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Stochastic homogenization of horospheric tree products

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Abstract.

We construct measures invariant with respect to equivalence relations which are graphed by horospheric products of trees. The construction is based on using conformal systems of boundary measures on treed equivalence relations. The existence of such an invariant measure allows us to establish amenability of horospheric products of random trees.

§ Introduction

The study of *graphed measured equivalence relations* has two origins. The first one is *ergodic*, namely, the orbit equivalence theory for measure class preserving actions of countable groups. The second one is *geometric*, because such equivalence relations naturally arise (as traces of the leaf partition) on transversals of foliated or laminated spaces endowed with holonomy (quasi-)invariant measures.

The departure point of Feldman and Moore in their famous paper [FM77] published in 1977 (and announced two years earlier [FM75]) was entirely ergodic: it was the idea that numerous properties of measure class preserving group actions can actually be expressed just in terms of the associated orbit equivalence relation. In their work Feldman and Moore did not consider any additional leafwise graph structures on equivalence relations. However, at about the same time Plante [Pla75] essentially introduced graphed equivalence relations (in terms of finitely generated holonomy pseudogroups) in the topological context of foliations.

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It was only in 1990 that Adams [Ada90] defined the notion of a graphed equivalence relation in the purely measure-theoretical setup and proved non-amenability of non-elementary treed equivalence relations with a finite invariant measure. Later this notion was used by the first author [Kai97] in order to clarify the relationship between the amenability of an equivalence relation and the amenability of its leafwise graphs and to give a new geometrical proof of the Connes–Feldman–Weiss theorem on the equivalence of hyperfiniteness and amenability. A new insight was brought in by Gaboriau [Gab00] by introducing the *cost* of an equivalence relation with an invariant probability measure as the lowest possible average value of the degree of vertices in leafwise connected graph structures. This invariant turned out to be very useful and has found numerous applications (e.g., see the recent survey [Fur09]).

From the probabilistic point of view a discrete equivalence relation with a quasi-invariant measure naturally arises from any stationary Markov chain with discrete transition probabilities (see [Kai98] for an explicit formula for the Radon–Nikodym cocycle). Finitely supported transition probabilities then produce a locally finite graph structure on this equivalence relation. In particular, there is a canonical one-to-one correspondence between *detailed balance* stationary measures of the leafwise simple random walk on the state space of a graphed equivalence relation (i.e., the ones with respect to which this random walk is reversible) and invariant measures of the equivalence relation. Namely, the density of a stationary measure with respect to the corresponding invariant measure is just the vertex degree function deg.

Another link with the probability theory is provided by the fact that a graphed equivalence relation on a probability space naturally gives rise to a map from this space to the space of rooted graphs $\mathcal{G}$ (i.e., a *random rooted graph*). It assigns to any point from the state space its leafwise graph with this very point as the distinguished vertex. *Stochastic homogenization* [Kai03] of a certain family of infinite graphs consists in finding a probability measure invariant with respect to an equivalence relation whose classes are endowed with graph structures from this family. The role of such a measure is then similar to the role of an invariant measure for a usual dynamical system.

The space of rooted graphs itself has an intrinsic graph structure: two rooted graphs are neighbours if they are isomorphic as unrooted graphs and their roots are neighbours in this common graph Γ. Moreover, if Γ is *rigid*, i.e., its isometry group is trivial, then the graph on its equivalence class (which is obtained by varying the root position) is precisely Γ itself. Denote by $\mathcal{G}_{\not\circlearrowleft} \subset \mathcal{G}$ the space of rooted rigid graphs. Then for rigid graphs the problem of stochastic homogenization reduces to finding an invariant probability measure on the corresponding subspace of $\mathcal{G}_{\emptyset}$.

It is easy to construct invariant measures on $\mathcal{G}_{\emptyset}$ by random perturbations of Cayley graphs of finitely generated infinite groups. However, there are random graphs whose origin has nothing to do with groups. The first example of stochastic homogenization in such situations is the invariant measure on the space of rigid rooted trees obtained from *augmented branching processes.* When studying random walks on Galton–Watson trees Lyons, Pemantle and Peres noticed that it is natural to modify the branching process by letting the progenitor to have one more additional offspring (so that all vertices statistically have the same number of neighbours). Then the arising probability measure on Galton–Watson trees augmented in this way is stationary with respect to the leafwise simple random walk [LPP95b]. Thus, by [Kai98], dividing this measure by the degree function deg produces an invariant measure (in Theorem 3.2 we also give a simple direct proof of this fact).

The main purpose of the present paper is to obtain a stochastic homogenization for yet another family of graphs: *horospheric products of trees.*

These graphs were first introduced by Diestel and Leader [DL01] in an attempt to answer a question of Woess [Woe91] on existence of vertex-transitive graph not quasi-isometric to Cayley graphs. Although the fact that the Diestel–Leader graphs indeed provide such an example was only recently proved by Eskin, Fisher and Whyte [EFW07], in the meantime the construction of Diestel and Leader attracted a lot of attention because of its numerous interesting features (see [Woe05, BNW08] and the references therein).

The starting point of this construction is the fact that, given an infinite tree T, any boundary point $\gamma \in \partial T$ determines the associated $\mathbb{Z}$-valued additive *Busemann cocycle* β_γ on T: for any two vertices $x, y \in T$ the value $\beta_\gamma(x, y)$ is, informally speaking, the "difference between the distances" from the points x and y to the point at infinity γ (this cocycle is actually well-defined for any $\mathsf{CAT}(0)$ space). The level sets of the Busemann cocycle β_γ consist of the points in T which are *equidistanced* from γ and are called *horospheres* (or *horocycles* in the case of the classical hyperbolic plane, whence the frequently used alternative term "horocyclic products").

The horospheric product of two pointed at infinity rooted trees (T, o, γ) and (T', o', γ') is then defined in the following way. Take the graph-theoretical product of the trees T, T', and consider its subgraph

Γ which has the same vertex set $T \times T'$, but contains only those edges of the product graph which are in the kernel of the cocycle $\beta_\gamma + \beta_{\gamma'}$. The horospheric product of the trees (T, o, γ) and (T', o', γ') is then the connected component of the graph Γ which contains the product origin (o, o').

Geometrically one can think about the horospheric products in the following way [KW02]. Draw the tree T' upside down next to T so that the respective horospheres are at the same levels. Connect the two roots o, o' with an elastic spring. It can move along each of the two trees, may expand infinitely, but must always remain in horizontal position. The vertex set of the horospheric product consists then of all admissible positions of the spring. From a position (x, x') with $\beta_\gamma(o, x) + \beta'_{\gamma'}(o', x') = 0$ the spring may move downwards to one of the "sons" of x and at the same time to the "father" of x', or upwards in an analogous way. Such a move corresponds to going to a neighbour (y, y') of (x, x').

It is natural to look for a stochastic homogenization of horospheric products based on stochastic homogenizations of trees, i.e., on treed equivalence relations with an invariant probability measure. However, such equivalence relations are non-amenable (unless elementary) [Ada90], and therefore there is no measurable way of assigning a single boundary point $\gamma \in \partial T_x$ to any point x from the base space of the equivalence relation [Kai04] (here T_x is the leafwise tree on the equivalence class of x). Thus, a stochastic homogenization of horospheric products should be preceded by a choice of an appropriate measurable *system of boundary measures* $\{\nu_x\}$ on ∂T_x. By analogy with Fuchsian and Kleinian groups (e.g., see [Pat76, Sul79]) we say that a system $\{\nu_x\}$ is *conformal* if it is quasi-invariant and its Radon–Nikodym derivatives satisfy the relation $d\nu_y/d\nu_x(\gamma) = \exp(-\lambda\beta_\gamma(x, y))$ for a certain $\lambda > 0$ (the *dimension* of the system).

Given a treed equivalence relation R on a space X, we define its *boundary bundle* as the set $\widetilde{X} = \{(x, \gamma) : x \in X, \gamma \in \partial T_x\}$ (this is an immediate analogue of the unit tangent bundle on negatively curved manifolds). If two points $x, y \in X$ are equivalent, then T_x and T_y coincide as unrooted trees, so that there is a natural identification of the boundaries ∂T_x and ∂T_y. Therefore, $\widetilde{X}$ also has a structure of a treed equivalence relation

$$\widetilde{R} = \{(x, y, \gamma) : (x, y) \in R, \gamma \in \partial T_x \cong \partial T_y\},$$

and $\widetilde{R}$ carries the $\mathbb{Z}$-valued additive cocycle $\widetilde{\beta} = \beta_\gamma(x, y)$. If μ is an R-invariant measure on X, and $\{\nu_x\}$ is a conformal system of boundary measures of dimension λ, then the result of the integration of the system

$\{\nu_x\}$ against the measure μ is the measure $\widetilde{\mu}$ on $\widetilde{X}$ such that its Radon–Nikodym cocycle with respect to the equivalence relation $\widetilde{R}$ is precisely $\exp(-\lambda\widetilde{\beta})$. Thus, the measure $\widetilde{\mu}$ is, in our context, a solution of a classical problem of ergodic theory: to find a measure with the prescribed Radon–Nikodym derivatives.

