazhdan's Property (T) in another model of random groups called the triangular model. For finitely generated groups, Property (T) is equivalent to the fixed-point property for Hilbert spaces. Furthermore, Gromov proved in [8] that random groups of the graph model have both Property (T) and the fixed-point property for the class $\overline{\mathcal{C}}_{reg}$ which will be defined in Example 3.5 and contains all regular CAT(0) spaces. Silberman [17] gave a detailed proof for Property (T) of random groups of the graph model.

On the other hand, there is a famous open problem about whether random groups are nonlinear. Here, a finitely generated group Γ is called nonlinear if Γ cannot be isomorphic to a subgroup of $GL(n, \mathbf{R})$. It is well-known that if a group Γ has the fixed-point property for the class of Hilbert spaces, symmetric spaces and Euclidean buildings, then Γ is nonlinear. Hence it is natural to ask if random groups of some model have the fixed-point property for some class of CAT(0) spaces containing these spaces.

In this paper, we report on our recent results with Izeki and Nayatani [11, 12] that random groups of the triangular model have the fixed-point property $F\mathcal{Y}_{\leq\delta_0}$ for any $\delta_0 < 1/2$ and random groups of the graph model have the fixed-point property $F\mathcal{Y}$ for any class $\mathcal{Y}$ of CAT(0) spaces with bounded δ. Here $F\mathcal{Y}_{\leq\delta_0}$ denotes the fixed-point property for the class $\mathcal{Y}_{\leq\delta_0}$ of CAT(0) spaces whose δ (defined in §3) is not more than δ_0, and a class $\mathcal{Y}$ of CAT(0) spaces is said to have bounded δ if there exists $\delta_0 < 1$ such that $\delta(Y) \leq \delta_0$ for any $Y \in \mathcal{Y}$.

Though there were not so many groups known, up to now, with such a strong fixed-point property, the above theorems state that such groups exist in abundance.

§2. Basic Definitions

2.1. Fixed-point property

Let Y be a metric space. A finitely generated group Γ is said to have the fixed-point property for Y if any isometric action of Γ on Y admits a global fixed point, and we denote this property by FY. If $\mathcal{Y}$ is a class of metric spaces, then Γ has $F\mathcal{Y}$ if Γ has the fixed-point property for any $Y \in \mathcal{Y}$.

A well-known example of a fixed-point property is Kazhdan's Property (T). Kazhdan's Property (T) was originally defined in a representation theoretic way as follows. A group Γ is said to have Property (T) if the trivial representation is an isolated point in the set of irreducible unitary representations. However, it is well-known that there is a geometric interpretation:

Theorem 2.1 ([3, 9]). *A finitely generated group Γ has the fixed-point property for Hilbert spaces if and only if Γ has Kazhdan's Property (T).*

Higher rank lattices such as $SL(n, \mathbf{Z})(n \geq 3)$ are known to be examples of groups with Property (T), but it is not easy to construct an example of a group with Property (T) which is not a lattice. Infinite abelian groups, or more generally, infinite amenable groups do not have Property (T). Furthermore, free groups $F_n(n \geq 2)$ do not have Property (T), since they have infinite abelian groups as their quotients, and any quotient group of a group with Property (T) would also have Property (T).

Property (T) induces some other fixed-point properties.

Theorem 2.2 ([18]). *If a finitely generated group Γ has Property (T), then Γ has the fixed-point property for trees.*

The fixed-point property for trees is called Serre's Property FA and is also important in discrete group theory because an isometric action on a tree is related to a decomposition of a group via Bass–Serre theory. If Γ has Property FA, then Γ does not split as an amalgamated free product nor an HNN extension.

Furthermore, finitely generated groups with Property (T) also have the fixed-point property for real and complex hyperbolic spaces. See [1] for these fact and more details about Kazhdan's Property (T).

2.2. CAT(0) spaces

Since our theorems are concerned with fixed-point properties for some classes of CAT(0) spaces, we recall here the notion of CAT(0) spaces briefly.

