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A note on rough differential equations with unbounded coefficients

Yuzuru Inahama

Abstract.

By using a cutoff technique, we make a rough path approach to SDEs with linear growth coefficients. As an application, we improve Fang–Zhang's large deviation for loop group valued Brownian motion.

§1. Introduction

In the rough path theory, T. Lyons extended the notion of integration along a path and ordinary integral differential equations (ODEs). See [19, 18]. The key point is that not only the path itself (i.e., the first level path), but also the iterated integral of the path (i.e., the second level path) is considered. Such a pair of the first and the second level paths are called a rough path. In this theory, the Itô map is deterministic and, moreover, is continuous. (Lyons' continuity theorem or also known as the universal limit theorem.) This is quite different from the usual Itô calculus, in which the Itô map is measurable, but not continuous. If we put a Brownian-like measure on the space of rough path, then we obtain a solution of the corresponding Stratonovich stochastic differential equation (SDE). Thus, in the rough path theory, diffusion processes are obtained as the image of continuous Itô maps.

Let $\sigma : \mathbf{R}^d \to \mathrm{Mat}(d,n) = \mathbf{R}^d \otimes (\mathbf{R}^n)^*$ and $b : \mathbf{R}^d \to \mathbf{R}^d$ be nice coefficients with certain regularity. In this section we consider the follwoing differential equation in $\mathbf{R}^d$: for a given $\mathbf{R}^n$-valued nice continuous path X,

$$dY_t = \sigma(Y_t)dX_t + b(Y_t)dt, \qquad Y_0 = 0. \tag{1}$$

The correspondence $X \mapsto Y = \Phi(X)$ is called the Itô map.

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Let us regard the equation (1) as an ODE in the rough path sense. It is well-known that, if σ and b are of C_b^3, then the solution $Y \in G\Omega_p(\mathbf{R}^d)$ exists for any $X \in G\Omega_p(\mathbf{R}^n)$ and that the Itô map Φ is locally Lipschitz continuous with resprect to this topology. Moreover, if W is the Brownian rough paths, then, the first level path of $Y = \Phi(W)$ is a solution of the corresponding (stratonovich-type) SDE.

It is well-known that the SDE has the strong solution even when the coefficients σ and b are of linear growth (and locally Lipschitz). Until recently, there seemed to be no published papers to discuss the continuity theorem for the case of unbounded coefficients, although three results have just been published. (See Section 10-7, Friz and Victoir [8], Lejay [17], Gubinelli and Lejay [10]. Basically, these results are for geometric rough paths in finite dimensional setting, except results for Lip$(2+\varepsilon)$-type coefficient in [17] are in Banach setting.) The aim of this paper is to show that, in some sense (almost surely with respect to the law of the Brownian rough paths), the solution can be written in the image of the Itô map in the rough path sense. (See Theorem 2.1).

As a slight modification of (1), we have the following equation:

$$dY_t^\varepsilon = \sigma(Y_t^\varepsilon)\varepsilon dW_t + a(\varepsilon)b(Y_t^\varepsilon)dt, \qquad Y_0^\varepsilon = 0. \tag{2}$$

Here, $\varepsilon > 0$ is a small parameter $\varepsilon > 0$ and $a : [0,\infty) \to \mathbf{R}$ is a nice function of ε. We will also prove the large deviation principle of the Freidlin–Wentzell type and Laplace's method for Y^ε. (See Proposition 2.2 and Theorem 2.4. This is a very classical problem. See [2, 3] for example. When σ and b are C_b^3 or C_b^∞, the rough path proof for the large deviation is given in [16] and the Laplace asymptotics is given in [1, 11, 13, 14].)

In the last section we apply the same arguments to a certain infinite dimensional diffusion, namely the Brownian motion on loop groups over a compact Lie groups, to prove a large deviation with respect to the topology induced by a Besov-type norm, which is stronger than the usual sup-norm. (See Corollary 3.2.) This large deviation with respect to the usual topology was first shown in [8] and then, in [12] with the rough path theory.

The method in this paper is somewhat ad-hoc because it is a combination of the rough path theory and the usual Itô calculus. So, one may think that the argument in this paper could be replaced with more sophisticated ones in such papers as [9, 17, 10]. However, the cutoff argument in this paper is not completely useless. For example, in this paper, the coefficient whose derivatives (of order 2,3) are not bounded is treated. (And we consider the Banach cases, too.)