Now, let $\{\nu'_{x'}\}$ be a conformal system of boundary measures of the same dimension λ on another treed equivalence relation with invariant measure (X', μ', R'), and let $\widetilde{\mu}'$ be the associated measure on the boundary bundle $\widetilde{X}'$. We shall say that the kernel $\boldsymbol{R}$ of the cocycle $c = \widetilde{\beta} + \widetilde{\beta}'$ is the *horospheric product* of the treed equivalence relations R and R'. The equivalence relation $\boldsymbol{R}$ is endowed with a natural graph structure such that its equivalence classes are precisely the horospheric products of trees from equivalence relations R and R'. Moreover, the product measure $\widetilde{\mu} \times \widetilde{\mu}'$ is $\boldsymbol{R}$-invariant, thus providing the sought for stochastic homogenization of horospheric products (Theorem 2.18). It follows from the fact that the logarithms of the Radon–Nikodym cocycles of the measures $\widetilde{\mu}, \widetilde{\mu}'$ are proportional to the respective Busemann cocycles $\widetilde{\beta}, \widetilde{\beta}'$ with the same proportionality coefficient $-\lambda$, so that the product measure is invariant with respect to the kernel of $\widetilde{\beta} + \widetilde{\beta}'$. Note that this construction is very similar to the construction of an invariant measure of the geodesic flow on a negatively curved manifold from a conformal measure [Kai90].

As an application we show in Theorem 2.20 that the horospheric product of almost any pair of pointed at infinity rooted trees arising in the above situation is amenable, (i.e., does not satisfy the strong isoperimetric inequality: there are subsets whose boundary is arbitrarily small compared with the subset itself). The proof is based on the fact that the equivalence relations $(\widetilde{X}, \widetilde{\mu}, \widetilde{R})$ and $(\widetilde{X}', \widetilde{\mu}', \widetilde{R}')$ are both amenable (as they are graphed by trees pointed at infinity). Therefore their product and its subrelation $\boldsymbol{R}$ are also amenable. On the other hand, since $\boldsymbol{R}$ has a finite invariant measure, amenability of $\boldsymbol{R}$ implies amenability of its leafwise graphs.

Another application is the existence of the associated finite stationary measure of the leafwise simple random walk and the ensuing possibility for a study of the asymptotical properties of simple random walks on individual horospheric products (the linear rate of escape, the harmonic measure, the Poisson boundary, the asymptotic entropy etc.). We shall return to this subject in another publication.

As an example in Section 3 we consider the horospheric products of augmented Galton–Watson trees. It is easy to see that the *branching* *measures* on their boundaries (i.e., the limits of the appropriately normalized uniform measures on the spheres around the root) form a conformal system of dimension $\lambda = \log m$, where m is the mean of the offspring distribution. Thus, horospheric products of augmented Galton–Watson trees corresponding to any two branching processes with the same mean are stochastically homogeneous.

There are numerous natural questions which arise in connection with our study. We hope to address them in the future, and here we shall just briefly mention some of them.

(1) In the present paper we do not consider at all the question about the ergodicity of arising measures. In fact one can show that in our setup the horospheric product of ergodic boundary bundles is also ergodic (the proof is based on an analogue of the famous *Hopf argument* used for proving ergodicity of the geodesic flow on negatively curved manifolds [Kai90]).

(2) Currently the augmented Galton–Watson measures (and similar measures arising from more general branching processes) are the only examples of "nice" invariant measures on the space of rooted trees. Although it was recently proved that any invariant measure can be obtained as an appropriate weak limit [Ele08], it would still be interesting to have other explicit examples.

(3) Which treed equivalence relations with an invariant measure admit a conformal system of boundary measures? When is such a system unique? In fact, conformal systems are closely related with the Hausdorff boundary measures (in perfect analogy with the Fuchsian and Kleinian case [Sul79]). One can show that under natural assumptions there is at most one conformal system of boundary measures which coincides with the system of the Hausdorff measures.

(4) Our point of view on boundary measures on random trees consists in considering *systems* of boundary measures corresponding to varying roots rather than a *single* measure (once again, in perfect analogy with the theory of boundary measures on negatively curved manifolds). In addition to the Busemann cocycle one can consider other natural cocycles (or potentials) on the boundary bundle and ask for existence of boundary systems with prescribed Radon–Nikodym derivatives, which leads to the notion of a *Gibbs system of boundary measures.* This notion, in particular, provides a unified approach to a number of results on multifractal properties of various boundary measures on Galton–Watson trees [MS04, Kin08]. It is also interesting to look at the ergodic properties of the arising invariant measures of the leafwise geodesic flow.

(5) It is still unknown whether a.e. leafwise graph in a graphed equivalence relation with a finite invariant measure has a precise exponential rate of growth [HK87]. As far as we know, this issue is completely open even for treed equivalence relations. A refinement of this question is the following problem: for which treed equivalence relations do the normalized uniform measures on spheres converge (like for the Galton–Watson and other trees arising from branching processes)?

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§1. Graphed equivalence relations

1.A. Equivalence relations

We shall first remind the basics from the theory of *discrete measured equivalence relations* created by Feldman and Moore [FM77]. Their starting point was the observation that many properties of a measure class preserving action of a countable group can actually be expressed just in terms of the corresponding orbit equivalence relation. We shall partially use the *groupoid* approach, see [Ren80, ADR00].

For an arbitrary equivalence relation $R \subset X \times X$ on a *state space* X the *composition*

$$(x,y)(y,z) = (x,z) \qquad \text{for} \quad (x,y),(y,z) \in R \tag{1.1}$$

determines a *groupoid structure* $\boldsymbol{G} = \boldsymbol{G}(R)$ with

- the *set of objects* X,
- the *set of morphisms* R,
- the *source map* $\boldsymbol{s} : (x,y) \mapsto x$,
- the *target map* $\boldsymbol{t} : (x,y) \mapsto y$,
- the *identity embedding* $\boldsymbol{\varepsilon} : x \mapsto (x,x)$,
- the *involution* $\boldsymbol{\theta} : (x,y) \mapsto (x,y)^{-1} = (y,x)$.

Denote by

$$[x] = [x]_R = R(x)$$

the *R-equivalence class* of a point $x \in X$. In other terminologies (which come from two important sources of equivalence relations: foliations and group actions) one also calls equivalences classes *leafs* or *orbits.*

An equivalence relation R on X is called *discrete measured* if

(i) It is *countable*, i.e., the classes $[x]$ are at most countable;

(ii) Its state space X is endowed with a structure of a *standard Borel space*, and it carries a σ-finite Borel measure μ, so that (X, μ) is a *Lebesgue measure space* (i.e., its non-atomic part is isomorphic to an interval with the Lebesgue measure on it);

(iii) It is measurable as a subset of $X \times X$ (endowed with the product Borel structure);

(iv) It preserves the class of the measure μ ($\equiv$ the measure μ is *quasi-invariant* with respect to R), which means that for any subset $A \subset X$ with $\mu(A) = 0$ its *saturation*

$$[A] = \bigcup_{x \in A} [x]$$

also has measure 0.

Below all the equivalence relations are assumed to be discrete measured with *infinite equivalence classes.* All the properties related to measure spaces will be understood *mod 0*, i.e., up to measure 0 subsets.

Any discrete measured equivalence relation can be presented as the orbit equivalence relation of a measure class preserving action of a certain countable group (although there are equivalence relations for which such an action can not be free [Fur99]). However, there are equivalence relations whose origin *a priory* has nothing to do with group actions (for instance, *treed equivalence relations* which we shall study below).