Definition 2.3 (CAT(0) space). *A complete metric space (Y, d) is called a CAT(0) space if it satisfies the following two conditions:*

(1) *Any two points in Y can be joined by a geodesic, that is, an isometric embedding of an interval.*

(2) *For any $x, y, z \in Y$ and any geodesic $\gamma : [0, 1] \to Y$ with $\gamma(0) = y, \gamma(1) = z$, we have for $0 \leq t \leq 1$,*

$$d(x, \gamma(t))^2 \leq (1-t)d(x, y)^2 + td(x, z)^2 - t(1-t)d(y, z)^2.$$

A complete metric space satisfying the first condition is called a geodesic space. Note that the inequality in the second condition becomes an equality for triangles in the Euclidean plane. So roughly speaking, CAT(0) space is a geodesic space in which any geodesic triangles are "thinner", or at least "not thicker", than triangles in the Euclidean plane.

Example 2.4. *The following are examples of CAT(0) spaces.*

(1) *Hilbert spaces,*
(2) *trees,*
(3) *Hadamard manifolds (i.e. a complete simply connected Riemannian manifold with sectional curvature* ≤ 0*)*
(4) *Euclidean buildings.*

For more about CAT(0) spaces, see [2].

§3. Izeki–Nayatani's invariant δ

We recall here the definition of the numerical invariant $\delta(Y)$ of a CAT(0) space Y introduced by Izeki and Nayatani [13], which is in some sense considered to measure the degree of singularity of a CAT(0) space Y.

At first we define the notion of a barycenter. We define a barycenter only for a finitely supported measure, as we do not need the more general case.

Definition 3.1 (barycenter). *Let Y be a CAT(0) space and let* $\mu = \sum_{i=1}^{m} t_i \operatorname{Dirac}_{y_i}$ *be a probability measure with finite support on Y. The barycenter* $\overline{\mu} \in Y$ *of μ is the unique minimizing point of the function*

$$y \mapsto \sum_{i=1}^{m} t_i d_Y(y, y_i)^2.$$

Definition 3.2 (Izeki–Nayatani's invariant δ). *Let Y be a CAT(0) space. Let μ be a finitely supported probability measure on Y, and let* $\overline{\mu} \in Y$ *be the barycenter of μ. Consider all maps* $\phi\colon \operatorname{supp}\mu \longrightarrow H$ *satisfying*

$$\|\phi(y_i)\| = d_Y(\overline{\mu}, y_i), \quad \|\phi(y_i) - \phi(y_j)\| \leq d_Y(y_i, y_j),$$

and set

$$\delta(Y, \mu) = \inf_{\phi} \left[\left\| \int_Y \phi \, d\mu \right\|^2 \Big/ \int_Y \|\phi\|^2 \, d\mu \right].$$

Here H denotes an infinite-dimensional Hilbert space. We then define

$$\delta(Y) = \sup_{\mu} \delta(Y, \mu),$$

where sup *is taken over all finitely supported probability measures on Y.*

This invariant $\delta(Y)$ takes value in $[0,1]$ because $\delta(Y,\mu)$ is in $[0,1]$ by the Cauchy–Schwartz inequality, and equals 0 when Y is a Hilbert space.

Example 3.3 ([13]). *The following are some known estimates for δ.*

(1) *If Y is a Hilbert space or a Hadamard manifold or a tree, then $\delta(Y) = 0$.*

(2) *For an integer $n \geq 2$ and a prime p, let $Y_{n,p}$ be the Euclidean building associated to $PGL(n, \mathbf{Q}_p)$. Then we have*

$$\delta(Y_{3,p}) \geq \frac{(\sqrt{p}-1)^2}{2(p-\sqrt{p}+1)}.$$

If $p = 2$, then

$$\delta(Y_{3,2}) \leq \frac{37-18\sqrt{2}}{28} = 0.4122\ldots.$$

Moreover, we know that for any integer $N \geq 2$, there exists a number $\delta_N < 1$ such that for any integer $2 \leq n \leq N$ and any prime p we have $\delta(Y_{n,p}) \leq \delta_N$. However, we do not know whether there exists a number $\delta_\infty < 1$ such that $\delta(Y_{n,p}) \leq \delta_\infty$ for any integer $n \geq 2$ and any prime p.

We can easily show that the invariant δ satisfies the following proposition.

Proposition 3.4. (1) *For any convex closed subspace Y' of a* CAT(0) *space Y, we have $\delta(Y') \leq \delta(Y)$.*

(2) *For the product of two* CAT(0) *spaces Y, Y', we have*

$$\delta(Y \times Y') = \max\{\delta(Y), \delta(Y')\}.$$

(3) *Let (Y_n, d_n) be a sequence of* CAT(0) *spaces, ω a non-principal ultrafilter on* $\mathbf{N}$ *and (Y_ω, d_ω) the ultralimit $(Y_\omega, d_\omega) = \omega\text{-}\lim_n (Y_n, d_n)$. Then,*

$$\delta(Y_\omega) \leq \omega\text{-}\lim_n \delta(Y_n)$$

holds ([11, Proposition 3.2]).