In particular, the main example (Malliavin's loop group valued process) is such a case. In this example, the coefficient is called "Nemetskii map" and, with respect to this Besov-type topology $\mathcal{X}_{\theta,q}$, it is of linear growth but not in general bounded. (See [6].) A simple computation shows that the derivative of a Nemetskii map is again a Nemetskii map of the same form, therefore not bounded either.

§2. A rough differential equation with coefficients of linear growth

2.1. A very simple review of the rough path theory

Now we briefly recall the definition and basic properties of the space of geometric rough path and the Brownian rough paths. (See Lyons and Qian [19].) For later use, we give definitions in infinite dimensional setting. Let $\mathcal{B}$ be a real Banach space. We denote by $\mathcal{B}\otimes\mathcal{B}$ the projective tensor product. In the following we only consider the projective norm on $\mathcal{B}\otimes\mathcal{B}$. When $\mathcal{B}=\mathbf{R}^n$, this coincides with the usual tensor product. Let $[0,1]$ be the time interval as usual and let $p\in(2,3)$ be the roughness, which will be fixed throughout the paper.

A continuous map $X=(1,X^1,X^2)$ from the simplex $\Delta=\{(s,t)\,|\,0\le s\le t\le 1\}$ to the truncated tensor algebra $T^{(2)}(\mathcal{B})=\mathbf{R}\oplus\mathcal{B}\oplus(\mathcal{B}\otimes\mathcal{B})$ is said to be a rough path of roughness p if, for every $s\le u\le t$, $X_{s,t}=X_{s,u}\otimes X_{u,t}$ and

$$\Big(\sup_D\sum_l |X^j_{t_{l-1},t_l}|^{p/j}\Big)^{j/p}<\infty \qquad \text{for } j=1,2$$

hold, where $D=\{0=t_0<t_1<\cdots<t_N=1\}$ runs over all finite partition of $[0,1]$. For two rough paths X and Y, p-variation distance is defined as follows:

$$d_p(X,Y)=\max_{j=1,2}\Big(\sup_D\sum_l |X^j_{t_{l-1},t_l}-Y^j_{t_{l-1},t_l}|^{p/j}\Big)^{j/p}.$$

Let $x=(x_t)_{0\le t\le 1}$ be a $\mathcal{B}$-valued continuous path of finite total variation and set $X^1_{s,t}:=x_t-x_s$, $X^2_{s,t}:=\int_s^t(x_u-x_s)\otimes dx_u$ for $(s,t)\in\triangle$. A rough path obtained in this way is called a smooth rough path (lying above x). A rough path obtained as the d_p-limit of a sequence of smooth rough paths is called a geometric rough path. The space of all the $\mathcal{B}$-valued geometric rough paths is denoted by $G\Omega_p(\mathcal{B})$. Thus, the space of $\mathcal{B}$-valued continuous paths of finite total variation is continuously imbedded in $G\Omega_p(\mathcal{B})$.

Let $\hat{\mathcal{B}}$ be another real Banach space and we denote by $L(\mathcal{B},\hat{\mathcal{B}})$ the set of all bounded linear maps from $\mathcal{B}$ to $\hat{\mathcal{B}}$. Let $\sigma : \hat{\mathcal{B}} \to L(\mathcal{B},\hat{\mathcal{B}})$ be C_b^3 in the Fréchet sense (i.e, $\nabla^j\sigma$ are continuous and bounded for all $j = 0, 1, 2, 3$). Consider the following differential equation:

$$dy_t = \sigma(y_t) \cdot dx_t \qquad \text{with } y_0 \in \hat{\mathcal{B}}.$$

If x is $\mathcal{B}$-valued continuous and of bounded variation, then a unique solution y exists, which is again continuous and of bounded variation. Let us denote $y = \Phi(x)$. Then, by Lyons' continuity theorem, the Itô map Φ naturally extends to a continuous map from $G\Omega_p(B)$ to $G\Omega_p(\hat{B})$. (i.e., $Y = \Phi(X)$. See Section 6.2, Lyons and Qian [19].)