1.B. The Radon–Nikodym cocycle

The fibers of the source map $\boldsymbol{s}$ satisfy the *transitivity relation*:

$$(x, y)\, \boldsymbol{s}^{-1}(y) = \boldsymbol{s}^{-1}(x) \qquad \forall\, (x, y) \in R$$

(the multiplication in the left-hand side is the groupoid composition (1.1)). Denote by $\#_x$ the *counting measure* on the fiber $\boldsymbol{s}^{-1}(x)$ of the source map (this fiber is in obvious one-to-one correspondence $(x, y) \mapsto y$ with the class $[x]$). The system of measures $\{\#_x\}_{x \in X}$ is then *left invariant* in the sense that

$$(x, y)\#_y = \#_x \qquad \forall\, (x, y) \in R\,,$$

so that it is a *source* (or *left*) *Haar system* for the groupoid $\boldsymbol{G}$. The result of the integration of the fiber measures $\#_x$ against the measure μ on the state space X is the σ-finite measure $\mu_{\#}$ defined as

$$d\mu_{\#}(x, y) = d\mu(x) d\#_x(y) = d\mu(x)\,,$$

which is called the *left counting measure* on R.

In the same way, denote by $\#^x$ the counting measure on the fiber $\boldsymbol{t}^{-1}(x)$ of the target map. The system $\{\#^x\}$ is *right invariant* in the sense that

$$\#^x(x,y) = \#^y \qquad \forall\,(x,y) \in R\,,$$

so that it is a *target* (or *right*) *Haar system* for the groupoid $\boldsymbol{G}$. The result of the integration of the fiber measures $\#^x$ against the measure μ on the state space X

$$d\mu^{\#}(x,y) = d\mu(y)d\#^y(x) = d\mu(y)$$

is called the *right counting measure* on R. Alternatively, the right counting measure $\mu^{\#}$ can be obtained from the left counting measure $\mu_{\#}$ (and *vice versa*) by applying the involution $\boldsymbol{\theta}$, so that $\mu^{\#} = \boldsymbol{\theta}\mu_{\#}$ and $\mu_{\#} = \boldsymbol{\theta}\mu^{\#}$.

It turns out that the measures $\mu_{\#}$ and $\mu^{\#}$ are equivalent if and only if the original measure μ is quasi-invariant with respect to R. In this case the Radon–Nikodym derivative

$$\Delta(x,y) = \frac{d\mu^{\#}}{d\mu_{\#}}(x,y)$$

is called the *Radon–Nikodym cocycle* of the measure μ with respect to R (it is a *multiplicative cocycle* in the sense that

$$\Delta(x,y)\Delta(y,z) = \Delta(x,z)$$

for any triple of equivalent points $x, y, z \in X$). If $\Delta \equiv 1$, then the measure μ is *R-invariant*, or, respectively, the equivalence relation R *preserves* the measure μ.

Equivalently, the measure μ is quasi-invariant with respect to R if and only if for any *partial transformation* φ of R (i.e., a measurable bijection between two measurable subsets $A, B \subset X$ whose graph is contained in R) the φ-image $\varphi(\mu|_A)$ of the restriction of μ to A is absolutely continuous with respect to the restriction $\mu|_B$ of μ to B, and

$$\Delta(x,y) = \frac{d\varphi^{-1}\mu}{d\mu}(x) = \frac{d\mu}{d\varphi\mu}(y)\,.$$

Thus, the Radon–Nikodym cocycle can also be considered as the "ratio of differentials"

$$\Delta(x,y) = \frac{d\mu(y)}{d\mu(x)}\,, \qquad (x,y) \in R\,.$$

This formalism is quite convenient and can always be made rigorous by passing to the appropriate partial transformations.

If $R = R_G$ is the orbit equivalence relation determined by a measure class preserving action of a countable group G on a measure space (X, μ), then

$$\Delta(x, gx) = \frac{dg^{-1}\mu}{d\mu}(x) .$$

1.C. Graph structures

Recall that a *graph* Γ is determined by its *set of vertices* (usually it is denoted in the same way as the graph itself) and its *set of edges*. We shall always deal with *non-oriented graphs without loops and multiple edges*, so that the set of edges can be identified with a symmetric subset of $\Gamma \times \Gamma \setminus \mathsf{diag}$.

Analogously, a (non-oriented) *graph structure* on a discrete measured equivalence relation (X, μ, R) is determined by a measurable symmetric subset $K \subset R \setminus \mathsf{diag}$. The result of the restriction of this graph structure to an equivalence class $[x]$ gives the *leafwise graph* denoted by $[x]^K$. We shall call (X, μ, R, K) a *graphed equivalence relation* [Ada90]. Actually, in a somewhat less explicit form (in terms of finitely generated pseudogroups) this definition is already present in [Pla75], [Ser79] and [GC85].

We shall always deal with the graph structures which are *locally finite*, i.e., any vertex has only finitely many neighbours, and denote by deg the integer valued function which assigns to any point $x \in X$ the degree (valency) of x in the graph $[x]^K$. Passing, if necessary, to a smaller equivalence relation, we may always assume that the graph structure is *leafwise connected*, i.e., a.e. leafwise graph $[x]^K$ is connected. The latter condition means that

$$R = \bigcup_{n \geq 1} K^n ,$$

with respect to the groupoid multiplication (1.1).

The simplest example of a locally finite leafwise connected graph structure arises in the situation when $R = R_G$ is the orbit equivalence relation of an action of a finitely generated countable group G. For a symmetric generating set S put

$$K = \{(x, y) \in R : y = sx \text{ for a certain } s \in S\} .$$

Then the leafwise graphs $[x]^K$ are isomorphic either to the (left) *Cayley graph* (G, S) (if the orbit Gx is free), or to the *Schreier graphs* determined by subgroups of G (if the orbit Gx is not free). Once again, although any measured equivalence relation can be generated by a group action, there is a lot of graph structures (for instance, treed equivalence relations considered below) which can not be obtained in this way (cf. the comment at the end of Section 1.A).

If the measure μ is R-invariant and finite, then the leafwise graphs $[x]^K$ have properties which make them similar to Cayley graphs of finitely generated groups. In particular, under this condition $\mathsf{deg} \cdot \mu$ (the measure μ multiplied by the density deg) is a stationary measure of the leafwise simple random walk along the classes of the graphed equivalence relation (X, μ, R, K) [Kai98]. Yet another property is related to *amenability* of the involved structures.

1.D. Amenability of groups, graphs and equivalence relations

There is a lot of definitions and applications of amenability, which illustrates importance and naturalness of this notion. Here we shall just briefly outline the properties which are used later on in this paper (see [Gre69], [ADR00] for the missing references and for further details).

Let us first remind that the class of *amenable groups* is, from the analytical point of view, the most natural extension of the class of finite groups. Indeed, finite groups can be characterized within the class of all (at most) countable groups by existence of finite invariant measures. There are two ways of "extending" the finiteness property to infinite groups. One can look either for fixed points in a bigger space, or for approximative invariance instead of precise one.

Von Neumann implemented the first idea and defined *amenable groups* as those which admit a translation invariant *mean*, i.e, a finitely additive probability measure (actually, the term "amenable" was introduced much later by Day). Means being highly non-constructive objects, the other option was explored by Reiter who introduced the following condition on a countable group G: there exists an *approximatively invariant sequence* of probability measures θ_n on G, i.e., such that

$$\|\theta_n - g\theta_n\| \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall\, g \in G\,,$$

where $\|\cdot\|$ denotes the total variation norm. Reiter proved that the above condition (nowadays known as *Reiter's condition*) is in fact equivalent to amenability as defined by von Neumann.

By specializing Reiter's condition to sequences of probability measures equidistributed on finite subsets of G one obtains *Følner's condition*: there exists a sequence of finite subsets $A_n \subset G$ such that

$$\frac{|gA_n \triangle A_n|}{|A_n|} \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall\, g \in G\,,$$

where $\triangle$ denotes the symmetric difference of two sets, and $|A|$ is the cardinality of a finite set A. This condition is also equivalent to amenability of the group G.

For finitely generated groups the above approximative invariance condition on a sequence of subsets $A_n \subset G$ takes especially simple form:

$$\frac{|\partial A_n|}{|A_n|} \underset{n\to\infty}{\longrightarrow} 0 , \tag{1.2}$$

where ∂A denotes the *boundary* of a set $A \subset G$ in the left Cayley graph determined by a finite symmetric generating set (i.e., ∂A is the set of all points from A which have a neighbour from the complement of A). This is an *isoperimetric* characterization of amenability. Its formulation does not require any group structure, and therefore it can be applied to arbitrary graphs. The graphs of *bounded geometry* (i.e., with uniformly bounded vertex degrees) which satisfy the above isoperimetric property are called *amenable.* In spectral terms amenable graphs are characterized as the graphs for which the spectral radius of the Markov operator of the simple random walk is 1, which is a generalization of Kesten's description of amenable groups.