For a fixed $\delta_0 \in [0,1]$, let $\mathcal{Y}_{\leq \delta_0}$ denote the class of CAT(0) spaces Y satisfying $\delta(Y) \leq \delta_0$. Then, the above proposition shows that the class $\mathcal{Y}_{\leq \delta_0}$ is closed under the operations of taking a direct product, a convex closed subspace and an ultralimit.

Let $\mathcal{Y}$ be a family of CAT(0) spaces. $\mathcal{Y}$ is said to have "bounded δ" if there exists $\delta_0 < 1$ such that $\delta(Y) \leq \delta_0$ for all $Y \in \mathcal{Y}$.

Example 3.5. (1) *Let $\mathcal{H}$ be the class of Hilbert spaces, then $\mathcal{H}$ has bounded δ, since we have $\delta(H) = 0$ for any $H \in \mathcal{H}$.*

(2) *Let $\overline{\mathcal{C}}_{reg}$ be the minimal class of CAT(0) spaces that contains all smooth CAT(0) spaces (i.e. Hadamard manifolds) and that is closed under taking closed convex subspaces and ultralimits. Then, $\overline{\mathcal{C}}_{reg}$ has bounded δ, since any $Y \in \overline{\mathcal{C}}_{reg}$ has $\delta(Y) = 0$.*

(3) *Let $\mathcal{Y}_{\leq\delta_0}$ be the class of CAT(0) spaces Y satisfying $\delta(Y) \leq \delta_0$ as above. If δ_0 is less than 1, then $\mathcal{Y}_{\leq\delta_0}$ has bounded δ by definition. $\overline{\mathcal{C}}_{reg}$ is a subclass of $\mathcal{Y}_{\leq\delta_0}$ for any δ_0.*

(4) *For any N, the class $\mathcal{Y} = \{Y_{n,p} | n \leq N\}$ of Euclidean buildings has bounded δ.*

By using the invariant δ, Izeki and Nayatani obtained the following fixed-point theorem in [13].

Theorem 3.6. *Let a discrete group Γ act properly discontinuously and cocompactly on a simplicial complex X with an admissible weight, and let Y be a CAT(0) space. If*

$$\mu_1(Lk_x)(1 - \delta(Y)) > 1/2$$

holds for any $x \in X$, then Γ has FY. Here, $\mu_1(Lk_x)$ is the second eigenvalue of the combinatorial Laplacian of the link of x.

If we take $X = Y = Y_{3,p}$ and let Γ be a cocompact lattice of $PGL(3, \mathbf{Q}_p)$ in the above theorem, then of course Γ does not have FY. Hence we have

$$\mu_1(Lk_x)(1 - \delta(Y_{3,p})) \leq 1/2.$$

So we get the former estimate of Example 3.3 (2)

$$\delta(Y_{3,p}) \geq 1 - \frac{1}{2\mu_1(Lk_x)} = \frac{(\sqrt{p} - 1)^2}{2(p - \sqrt{p} + 1)}$$

by using the computation of the μ_1 of generalized polygons by Feit–Higman [4] since the link of any point $x \in Y_{3,p}$ can be identified with the generalized triangle associated to the finite projective plane $P^2(\mathbf{F}_p)$.

Moreover, if we take $X = Y_{3,p}$ and let Y be any CAT(0) space with

$$\delta(Y) < \frac{(\sqrt{p} - 1)^2}{2(p - \sqrt{p} + 1)}$$

and let Γ be a cocompact lattice of $PGL(3, \mathbf{Q}_p)$ in the above theorem, then we have

$$\mu_1(Lk_x)(1 - \delta(Y)) > 1/2,$$

thus we get the fixed-point property for Y. This means that Γ has the fixed-point property for any CAT(0) space Y with

$$\delta(Y) < \frac{(\sqrt{p}-1)^2}{2(p-\sqrt{p}+1)}.$$

In particular, the group Γ has the fixed-point property for $\overline{\mathcal{C}}_{reg}$.