A Gaussian measure μ on $\mathcal{B}$ is said to be exact (with respect to the projective norm on $\mathcal{B}\otimes\mathcal{B}$) if there exist constants $\alpha < 1$ and $c > 0$ such that

$$\mathbb{E}\Big|\sum_{i=1}^{N} \eta_{2j-1} \otimes \eta_{2j}\Big|_{\mathcal{B}\otimes\mathcal{B}} \leq cN^{\alpha} \qquad \text{for all } N \in \mathbf{N}.$$

Here, $\{\eta_j\}_{j=1,2,\dots}$ are $\mathcal{B}$-valued i.i.d., with the law of η_1 being μ. When $\dim\mathcal{B} < \infty$, we can easily see that the standard normal distribution is exact with $\alpha = 1/2$.

Let $\mathbb{P}$ be the law of $\mathcal{B}$-valued Brownian motion associated with μ. This is a probability measure on the path space $\mathcal{P}(\mathcal{B}) = \{w : [0,1] \to \mathcal{B} | w \text{ is continuous and } w_0 = 0\}$.

Set

$$(3) \qquad \mathcal{S} = \{w \in \mathcal{P}(\mathcal{B}) |\ \{W(m)\}_{n=1,2,\dots} \text{ is Cauchy in } G\Omega_p(\mathcal{B})\ \},$$

where $w(n)$ is the piecewise linear approximation for w associated with the partition $\{k/2^m\}_{k=1,\dots,2^m}$ of $[0,1]$ and $W(m)$ is the smooth rough path lying above it. Then, it is proved in Ledoux, Lyons and Qian [15] that under exactness condition, Brownian rough paths $W := \lim_{m\to\infty} W(m)$ exist, i.e., $\mathbb{P}(\mathcal{S}^c) = 0$.

When $\dim\mathcal{B}, \dim\hat{\mathcal{B}} < \infty$, Lyons' continuity theorem and the Wong–Zakai approximation imply that $y_t := y_0 + \Phi(W)_1(0,t)$ satisfies the following Stratonovich stochastic differential equation (SDE):

$$dy_t = \sigma(y_t) \circ dw_t \qquad \text{with } y_0 \in \hat{B}.$$

(SDEs with a drift term can be treated in a simlar way.)

2.2. Almost sure existence of the solution

In this and in the next subsetions, we work in a finite dimensional setting and we assume that σ and b are C^3 and at most of linear growth.

In this case, the SDE

$$(4) \qquad dy_t = \sigma(y_t) \circ dw_t + b(y_t)dt = \sigma(y_t)dw_t + \hat{b}(y_t)dt, \qquad y_0 = 0.$$

has the strong solution (i.e., the existence and the pathwise uniqueness hold for y). Here,

$$\hat{b}(y) = b(y) + \frac{1}{2}\mathrm{Tr}\big[\nabla\sigma(y)\langle\sigma(y)*, *\rangle\big], \qquad y \in \mathbf{R}^d$$

as usual and $(w_t)_{t\in[0,1]}$ is the standard n-dimensional Brownian motion. Note also that, if $X \in \mathrm{BV}(\mathbf{R}^n)$, the corresponding ODE (1) has a unique solution Y, and the mapping $X \in \mathrm{BV}(\mathbf{R}^n) \mapsto Y \in \mathrm{BV}(\mathbf{R}^d)$ is continuous.

In the sequel, we will use the following notations. Let $\chi : \mathbf{R} \to [0,1]$ be a C^3 function such that $\chi(r) = 1$ for $r \le 0$ and $\chi(r) = 0$ for $r \ge 1$. (Note that χ can be chosen to be C^∞.) We set, for $n = 1, 2, \ldots$, $\chi_n(r) = \chi(r-n)$ and $\sigma_n(y) = \sigma(y)\chi_n(|y|)$, $b_n(y) = b(y)\chi_n(|y|)$, etc. Clearly, supports of σ_n and b_n are contained in $\{y \in \mathbf{R}^d \mid |y| \le n+1\}$.