In a different direction the notion of amenability has been extended to group actions, equivalence relations, and, more generally, to groupoids. Zimmer was the first to notice that non-amenable groups may have actions which are similar to actions of amenable groups. His original (rather heavy) definition of amenable actions in terms of a fixed point property for Banach bundles was almost immediately reformulated by Renault by using a modification of Reiter's condition (although the work of Renault remained virtually unknown for quite a while, cf. [Kai97]). In particular, for discrete measured equivalence relations this definition takes the following form: an equivalence relation (X, μ, R) is *amenable* if there exists a sequence of measurable maps assigning to any point $x \in X$ a probability measure θ_n^x on the equivalence class of x such that

$$\|\theta_n^x - \theta_n^y\| \to 0 \qquad \text{for } \mu_\#\text{-a.e. } (x, y) \in R . \tag{1.3}$$

Thus, for a graphed equivalence relation (X, μ, R, K) there are two notions of amenability. The "global" amenability is the amenability of the equivalence relation (X, μ, R) in the sense of (1.3) and does not depend on the graph structure K, whereas the "local" or "leafwise" amenability means that μ-a.e. graph $[x]^K$ is amenable in the sense of (1.2). In general these conditions are not equivalent (see [Kai97] for a complete description of their relationship). However, for the purposes of the present paper we only need the following implication: if the equivalence relation (X, μ, R) is amenable, the measure μ is finite invariant, and the degrees of leafwise graphs of the structure K are uniformly bounded, then μ-a.e. graph $[x]^K$ is also amenable [GC85].

1.E. Random graphs and stochastic homogenization

A *rooted* ($\equiv$ *pointed*) graph $(\Gamma, o) = \Gamma_o$ is a graph Γ endowed with a reference vertex $o \in \Gamma$. Two rooted graphs Γ_o and $\Gamma'_{o'}$ are isomorphic ($\equiv$ isometric) if there is an isomorphism ($\equiv$ isometry with respect to the graph metric) $\varphi : \Gamma \to \Gamma'$ such that $\varphi(o) = o'$. A graph Γ is *rigid* if its isometry group $\mathsf{Iso}(\Gamma)$ is trivial; we shall also say that a rooted graph Γ_o is rigid if its underlying graph Γ is rigid.

We shall denote by $\mathcal{G}$ the space of (isometry classes of) infinite locally finite connected rooted graphs, and by $\mathcal{G}_{\not\emptyset}$ the subspace of $\mathcal{G}$ which consists of rigid rooted graphs. The space $\mathcal{G}$ can be given a complete separable metric by putting

$$d(\Gamma_o, \Gamma'_{o'}) = 2^{-r} ,$$

where $r \geq 0$ is the maximal integer such that the r-balls centered at the roots o, o' of the graphs Γ, Γ', respectively, are isometric as finite rooted graphs. Thus, $\mathcal{G}$ is a *Polish space*, and therefore its Borel structure is standard.

Given a graphed equivalence relation (X, μ, R, K), any point $x \in X$ determines the graph $[x]^K$. Let us denote by $[x]^K_\bullet = ([x]^K, x)$ the graph $[x]^K$ rooted at the point x. Thus, we have the map

$$X \to \mathcal{G} , \quad x \mapsto [x]^K_\bullet .$$

In particular, if μ is a probability measure, then its image under the above map is a probability measure on the space of rooted graphs $\mathcal{G}$, i.e., a *random rooted graph.*

Conversely, the space $\mathcal{G}$ is endowed with a natural equivalence relation $\mathcal{R}$: two rooted graphs Γ_o and $\Gamma'_{o'}$ are equivalent if the underlying graphs Γ and Γ' are isomorphic. It gives rise to a natural graph structure $\mathcal{K}$ on $\mathcal{R}$ [Kai98]:

$$\mathcal{K} = \left\{ \left(\Gamma_o, \Gamma_{o'}\right) : o \text{ and } o' \text{ are neighbours in } \Gamma \right\} .$$

If the group of isometries of Γ is non-trivial, then the graph $[\Gamma_o]^{\mathcal{K}}$ is the quotient of the graph Γ with respect to the action of the isometry group (in particular, it may contain loops). However, if $\mathsf{Iso}(\Gamma)$ is trivial, then $[\Gamma_o]^{\mathcal{K}}$ is isomorphic to Γ. Thus, the restriction of the equivalence relation $\mathcal{R}$ to $\mathcal{G}_{\not\emptyset}$ (which we shall also denote by $\mathcal{R}$) has the following property:

> the graph structure of the equivalence class of any rooted graph $\Gamma_o \in \mathcal{G}_{\not\emptyset}$ is isomorphic to Γ itself.

Definition 1.4 ([Kai03]). The random rooted graph determined by a probability measure μ on the space $\mathcal{G}_{\oslash}$ is *stochastically homogeneous* if the measure μ is invariant with respect to the equivalence relation $\mathcal{R}$.

Below we shall give examples of stochastic homogenization of trees and their horospheric products.

§2. Horospheric products of trees

2.A. Trees

Recall that a *tree* is a connected graph without cycles. Any two vertices x, y in a tree T can be joined with a unique *geodesic segment* $[x, y]$. Any locally finite tree T has a natural *compactification* $\overline{T} = T \sqcup \partial T$ obtained in the following way: a sequence of vertices x_n which goes to infinity in T converges in this compactification if and only if for a certain ($\equiv$ any) reference point $o \in T$ the geodesic segments $[o, x_n]$ converge pointwise. Thus, for any reference point $o \in T$ the *boundary* ∂T can be identified with the space of geodesic rays issued from o (and endowed with the topology of pointwise convergence). There are many other equivalent descriptions of the boundary ∂T (and of the compactification $\overline{T}$), in particular, as the *space of ends* of T and as the *hyperbolic boundary* of T.

A tree T with a distinguished boundary point $\gamma \in \partial T$ is called *pointed at infinity* ($\equiv$ remotely rooted; in the terminology of Cartier [Car72] the point γ is called a "mythological progenitor"). A triple $T_o^\gamma = (T, o, \gamma)$ with $o \in T$ and $\gamma \in \partial T$ is a *rooted tree pointed at infinity.*

Any two geodesic rays converging to the same boundary point eventually coincide, so that any boundary point $\gamma \in \partial T$ determines the associated additive $\mathbb{Z}$-valued *Busemann cocycle* on $T \times T$. It is defined as

$$\beta_\gamma(x, y) = d(y, o) - d(x, o) \ , \tag{2.1}$$

where d is the graph distance on T, and o is the *confluence* of the geodesic rays $[x, \gamma)$ and $[y, \gamma)$, see Figure 1.

The Busemann cocycle can also be defined as

$$\beta_\gamma(x, y) = \lim_{z \to \gamma} \left[d(y, z) - d(x, z) \right] \ ,$$

so that it is a "regularization" of the formal expression $d(y, \gamma) - d(x, \gamma)$. In the presence of a reference point $o \in T$ one can also talk about the *Busemann function*

$$b_\gamma(x) = \beta_\gamma(o, x) \ .$$

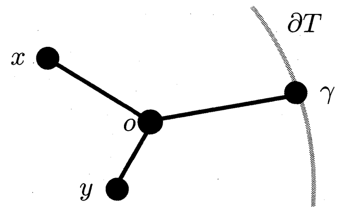

Fig. 1

The level sets

$$H_k = \{x \in T : b_\gamma(x) = k\}$$

of the Busemann function ($\equiv$ of the Busemann cocycle) are called *horospheres* centered at the boundary point γ, see Figure 2.