§4. Random groups of the triangular model

Żuk considered in [19] a model of random groups which is now called the triangular model as follows: For $0 \leq d \leq 1$ and a fixed constant $c > 1$, let $P_{\mathcal{M}}(m, d)$ be the set of presentations $P = \langle S | R \rangle$, where $S = \{s_1^{\pm 1}, \ldots, s_m^{\pm 1}\}$ and R is a set of words of length 3 with respect to S satisfying $c^{-1}(2m-1)^{3d} \leq \#R \leq c(2m-1)^{3d}$. Let $\Gamma(P)$ denote the group defined by the presentation P. Then, Żuk showed the following theorem.

Theorem 4.1 ([19]). *If $d > 1/3$,*

$$\lim_{m\to\infty} \frac{\#\{P \in P_{\mathcal{M}}(m,d) \mid \Gamma(P) \text{ has Kazhdan's Property (T) }\}}{\#P_{\mathcal{M}}(m,d)} = 1.$$

In order to prove this theorem, Żuk obtained a spectral criterion for a finitely generated group given by a presentation to have Kazhdan's Property (T). The criterion is stated in terms of the second eigenvalue of the discrete Laplacian of a certain finite graph, canonically associated with the presentation of the group and denoted by $L'(S)$ in [19]; if this invariant is greater than 1/2, then the group has Property (T).

Our first theorem states that under the same conditions as Żuk's theorem, random groups have the fixed-point property for the class $\mathcal{Y}_{\leq \delta_0}$ ($\delta_0 < 1/2$) of metric spaces.

Theorem 4.2 ([11]). *For $\delta_0 < 1/2$ and $d > 1/3$, we have*

$$\lim_{m\to\infty} \frac{\#\{P \in P_{\mathcal{M}}(m,d) \mid \Gamma(P) \text{ has } F\mathcal{Y}_{\leq \delta_0}\}}{\#P_{\mathcal{M}}(m,d)} = 1.$$

As we saw in the example 3.5, the class $\mathcal{Y}_{\leq \delta_0}$ contains all Hilbert spaces, Hadamard manifolds, trees and some Euclidean buildings. Since Property (T) is equivalent to the fixed-point property for Hilbert spaces, our theorem is stronger than Żuk's.

§5. Random groups of the graph model

In this section, we recall the setting of random groups of the graph model considered in [8] and state our main theorem.

Let $k \geq 2$ be an integer and let Γ be a free group generated by $S = \{s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$. We fix a finite graph $G = (V, E)$. A map $\alpha : E \to S$ is called an S-labeling if $\alpha((u,v)) = \alpha((v,u))^{-1}$, and let $\mathcal{A} = \{\alpha : E \to S\}$ be the set of S-labelings for G. For a closed path $\vec{c} = (\vec{e_1}, \ldots, \vec{e_r})$, we get a word on S by $\alpha(\vec{c}) = \alpha(\vec{e_1}) \ldots \alpha(\vec{e_r})$. By considering these words as relations, we get a finitely generated group Γ_α. Precisely, by setting

$$R_\alpha := \{\alpha(\vec{c}) | \vec{c} \text{ a closed path in } G\},$$

we define $\Gamma_\alpha = \Gamma/\overline{R_\alpha} = \langle S | R_\alpha \rangle$. Here, $\overline{R_\alpha}$ denotes the normal closure of R_α. As we defined a group for any S-labeling α, we get a model of random groups by giving a uniform probability measure on $\mathcal{A}$.

Definition 5.1. *For a sequence of finite graphs $\{G_i\}_{i=1}^\infty$ we say that random groups of the graph model have property P if*

$$\lim_{i \to \infty} \frac{\#\{\alpha \in \mathcal{A}_i | \Gamma_\alpha \text{ has property } P\}}{\#\mathcal{A}_i} = 1.$$

Here, $\mathcal{A}_i$ is the set of S-labelings for G_i.

Then our main theorem is the following.