Theorem 2.1. *Assume that σ and b are C^3 and σ, b, and $\hat{b}$ are of at most linear growth. Let W be the Brownian rough paths lying above the standard n-dimensional Brownian motion $(w_t)_{t\in[0,1]}$. Then, there are an open subset $\mathcal{O} \subset G\Omega_p(\mathbf{R}^n)$ and a continuous map $\Phi : \mathcal{O} \to G\Omega_p(\mathbf{R}^d)$ such that the following hold:*
(i) *$\mathcal{O}$ is of measure 1 with respect to the law of the Brownian rough paths.*
(ii) *$\mathrm{BV}(\mathbf{R}^n) \subset \mathcal{O}$ and, for $X \in \mathrm{BV}(\mathbf{R}^n)$, $t \mapsto \Phi(X)^1_{0,t}$ solves the ODE (1).*
(iii) *$t \mapsto \Phi(W)^1_{0,t}$ defined on the probability space $\mathcal{O}$ solves the SDE (4).*

Proof. Let $\mathcal{P}^n = \{w : C([0,1], \mathbf{R}^n) \mid w_0 = 0\}$. For $w \in \mathcal{P}^n$, $w(m) \in BV(\mathbf{R}^n)$ denotes the mth dyadic piecewise linear approximation of w and $W(m) \in G\Omega_p(\mathbf{R}^n)$ denote the geometric rough path lying above $w(m)$. Set $\mathcal{S}_1 = \{w \in \mathcal{P}^n \mid W(m)$ is Cauchy in $G\Omega_p(\mathbf{R}^n)\}$. This set is of full measure with respect to the standard Wiener measure. We write $W := \lim_{n\to\infty} W(m)$. By this injection $w \mapsto W$, $\mathcal{S}_1$ can also be regarded as a subset of $G\Omega_p(\mathbf{R}^n)$. Also set

$$\mathcal{S}_2 = \{w \in \mathcal{P}^n \mid \sup\nolimits_{0\le t\le 1} |y(m)_t| \text{ is bounded in } m\}.$$

And $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$. Here, $y(m) \in \mathrm{BV}(\mathbf{R}^d)$ is the solution of (1) for $X = w(m) \in \mathrm{BV}(\mathbf{R}^n)$.

Firstly, we show that $\mathcal{S}$ is of full measure. In order to show it, it is sufficient to show that

$$\mathbb{P}(w \in \mathcal{P}^n \mid \lim_{m\to\infty} \sup_{0\le t\le 1} |y(m)_t - y_t| = 0) = 1. \tag{5}$$

(A weak form of the Wong–Zakai approximation theorem.)

Let $k = 1, 2, \ldots$ and $(y_t)_{t\in[0,1]}$ be as in (4). Define a stopping time τ_k by $\tau_k = \inf\{t \ge 0 \mid |y_t| \ge k\} \wedge 1$. Consider the following equation:

$$dy_t^{(k)} = \sigma_k(y_t^{(k)}) \circ dw_t + b_k(y_t^{(k)})dt, \qquad y_0^{(k)} = 0. \tag{6}$$

Set a stopping time $\hat{\tau}_k$ by $\hat{\tau}_k = \inf\{t \ge 0 \mid |y_t^{(k)}| \ge k\} \wedge 1$. From (4) and (6), we have

$$\begin{aligned} y_{t\wedge\tau_k\wedge\hat{\tau}_k} - y^{(k)}_{t\wedge\tau_k\wedge\hat{\tau}_k} &= \int_0^{t\wedge\tau_k\wedge\hat{\tau}_k} \sigma(y_s)dw_s - \int_0^{t\wedge\tau_k\wedge\hat{\tau}_k} \sigma_k(y_s^{(k)})dw_s \\ &\quad + \int_0^{t\wedge\tau_k\wedge\hat{\tau}_k} \hat{b}(y_s)ds - \int_0^{t\wedge\tau_k\wedge\hat{\tau}_k} \hat{b}_k(y_s^{(k)})ds \\ &= \int_0^{t\wedge\tau_k\wedge\hat{\tau}_k} [\sigma(y_s) - \sigma(y_s^{(k)})]dw_s \\ &\quad + \int_0^{t\wedge\tau_k\wedge\hat{\tau}_k} [\hat{b}(y_s) - \hat{b}(y_s^{(k)})]ds. \end{aligned}$$

It is a routine to obtain with Burkholder's and Gronwall's inequalities that, almost surely, $y_{t\wedge\tau_k\wedge\hat{\tau}_k} = y^{(k)}_{t\wedge\tau_k\wedge\hat{\tau}_k}$ for all t. Note that $\{\tau_k = 1\} = \{|y_t| \le k, \quad \forall t \in [0,1]\}$ and $y = y^{(k)}$ on this set. Since the coefficients of (6) are C_b^2, the Wong–Zakai approximation theorem holds for $y^{(k)}$. Noting that the union $\cup_{k=1}^{\infty}\{\tau_k = 1\}$ is of measure 1, we see that (5) holds.