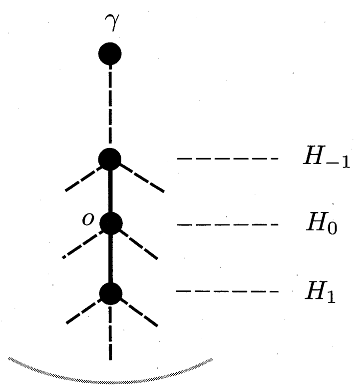

Fig. 2

2.B. Horospheric products

Definition 2.2. Let $T = (T, o, \gamma)$ and $T' = (T', o', \gamma')$ be two rooted trees pointed at infinity, and let $b = \beta_\gamma(o, \cdot)$, $b' = \beta_{\gamma'}(o', \cdot)$ be the corresponding Busemann functions. The *horospheric product* $T \upharpoonleft\!\downharpoonright T'$ is the graph with the vertex set

$$\{(x, x') \in T \times T' : b(x) + b'(x') = 0\}$$

and the edge set

$$\left\{ \left((x,x'),(y,y') \right) : (x,y) \text{ and } (x',y') \text{ are edges in } T,T', \text{ respectively} \right\} .$$

Remark 2.3. In the cocycle language, the product $T \times T'$ is endowed with the $\mathbb{Z}$-valued additive cocycle $c = \beta_\gamma + \beta_{\gamma'}$. Its *kernel* $\ker c = c^{-1}(0) \subset T \times T'$ consists of connected components (with respect to the product graph structure) which are the horospheric products corresponding to different choices of the roots o, o'.

Geometrically one can think about the horospheric products in the following way [KW02]. Draw the tree T' upside down next to T so that the respective horospheres $H_k(T)$ and $H_{-k}(T')$ are at the same level. Connect the two origins o, o' with an elastic spring. It can move along each of the two trees, may expand infinitely, but must always remain in horizontal position. The vertex set of $T \upharpoonleft\!\downharpoonright T'$ consists then of all admissible positions of the spring. From a position (x,x') with $b(x)+b'(x')=0$ the spring may move downwards to one of the "sons" of x and at the same time to the "father" of x', or upwards in an analogous way. Such a move corresponds to going to a neighbour (y,y') of (x,x'), see Figure 3.

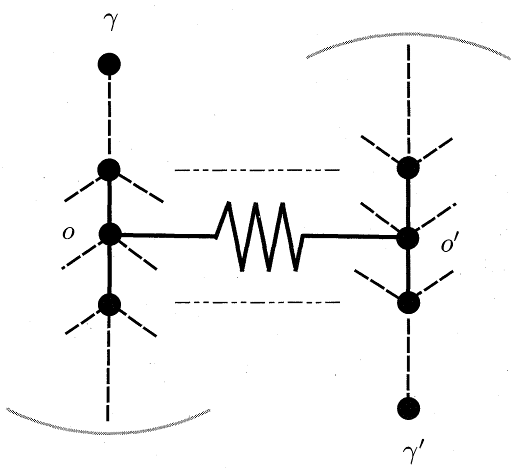

Fig. 3

Remark 2.4. This construction (in a different terminology) was first introduced by Diestel and Leader [DL01] in an attempt to answer a question of Woess [Woe91, Problem 1]: *is there a locally finite* *vertex-transitive graph which is not quasi-isometric with a Cayley graph of some finitely generated group?* They suggested the product $\mathbb{T}_3 \updownarrows \mathbb{T}_4$ of homogeneous trees of degrees 3 and 4, respectively, as a possible example. That this is indeed so was only recently proved by Eskin, Fisher and Whyte [EFW07]. In the meantime the construction of Diestel and Leader attracted a lot of attention because of its numerous interesting features (see [Woe05, BNW08] and the references therein). The arising graphs are also known under the names of *Diestel–Leader graphs* (when the multipliers are homogeneous trees) and of *horocyclic products* (however, we feel that the adjective "horospheric" is more appropriate, because the level sets of Busemann functions in trees are anything but cycles).

2.C. Treed equivalence relations

A graphed equivalence relation (X, μ, R, K) is *treed* if a.e. leafwise graph is a tree [Ada90]. We shall denote by $T_x = [x]_\bullet^K$ the leafwise tree of a point x rooted at x. The *boundary bundle* of a treed equivalence relation (X, μ, R, K) is the set

$$\widetilde{X} = \{(x, \gamma) : x \in X, \gamma \in \partial T_x\} = \bigcup_{x \in X} \{x\} \times \partial T_x \ ,$$

so that it is an analogue of the unit tangent bundle of a negatively curved manifold (since we are talking about equivalence relations, a better analogue is actually the unit tangent bundle of a foliation with negatively curved leaves, like, for instance, the stable foliation of the geodesic flow on a negatively curved manifold, see Remark 2.11 below).

We shall endow $\widetilde{X}$ with the equivalence relation (the *boundary extension* of R)

$$\begin{aligned} \widetilde{R} &= \big\{ \big((x, \gamma), (y, \gamma)\big) : (x, y) \in R, \ \gamma \in \partial T_x \cong \partial T_y \big\} \\ &\cong \{(x, y, \gamma) : (x, y) \in R, \ \gamma \in \partial T_x \cong \partial T_y\} \end{aligned}$$

(if two points $x, y \in X$ are equivalent, then T_x and T_y coincide as unrooted trees, so that there is a natural identification of the boundaries ∂T_x and ∂T_y) and with the treed graph structure $\widetilde{K}$ inherited from X.

The equivalence relation $\widetilde{R}$ is endowed with the $\mathbb{Z}$-valued additive cocycle (which we shall also call *Busemann* along with the cocycle (2.1))

$$\widetilde{\beta} : (x, y, \gamma) \mapsto \beta_\gamma(x, y) \ . \tag{2.5}$$

It will play an important role in the sequel.

In order to endow $\widetilde{X}$ with a Borel structure, we shall fix, once and forever, a Borel identification of the space X with the unit interval. This identification provides us with a linear order on any subset of X. In particular, the set of neighbours of any point $x \in X$ can be canonically identified with the set $\{1, 2, \dots, d\}$, where $d = \deg x$. Thus, in the case of a treed equivalence relation we can record any leafwise geodesic $x = x_0, x_1, x_2, \dots$ issued from a point $x \in X$ as a sequence $n_1, n_2, \dots$, where n_i is the position of x_i among the neighbours of x_{i-1}. In this way we obtain, for any $x \in X$, a one-to-one map τ_x from ∂X to a Borel subset of $\mathbb{N}^{\mathbb{N}}$ (this is similar to the well-known *Ulam–Harris notation*). Note that the maps τ_x *do* depend on x (not only on its equivalence class!), although the boundaries $\partial T_x, \partial T_y$ can be identified for any two equivalent points $x, y \in X$.

Finally, let us introduce a measure on the boundary extension $\widetilde{X}$ which would be quasi-invariant with respect to the equivalence relation $\widetilde{R}$. Since $\widetilde{X}$ is fibered over X, it is natural to construct such a measure by integrating a system of measures on the fibers against the measure μ on the base.

Definition 2.6. Given a treed equivalence relation (X, μ, R, K), a system of finite measures $\{\nu_x\}_{x \in X}$ on the boundaries ∂T_x of leafwise trees T_x is *measurable* if the map

$$x \mapsto \tau_x(\nu_x)$$

from X to the space of measures on $\mathbb{N}^{\mathbb{N}}$ is (weakly) measurable (i.e., for any measurable function f on $\mathbb{N}^{\mathbb{N}}$ the integrals $\langle f, \tau_x(\nu_x) \rangle$ depend on x measurably). A measurable system of boundary measures $\{\nu_x\}$ is *quasi-invariant* if for $\mu_{\#}$-a.e. pair $(x, y) \in R$ the measures ν_x and ν_y are equivalent. A measurable system of boundary measures $\{\nu_x\}$ gives rise to the measure

$$d\widetilde{\mu}(x, \gamma) = d\mu(x) d\nu_x(\gamma)$$

on the boundary bundle $\widetilde{X}$ which is called a *boundary extension* of μ.

Remark 2.7. Obviously,

$$\|\widetilde{\mu}\| = \int \|\nu_x\| d\mu(x) ,$$

and the measure $\widetilde{\mu}$ is finite if and only if the above integral is finite.

Remark 2.8. Another definition of a measurable boundary system of measures over a graphed equivalence relation with hyperbolic leaves is given in [Kai04] (in terms of separable measurable bundles of Banach spaces). One can establish the equivalence of these two definitions for treed equivalence relations in a rather straightforward (if tedious) way.