Theorem 5.2. *Let $\mathcal{Y}$ be a family of CAT(0) spaces with bounded δ. Let $\{G_i\}_{i=1}^\infty$ be a sequence of finite connected graphs whose number of vertices tends to infinity, and satisfying*

$$\begin{cases} 2 \leq deg(u) \leq d_0 \quad (u \in G_i), \\ girth(G_i) > i, \, diam(G_i) < 100i, \\ \mu_1(G_i) \geq \lambda_0 > 0, \\ \sharp\{\text{embeded paths in } G_i \text{ of length} < l/2\} \leq const \cdot \beta^{l/2}, \end{cases}$$

for some $\beta > 1$ sufficiently close to 1. Then we have

$$\lim_{i \to \infty} \frac{\#\{\alpha \in \mathcal{A}_i | \Gamma_\alpha \text{ is non-elementary hyperbolic and has } F\mathcal{Y}\}}{\#\mathcal{A}_i} = 1.$$

That is, random groups of the graph model have $F\mathcal{Y}$. Here $deg(u)$ is the degree of a vertex u, $diam(G)$ is the diameter of G, $girth(G)$ is the minimal length of closed paths in the graph G, and $\mu_1(G)$ is the first non-zero eigenvalue of the combinatorial Laplacian of G.

Note that the fixed-point property F$\mathcal{Y}$ we get here is stronger than F$\mathcal{Y}_{\leq\delta_0}$ ($\delta_0 < 1/2$), which we got in the triangular model.

As we do not know any example of a CAT(0) space Y with $\delta(Y) = 1$, there is the question of whether there really exists a CAT(0) space Y that satisfies $\delta(Y) = 1$. If there is no CAT(0) space with $\delta(Y) = 1$, we can easily show that there exists a constant $C < 1$ such that $\delta(Y) < C$ for any CAT(0) space Y. Then, by considering the class $\mathcal{Y}_{CAT(0)}$ of all CAT(0) spaces, we would get a hyperbolic group with F$\mathcal{Y}_{CAT(0)}$.

For the proof of Theorem 5.2, we used a fixed-point theorem via n-step energy estimation. For the notion of n-step energy and the fixed-point theorem, see [10] in this volume.

References

[1] B. Bekka, P. de la Harpe and A. Valette, Kazhdan's Property (T), New Math. Monogr., **11**, Cambridge Univ. Press, Cambridge, 2008.

[2] M. R. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

[3] P. Delorme, 1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations, Bull. Soc. Math. France, **105** (1977), 281–336.

[4] W. Feit and G. Higman, The nonexistence of certain generalized polygons, J. Algebra, **1** (1964), 114–131.

[5] É. Ghys, Groupes aléatoires, d'après Misha Gromov,..., Séminaire Bourbaki, **55**, 2002/03, n 916.

[6] M. Gromov, Hyperbolic groups, In: Essays in Group Theory, Math. Sci. Res. Inst. Publ., **8**, Springer-Verlag, 1987, pp. 75-263.

[7] M. Gromov, Asymptotic Invariants of Infinite Groups, London Math. Soc. Lecture Note Ser., **182**, Cambridge Univ. Press, 1993.

[8] M. Gromov, Random walk in random groups, Geom. Funct. Anal., **13** (2003), 73–146.

[9] A. Guichardet, Sur la cohomologie des groupes topologiques. II, Bull. Sci. Math. (2), **96** (1972), 305–332.

[10] H. Izeki, A fixed-point property of finitely generated groups and an energy of equivariant maps, in this volume.

[11] H. Izeki, T. Kondo and S. Nayatani, Fixed-point property of random groups, Ann. Global Anal. Geom., **35** (2009), 363–379.

[12] H. Izeki, T. Kondo and S. Nayatani, N-step energy of maps and fixed-point property of random groups, preprint.

[13] H. Izeki and S. Nayatani, Combinatorial harmonic maps and discrete-group actions on Hadamard spaces, Geom. Dedicata, **114** (2005), 147–188.

[14] Y. Ollivier, Sharp phase transition theorems for hyperbolicity of random groups, Geom. Funct. Anal., **14** (2004), 595–679.

[15] Y. Ollivier, A January 2005 invitation to random groups, Ensaios Mat., **10**, Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.

[16] A. Yu. Ol'shanskiĭ, Almost every group is hyperbolic, Internat. J. Algebra Comput., **2** (1992), 1–17.

[17] L. Silberman, Addendum to "random walks in random groups", Geom. Funct. Anal., **13** (2003), 147–177.

[18] Y. Watatani, Property T of Kazhdan implies property FA of Serre, Math. Japon., **27** (1982), 97–103.

[19] A. Żuk, Property (T) and Kazhdan constants for discrete groups, Geom. Funct. Anal., **13** (2003), 643–670.

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