Secondly, we show that $\mathrm{BV}(\mathbf{R}^n) \subset \mathcal{S}$. Since $\mathrm{BV}(\mathbf{R}^n) \subset \mathcal{S}_1$ is known, we show that $\mathrm{BV}(\mathbf{R}^n) \subset \mathcal{S}_2$. Let $X \in \mathrm{BV}(\mathbf{R}^n)$ and let Y be the solution of (1). Denote by $\|X\|_{1,[0,t]}$ be the total variation of X restricted on the subinterval $[0,t]$. In particular, $\|X\|_{1,[0,1]} = \|X\|_{\mathrm{BV}}$. By the assumption, there exists a positive constant $K > 0$ such that $|\sigma(y)| + |b(y)| \le K(1 + |y|)$ for all $y \in \mathbf{R}^d$. Therefore,

$$\begin{aligned} |Y_t| &\le K\int_0^t (1 + |Y_s|)d(\|X\|_{1,[0,s]} + s) \\ &= K(\|X\|_{\mathrm{BV}} + 1) + K\int_0^t |Y_s|d(\|X\|_{1,[0,s]} + s). \end{aligned}$$

By Gronwall's inequality, we have from this that

$$|Y_t| \le K(\|X\|_{\mathrm{BV}} + 1)e^{\|X\|_{1,[0,t]}+t} \le K(\|X\|_{\mathrm{BV}} + 1)e^{\|X\|_{\mathrm{BV}}+1}.$$

Noting that $\|X(k)\|_{\mathrm{BV}} \le \|X\|_{\mathrm{BV}}$ for all $k = 1, 2, \ldots$, we see that $X \in \mathcal{S}_2$.

Finally, we construct Φ. Set, for $r = 1, 2, \ldots$, $\mathcal{S}^r = \mathcal{S}_1 \cap \mathcal{S}_2^r$, where

$$\mathcal{S}_2^r = \{w \in \mathcal{P}^n \mid \sup_{0\le t\le 1} |y(k)_t| \le r - 1/2 \text{ for all } k\}. \tag{7}$$

Let $x \in \mathcal{S}_1 \cap \mathcal{S}_2^r$ and let $X \in G\Omega_p(\mathbf{R}^n)$ be the lift of x. Consider the following ODE:

$$dY_t^{(r)} = \sigma_r(Y_t^{(r)})dX_t + b_r(Y_t^{(r)})dt, \qquad Y_0^{(r)} = 0. \tag{8}$$

Since σ_r and b_r are C_b^3, we may think of the rough ODE which corresponds (8) as follows. Set $\hat{\sigma}_r : \mathbf{R}^d \to \mathrm{Mat}(d, n+1)$ by

$$\hat{\sigma}_r(y)\langle \xi, s\rangle = \sigma_r(y)\xi + b_r(y)s, \qquad \xi \in \mathbf{R}^n, s \in \mathbf{R}, y \in \mathbf{R}^d.$$

Let $\Psi_r : G\Omega_p(\mathbf{R}^{n+1}) \to G\Omega_p(\mathbf{R}^d)$ be the Itô map which corresponds to $\hat{\sigma}_r$. Since $\hat{\sigma}_r$ is C_b^3, Ψ_r is locally Lipschitz continuous. Let $\iota : G\Omega_p(\mathbf{R}^n) \times \mathrm{BV}(\mathbf{R}) \to G\Omega_p(\mathbf{R}^{n+1})$ be the continuous map, which naturally extends $\mathrm{BV}(\mathbf{R}^n) \times \mathrm{BV}(\mathbf{R}) \cong \mathrm{BV}(\mathbf{R}^{n+1})$. Then, $\Phi_r(X) := \Psi_r(\iota(X, \lambda))$ is the solution of the rough ODE (8), where $\lambda_t = t$. Since $\Phi_r : G\Omega_p(\mathbf{R}^n) \to G\Omega_p(\mathbf{R}^d)$ is continuous, there is an open neighborhood $\mathcal{U}_X$ of X such that $\sup_{0\le t\le 1} |\Phi_r(\hat{X})^1_{0,t}| \le r$ for any $\hat{X} \in \mathcal{U}_X$. We set $\mathcal{O}_r = \cup_{x\in\mathcal{S}_1\cap\mathcal{S}_2^r}\mathcal{U}_X$. If $\hat{X} \in \mathcal{O}_r \cap \mathcal{O}_{r'}$ for $r < r'$, then by the uniqueness of the ODE (8) for r', we have $\Phi_r(\hat{X}) = \Phi_{r'}(\hat{X})$. Hence, we may set $\mathcal{O} = \cup_{r=1}^\infty \mathcal{O}_r$ and $\Phi : \mathcal{O} \to G\Omega_p(\mathbf{R}^d)$ by $\Phi|_{\mathcal{O}_r} = \Phi_r$. By the continuity of Φ and the (weak) Wong–Zakai theorem for the SDE (4), we see that the law of $\Phi(W)$ ($W \in \mathcal{O}$) is the solution of (4). Q.E.D.