Proposition 2.9. *Let (X,μ,R,K) be a treed equivalence relation. Quasi-invariance of a measurable system of boundary measures $\{\nu_x\}$ is equivalent to quasi-invariance of the measure $\widetilde{\mu}$ with respect to the equivalence relation $\widetilde{R}$, and the Radon–Nikodym cocycle $\widetilde{\Delta}$ of the measure $\widetilde{\mu}$ with respect to $\widetilde{R}$ is connected with the Radon–Nikodym cocycle Δ of the measure μ with respect to the equivalence relation R by the formula*

$$\widetilde{\Delta}(x,y,\gamma) = \Delta(x,y)\frac{d\nu_y}{d\nu_x}(\gamma) .$$

Proof. In the language of "differentials" (see Section 1.B) of the involved measures

$$\widetilde{\Delta}(x,y,\gamma) = \frac{d\widetilde{\mu}(y,\gamma)}{d\widetilde{\mu}(x,\gamma)} = \frac{d\mu(y)d\nu_y(\gamma)}{d\mu(x)d\nu_x(\gamma)} = \frac{d\mu(y)}{d\mu(x)}\cdot\frac{d\nu_y(\gamma)}{d\nu_x(\gamma)} = \Delta(x,y)\frac{d\nu_y(\gamma)}{d\nu_x(\gamma)} .$$

Q.E.D.

Corollary 2.10. *If the measure μ is R-invariant, then the Radon–Nikodym cocycle of the measure $\widetilde{\mu}$ with respect to the equivalence relation $\widetilde{R}$ coincides with the pairwise Radon–Nikodym derivatives of the boundary system of measures $\{\nu_x\}$:*

$$\widetilde{\Delta}(x,y,\gamma) = \frac{d\nu_y}{d\nu_x}(\gamma) .$$

Remark 2.11. For negatively curved manifolds similar boundary extensions (where the boundary is the visibility sphere of the universal covering manifold) naturally arise in the study of invariant measures of the geodesic flow (e.g., see [Kai90, Led95]). Yet another boundary extension of a different kind can be associated with the measure-theoretical Poisson boundaries rather than with the topological ones (see [Kai95]).

Definition 2.12. A measurable system of boundary measures $\{\nu_x\}$ over a treed equivalence relation (X,μ,R,K) is called *conformal of dimension* $\lambda > 0$ if it is quasi-invariant and its Radon–Nikodym derivatives satisfy the relation

$$\frac{d\nu_y}{d\nu_x}(\gamma) = e^{-\lambda\beta_\gamma(x,y)} \quad \text{for } \widetilde{\mu}_\#\text{-a.e. } (x,y,\gamma)\in\widetilde{R} . \tag{2.13}$$

Remark 2.14. This definition is analogous to the definition of conformal streams (measures, densities, see [Pat76, Sul79, KL05]) for negatively curved manifolds. Note that in the group case the identity (2.13) only makes sense in combination with the requirement of equivariance of the map $x \mapsto \nu_x$ (as otherwise one gets a conformal system from an arbitrary measure by multiplying it by the exponent of the Busemann cocycle). In our situation the equivariance condition is replaced with the requirement that the system of measures $\{\nu_x\}$ be measurable (in perfect agreement with the general spirit of the theory of equivalence relations which consists in replacing group invariance with measurablity).

Remark 2.15. If the measure μ is invariant, then, in view of Corollary 2.10, a system $\{\nu_x\}$ is conformal if and only if the logarithm of the Radon–Nikodym cocycle of the measure $\widetilde{\mu}$ with respect to the equivalence relation $\widetilde{R}$ is proportional to the Busemann cocycle on $\widetilde{R}$ with the proportionality coefficient $-\lambda$.

Remark 2.16. If the measure μ is finite invariant, and μ-a.e. tree T_x has at least 3 ends, then in fact a.e. tree has a continuum of ends and the equivalence relation (X, R, μ) is non-amenable [Ada90], which implies that in this situation there are no invariant measurable systems of boundary measures (i.e., there are no conformal systems of dimension 0) [Kai04].

2.D. Horospheric product of treed equivalence relations

Let us first remind that the *product* of two equivalence relations (X, R) and (X', R') is the equivalence relation

$$R \times R' = \{((x, x'), (y, y')) : (x, y) \in R, (x', y') \in R'\}$$

on the state space $X \times X'$. If the relations R, R' are endowed with the respective graph structures K, K', then the product relation carries the natural product graph structure $K \times K'$ (an edge in the product is the product of edges in the multipliers). Finally, if μ (resp., μ') is a R- (resp., R'-) quasi-invariant measure on X (resp., X') with the Radon–Nikodym cocycle Δ (resp., Δ'), then the product measure $\mu \times \mu'$ is $R \times R'$-quasi-invariant, and its Radon–Nikodym cocycle is $\Delta \times \Delta'$.

Let now $(X, R, K), (X', R', K')$ be two treed equivalence relations, and let $(\widetilde{X}, \widetilde{R}, \widetilde{K})$ and $(\widetilde{X}', \widetilde{R}', \widetilde{K}')$ be their respective boundary extensions endowed with the Busemann cocycles $\widetilde{\beta}, \widetilde{\beta}'$ (2.5), so that their product $(\widetilde{X} \times \widetilde{X}', \widetilde{R} \times \widetilde{R}', \widetilde{K} \times \widetilde{K}')$ carries the cocycle $c = \widetilde{\beta} + \widetilde{\beta}'$.

Definition 2.17. The *horospheric product* of treed equivalence relations (X, R, K) and (X', R', K') is the equivalence relation $\boldsymbol{R} = \ker c \subset \widetilde{R} \times \widetilde{R}'$ on the product $\widetilde{X} \times \widetilde{X}'$. It is endowed with the graph structure $\boldsymbol{K} = \boldsymbol{R} \cap \widetilde{K} \times \widetilde{K}'$.

Thus, the $\boldsymbol{R}$-equivalence class of a point $(x, \gamma, x', \gamma') \in \widetilde{X} \times \widetilde{X}'$ endowed with the graph structure $\boldsymbol{K}$ is precisely the horospheric product of the pointed at infinity rooted trees $([x]_\bullet^K, \gamma)$ and $([x']_\bullet^{K'}, \gamma')$ in the sense of Definition 2.2.

Let us now endow the treed equivalence relations (X, R, K) and (X', R', K') with respective quasi-invariant measures μ, μ', let $\{\nu_x\}, \{\nu'_{x'}\}$ be respective quasi-invariant measurable systems of boundary measures, and let $\widetilde{\mu}, \widetilde{\mu}'$ be the corresponding quasi-invariant measures for the boundary extensions $(\widetilde{X}, \widetilde{R}, \widetilde{K})$ and $(\widetilde{X}', \widetilde{R}', \widetilde{K}')$. Then the product measure $\widetilde{\mu} \times \widetilde{\mu}'$ is $\boldsymbol{R}$-quasi-invariant (since $\boldsymbol{R} \subset \widetilde{R} \times \widetilde{R}'$), and its Radon–Nikodym cocycle $\boldsymbol{\Delta}$ is the restriction of the Radon–Nikodym cocycle of the measure $\widetilde{\mu} \times \widetilde{\mu}'$ from $\widetilde{R} \times \widetilde{R}'$ to $\boldsymbol{R}$. Thus, we obtain

Theorem 2.18. *Let (X, μ, R, K) and (X', μ', R', K') be treed equivalence relations with finite invariant measures, and let $\{\nu_x\}, \{\nu'_{x'}\}$ be respective conformal measurable systems of boundary measures of the same dimension $\lambda > 0$. Then the resulting measure $\widetilde{\mu} \times \widetilde{\mu}'$ on the horospheric product of these treed equivalence relations $(\widetilde{X} \times \widetilde{X}', \widetilde{\mu} \times \widetilde{\mu}', \boldsymbol{R}, \boldsymbol{K})$ is $\boldsymbol{R}$-invariant.*

Proof. By Corollary 2.10 and Definition 2.12 the Radon–Nikodym cocycle of the measure $\widetilde{\mu} \times \widetilde{\mu}'$ with respect to the equivalence relation $\widetilde{R} \times \widetilde{R}'$ is

$$e^{-\lambda(\widetilde{\beta}+\widetilde{\beta}')} = e^{-\lambda c} ,$$

where $c = \widetilde{\beta} + \widetilde{\beta}'$ is precisely the cocycle whose kernel determines the horospheric product of our treed equivalence relations. Q.E.D.

Remark 2.19. Our construction of an invariant measure from two conformal systems of boundary measures of the same dimension is based on the same idea as the construction of an invariant measure of the geodesic flow on a negatively curved manifold from a single conformal measure (see [Kai90] and the Appendix in [KL05]). The only difference is that in our situation we deal with two boundary systems rather than one in the classical case.