2.3. The large deviation principle of the Freidlin–Wentzell type

For $\varepsilon \in [0, 1]$ and a continuous function $a : [0, 1] \to \mathbf{R}$, let us consider the following SDE:

$$dy_t^\varepsilon = \sigma(y_t^\varepsilon) \circ \varepsilon dw_t + a(\varepsilon)b(y_t^\varepsilon)dt, \qquad y_0^\varepsilon = 0. \tag{9}$$

A typical example of a is $a(\varepsilon) = 1$ or $a(\varepsilon) = \varepsilon^2$. In this subsection, by using the rough path theory, we prove the large deviation principle for the law of y^ε as $\varepsilon \searrow 0$.

Let $\mathcal{H}^n (\subset \mathcal{P}^n)$ be the Cameron–Martin space. For $h \in \mathcal{H}^n$, we consider the following ODE in the usual sense:

$$dY_t^h = \sigma(Y_t^h)dh_t + a(0)b(Y_t^h)dt, \qquad Y_0^h = 0. \tag{10}$$

It is obvious that $Y^h \in \mathcal{H}^d$. For $z \in \mathcal{P}^d$, we set

$$\hat{I}(z) = \begin{cases} \inf\{\|h\|^2_{\mathcal{H}^n}/2 \mid z = Y^h\}, & \text{if } z = Y^h \text{ for some } h \in \mathcal{H}^n, \\ \infty, & \textit{otherwise}. \end{cases}$$

In the following theorem, there is no condition on the growth order of the second and the third derivatives of σ and b.

Proposition 2.2. *Assume that σ and b are C^3 and $\sigma, b, \hat{b}$ are of at most linear growth. Let y^ε be as in the SDE (9) and let $\hat{I}$ be as above. Then, the law of y^ε satisfies the large deviation principle as $\varepsilon \searrow 0$ with a good rate function $\hat{I}$.*

Before proving this proposition, we give a simple lemma. It is well-known that the large deviation is transferred by a continuous map. That is called the contraction principle. The following is a slight modification of it.

Lemma 2.3. *Let S and $\hat{S}$ be polish spaces and $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on S which satisfies the large deviation principle with a good rate function I. Set $H = \{a \in S \mid I(a) < \infty\}$. Assume that $f : S \to \hat{S}$ is a measurable map and U is an open subset of S such that $f|_U$ is continuous and $H \subset U$. Then, $\{\mu_\varepsilon \circ f^{-1}\}_{\varepsilon>0}$ satisfies the large deviation principle with a good rate function $\hat{I}$, where $\hat{I}(b) = \inf\{I(a) \mid a \in f^{-1}(\{b\})\}$.*