Theorem 2.20. *Let (X, μ, R, K) and (X', μ', R', K') be treed equivalence relations with finite invariant measures, and let $\{\nu_x\}, \{\nu'_{x'}\}$ be respective conformal measurable systems of boundary measures of the same dimension $\lambda > 0$. If the measures $\widetilde{\mu}, \widetilde{\mu}'$ are both finite, and if the valencies of the structures K, K' are uniformly bounded, then for $\widetilde{\mu} \times \widetilde{\mu}'$-a.e. (x, γ, x', γ') the horospheric product of the pointed at infinity rooted trees $([x]_\bullet^K, \gamma)$ and $([x']_\bullet^{K'}, \gamma')$ is amenable.*

Proof. The treed equivalence relations $(\widetilde{X}, \widetilde{\mu}, \widetilde{R}, \widetilde{K})$ and $(\widetilde{X}', \widetilde{\mu}', \widetilde{R}', \widetilde{K}')$ are both pointed at infinity, and therefore amenable [Kai04]. Thus, their product is also amenable together with the subrelation $\boldsymbol{R}$ (this follows, for instance, from the description of amenable equivalence relations as the ones which are orbit equivalent to $\mathbb{Z}$-actions [CFW81]). On the other hand, since the measure $\widetilde{\mu} \times \widetilde{\mu}'$ is finite and $\boldsymbol{R}$-invariant, amenability of $\boldsymbol{R}$ implies amenability of a.e. associated leafwise graph (see Section 1.D). Q.E.D.

§3. Galton–Watson trees

In this Section we shall discuss an example of invariant measures on treed equivalence relations and the associated horospheric products arising from branching processes.

3.A. Augmented process

Let $p = \{p_k\}$ be a probability distribution on the set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. It gives rise to the random rooted tree $\boldsymbol{T_o}$, which is the "genealogical tree" of the associated *Galton–Watson branching process*: the number of offspring of the progenitor $\boldsymbol{o}$ (the root of the tree) is distributed according to the law p, each of them also produces its own offspring according to the same law and independently of all the rest, etc. (see Figure 4). We shall denote by $\mathbf{P} = \mathbf{P}(p)$ the corresponding probability measure on the space of locally finite rooted trees $\mathcal{T} \subset \mathcal{G}$.

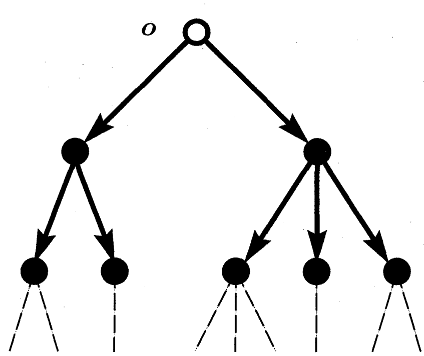

Fig. 4

For simplicity we shall assume that

(3.1)
(i) $p_0 = 0$, so that the extinction probability is 0, and the tree $\boldsymbol{T_o}$ is a.s. infinite and has no leaves;
(ii) the support of the distribution p contains more than one point, so that the tree $\boldsymbol{T_o}$ is a.s. rigid and has a continuum of ends.

The measure $\mathbf{P}$ on the space $\mathcal{T}_\emptyset \subset \mathcal{G}_\emptyset$ of rooted rigid trees is *not* quasi-invariant with respect to the natural equivalence relation $\mathcal{R}$ (or, rather, its restriction to $\mathcal{T}_\emptyset$ which we also denote by $\mathcal{R}$, see Section 1.E). The reason for this is the fact that the root $\boldsymbol{o}$ is different from other vertices of $\boldsymbol{T}$, because statistically it has one neighbour less (we skip the rigorous argument). However, a little modification of the Galton–Watson process (which we describe below) provides an $\mathcal{R}$-invariant measure on $\mathcal{T}_\emptyset$.

The *augmented Galton–Watson process* introduced in [LPP95b] is defined in the same way as the original Galton–Watson process with the only difference that the number of offspring of the progenitor (only) has the distribution $p'_k = p_{k-1}$ (i.e., the root has $k+1$ offspring with probability p_k), and these offspring all have independent standard Galton–Watson descendant trees with offspring distribution $\{p_k\}$. In other words, the number of offspring of the progenitor is "by force" increased by one. Denote by $\mathbf{P}' = \mathbf{P}'(p)$ the associated probability measure on the space of rigid rooted trees $\mathcal{T}_\emptyset$.

3.B. Invariant measure

Theorem 3.2. *The measure* $\mu = \dfrac{1}{\mathsf{deg}}\mathbf{P}'$ *on* $\mathcal{T}_\emptyset$ *is* $\mathcal{R}$*-invariant.*

This theorem was proved in [Kai98] by using the fact that $\mathbf{P}'$ is a stationary measure of the leafwise simple random walk on $(\mathcal{T}_\emptyset, \mathcal{R})$ [LPP95b] and a relation between stationary and invariant measures established in [Kai98] (cf. Section 1.C). For the sake of completeness we shall give a simple direct proof of Theorem 3.2.

Proof of Theorem 3.2. Let

$$\mathcal{R}_1 = \{(x,y) \in \mathcal{R} : d(x,y) = 1\} \subset \mathcal{T}_\emptyset \times \mathcal{T}_\emptyset \ ,$$

where d is the graph metric of the canonical graph structure on the equivalence relation $\mathcal{R}$. One can think about $\mathcal{R}_1$ as the set of *doubly rooted rigid trees*; its elements are triples (T, o, o'), where T is a rigid tree, $o \in T$ is its *principal root*, and its *secondary root* $o' \in T$ is at distance 1 from o. Denote by $\mathbf{P}_{\leftrightarrow}$ the probability measure on $\mathcal{R}_1$ obtained in the following way: consider the principal and the secondary roots as the progenitors of two independent Galton–Watson trees with the distribution p, and then join these roots with an edge, see Figure 5.

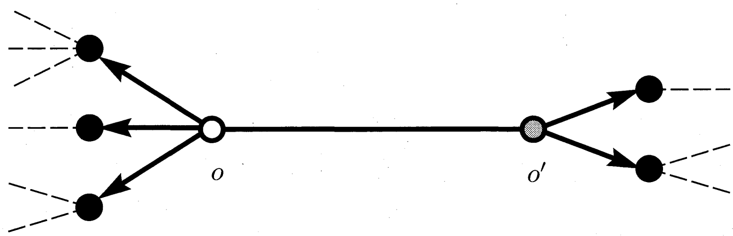

Fig. 5

The measure $\mathbf{P}_{\leftrightarrow}$ coincides with the restriction to $\mathcal{R}_1$ of the left counting measure $\mu_{\#}$ associated with the measure μ. Since $\mathbf{P}_{\leftrightarrow}$ is obviously invariant with respect to the involution $\boldsymbol{\theta}$ (which consists in exchanging the principal and the secondary roots, see Figure 6),

(3.3) any partial transformation of the equivalence relation $\mathcal{R}$ whose graph is contained in $\mathcal{R}_1$ preserves the measure μ.

This property easily implies that the measure μ is preserved by *all* partial transformations of $\mathcal{R}$, i.e, is $\mathcal{R}$-invariant. Indeed, let A be a μ-negligible subset of $\mathcal{T}_{\not\emptyset}$. Then by (3.3) its 1-neighbourhood (with respect to the leafwise graph distance) is μ-negligible as well, and so on, so that the $\mathcal{R}$-saturation of A is also μ-negligible. Thus, μ is $\mathcal{R}$-quasi-invariant. By (3.3) its Radon–Nikodym cocycle Δ is identically 1 on $\mathcal{R}_1$; therefore by the cocycle identity $\Delta \equiv 1$ on $\mathcal{R}$. Q.E.D.

3.C. Boundary measure

Given a rooted tree T_o denote by $S_o^n \subset T$ the n-sphere centered at the root o. If the distribution p has a finite first moment

$$m = \sum k\, p_k \ ,$$

then by (3.1) $m > 1$ (the associated branching process is *supercritical*), and, as it was noticed already by Doob (this was one of the first applications of the martingale theory), for $\mathbf{P}$-a.e. Galton–Watson tree T_o there exists a limit

$$\lim_{n\to\infty} \frac{|S_o^n|}{m^n} = L \ . \tag{3.4}$$

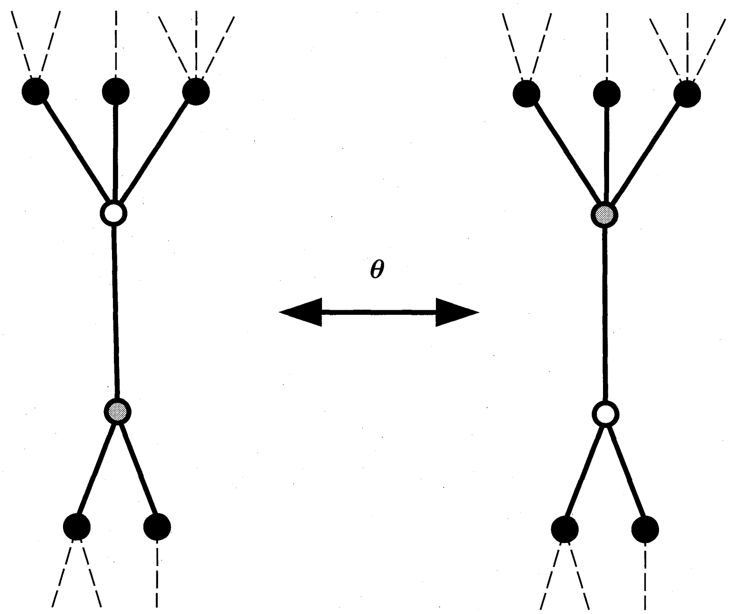

Fig. 6

Earlier works containing sufficient conditions for a.s. positivity of the limit (3.4) [Har48, Lev59] culminated in the following

Theorem 3.5 (Kesten–Stigum [KS66, AN72, LPP95a]). *Under the assumption* $p_0 = 0$ *either*

(i) $\sum k \log k \, p_k < \infty$,
(ii) $L > 0$ $\mathbf{P}$-*a.s.*,
(iii) $\mathbf{E}L = 1$, *where* $\mathbf{E}$ *denotes the expectation with respect to the measure* $\mathbf{P}$;

or

(i′) $\sum k \log k \, p_k = \infty$,
(ii′) $L = 0$ $\mathbf{P}$-*a.s.*

The idea that existence and positivity of the limit (3.4) can be used in order to define a measure on the boundary of the Galton–Watson tree is very natural, and apparently for the first time appeared in [Hol73]. Nowadays this boundary measure is usually known under the name of the *branching measure* (for instance, see [Liu01] and the references therein).

We shall need this result in a slightly modified form: for the augmented Galton–Watson trees instead of the usual ones.

Theorem 3.6. *Denote by* $\#_o^n$ *the counting measure on the* n*-sphere* S_o^n *of a rooted tree* T_o*. If the distribution* $p = (p_k)$ *satisfies condition* (i) *from Theorem 3.5 then the limit measure (with respect to the weak* topology on the compactification* $\overline{T}$*)*

$$\nu = \lim_n \frac{\#_o^n}{m^n} \tag{3.7}$$

exists for $\mathbf{P}'$*-a.e. tree* T_o*, and the expectation of its norm is*

$$\mathbf{E}'\|\nu\| = 1 + \frac{1}{m}\,. \tag{3.8}$$

Proof. For a point $x \in T \setminus \{o\}$ denote by $\mathscr{S}_o^x \subset \partial T$ the *shadow* of the point x as viewed from the root o, i.e., the set of endpoints of all geodesic rays issued from o and passing through x, see Figure 7.

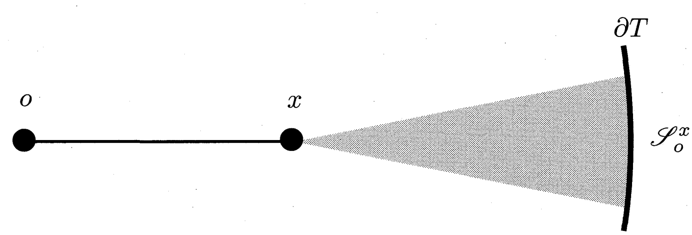

Fig. 7

Weak convergence of the sequence (3.7) is equivalent to convergence of the sequences

$$\frac{\#_o^n(\mathscr{S}_o^x)}{m^n} = \frac{|S_o^n \cap \mathscr{S}_o^x|}{m^n}$$

for all $x \in T$, which follows from (3.4), because $\mathscr{S}_o^x \cap T$ is a $\mathbf{P}$-distributed Galton–Watson tree growing from the root x. Formula (3.8) is then the result of "rescaling" property (iii) from Theorem 3.5: the expected number of offspring of the root is $m+1$, whereas for each of them the expected mass of the boundary measure is $1/m$. Q.E.D.

Now we can endow the treed equivalence relation $(\mathcal{T}_\emptyset, \mathbf{P}', \mathcal{R}, \mathcal{K})$ with the measurable system of boundary measures $\{\nu_x\}$, $x \in \mathcal{T}_\emptyset$ arising from

Theorem 3.6. The fact that the measures ν_x come from the counting measures on spheres rescaled by powers of the constant m immediately implies

Theorem 3.9. *Under conditions of Theorem 3.6 the system of boundary measures $\{\nu_x\}$ on the treed equivalence relation $(\mathcal{T}_{\not{\emptyset}}, \mathbf{P}', \mathcal{R}, \mathcal{K})$ is conformal with the exponent $\lambda = \log m$.*

In view of Theorem 2.18 we now obtain

Theorem 3.10. *Let $p = (p_k)$ and $q = (q_k)$ be two distributions satisfying conditions* (3.1) *and condition* (i) *from Theorem 3.5, and such that they have the same mean*

$$m = \sum k\, p_k = \sum k\, q_k \ .$$

Denote by $\mathbf{P}'$ and $\mathbf{Q}'$ the respective augmented Galton–Watson measures on $\mathcal{T}_{\not{\emptyset}}$, and let measures $\widetilde{\mathbf{P}}'$ and $\widetilde{\mathbf{Q}}'$ on the boundary bundle $\widetilde{\mathcal{T}}_{\not{\emptyset}}$ be the boundary extensions of $\mathbf{P}'$ and $\mathbf{Q}'$ determined by the respective systems of boundary measures from Theorem 3.6. Then the image of the product measure $\widetilde{\mathbf{P}}' \times \widetilde{\mathbf{Q}}'$ under the map $(T, T') \mapsto T \upharpoonleft\!\downharpoonright T'$ is an $\mathcal{R}$-invariant finite measure on $\mathcal{G}_{\not{\emptyset}}$.

By Theorem 2.20 it implies

Corollary 3.11. *If the distributions $\{p_k\}$ and $\{q_k\}$ are in addition finitely supported, then the horospheric product $T \upharpoonleft\!\downharpoonright T'$ of $\widetilde{\mathbf{P}}' \times \widetilde{\mathbf{Q}}'$-a.e. pair of pointed at infinity rooted trees (T, T') is amenable.*

Remark 3.12. It is well-known that the horospheric product of two homogeneous trees is amenable if and only if they have the same degrees (e.g., see [Woe05]). This Corollary can be considered as an analogue of this result. Actually, for proving it one does not need all the machinery above. Indeed, if the distributions $\{p_k\}$ and $\{q_k\}$ have a common point t in their supports, then the corresponding Galton–Watson trees with probability 1 contain as subgraphs arbitrarily large balls of the homogeneous tree $\mathbb{T}_{t+1}$. An easy estimate then shows that products of these subgraphs will produce Følner sets in the horospheric product (cf. [Sob09]). In fact, a similar argument works also in the situation when the convex hulls of these supports intersect, i.e.,

$$[\min\{p_k\}, \max\{p_k\}] \cap [\min\{q_k\}, \max\{q_k\}] \neq \varnothing \ .$$

[It would be interesting to study the applicability of this argument to other invariant measures on $\mathcal{T}_{\not{\emptyset}}$.] On the other hand, if these intervals do not intersect, then, *mutatis mutandis*, $\min\{p_k\} \geq \max\{q_k\} + 1$, which means that in the horospheric product all vertex degrees of one multiplier will be a.s. strictly less than all vertex degrees of the other multiplier. By comparing the simple random walk on the horospheric product of two such trees with an appropriate biased simple random walk on $\mathbb{Z}$, one can conclude that in this case the return probabilities decay exponentially, and therefore the horospheric product is non-amenable.

Remark 3.13. A connected graph of bounded geometry Γ is called *strongly amenable* if it admits a Følner sequence consisting of connected sets which all contain a chosen reference vertex. Otherwise Γ is said to be *weakly non-amenable* or to have the *anchored expansion property* [Tho92, BLS99, HSS00]. It was proved in [Sob09] that horospheric products of percolation subtrees in a homogeneous tree are a.s. strongly amenable. It would be interesting to address this problem for more general random horospheric products.

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