Proof. **(i)** Let $O \subset \hat{S}$ be an open set and assume that $c := \inf_{y \in O} \hat{I}(y) < \infty$. For any $\delta > 0$, there exists $b \in O$ such that

$$c + \delta > \hat{I}(b) = \inf\{I(a) \mid a \in f^{-1}(\{b\})\}.$$

$f^{-1}(O)$ includes $f^{-1}(O) \cap U$, which is open in U, and hence in S. So, there exists $a \in U \cap f^{-1}(\{b\})$ such that $c + \delta > I(a)$. From this we see that

$$\begin{aligned} -(c+\delta) &\le -I(a) \le -\inf\{I(x) \mid x \in U \cap f^{-1}(O)\} \\ &\le \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \mu_\varepsilon(U \cap f^{-1}(O)) \le \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \mu_\varepsilon(f^{-1}(O)), \end{aligned}$$

where we used the large deviation for $\{\mu_\varepsilon\}_{\varepsilon>0}$ in the third inequality. Letting $\delta \searrow 0$, we obtain the desired inequality.
(ii) Let $F \subset \hat{S}$ be a closed set. Then, $U \cap f^{-1}(F)$ is closed in U with respect to the relative topology, that is, $U \cap \tilde{F} = U \cap f^{-1}(F)$ for some closed subset $\tilde{F}$ of S. Since $U^c \cap f^{-1}(F) \subset U^c$, $f^{-1}(F) \subset \tilde{F} \cap U^c$,

where the right hand side is closed in S. By using the large deviation for $\{\mu_\varepsilon\}_{\varepsilon>0}$, we have

$$\limsup_{\varepsilon\searrow 0}\varepsilon^2\log\mu_\varepsilon(f^{-1}(F))\le\limsup_{\varepsilon\searrow 0}\varepsilon^2\log\mu_\varepsilon(\tilde{F}\cap U^c)$$
$$\le-\inf\{I(x)\mid x\in\tilde{F}\cap U^c\}=-\inf\{I(x)\mid x\in f^{-1}(F)\}.$$

It is a routine to prove the goodness of $\hat{I}$. Q.E.D.

Proof of Proposition 2.2. Set $\lambda_t = t$ and $A=\{a\lambda\mid a\in\mathbf{R}\}\subset \mathrm{BV}(\mathbf{R})$. Consider the following ODE for $w\in\mathcal{P}^n$:

(11) $$dy(k,a)_t=\sigma(y(k,a)_t)dw(k)_t+b(y(k,a)_t)d(a\lambda_t),\qquad y(k,a)_0=0.$$

For $a\in\mathbf{R}$, set $\mathcal{S}(a)=\mathcal{S}_1\cap\mathcal{S}_2(a)$, where

$$\mathcal{S}_2(a)=\{w\in\mathcal{P}^n\mid \sup_{0\le t\le 1}|y(k,a)_t|\text{ is bounded in }k\}.$$

Set $\mathcal{S}_2(a)^r$ in the same way as in (7) and set $\mathcal{S}(a)^r=\mathcal{S}_1\cap\mathcal{S}_2(a)^r$.

For $x\in\mathcal{S}(a)^r$ and its lift X, there exist open set $\mathcal{U}_{X,a}\subset G\Omega_p(\mathbf{R}^n)$ and $\mathcal{V}_{X,a}\subset A$ such that $\sup_{0\le t\le 1}|\Psi_r\circ\iota(X',\lambda')^1_{0,t}|\le r$ for any $(X',\lambda')\in\mathcal{U}_{X,a}\times\mathcal{V}_{X,a}$. Then, as before, set

$$\mathcal{O}_r=\cup\{\mathcal{U}_{X,a}\times\mathcal{V}_{X,a}\mid x\in\mathcal{S}(a)^r,a\in\mathbf{R}\}\quad\text{and}\quad\mathcal{O}=\cup_{r=1}^{\infty}\mathcal{O}_r.$$

By setting $\Phi(X',\lambda')=\Psi_r\circ\iota(X',\lambda')$ for $(X',\lambda')\in\mathcal{O}_r$, we have a well-defined continuous map $\Phi:\mathcal{O}\to G\Omega_p(\mathbf{R}^d)$.

For $\varepsilon\in[0,1]$, let $\mathbb{P}_\varepsilon$ be the law of the scaled Brownian rough paths εW and let $\delta_{a(\varepsilon)\lambda}$ be a point mass at $a(\varepsilon)\lambda\in A$. Then, the product measuse $\mathbb{P}_\varepsilon\times\delta_{a(\varepsilon)\lambda}$ is a probability measure on $G\Omega_p(\mathbf{R}^n)\times A$. In the same way as in the proof of Theorem 2.1, we can prove that, for any ε, (i) $\mathbb{P}_\varepsilon\times\delta_{a(\varepsilon)\lambda}(\mathcal{O})=1$, and (ii) the law of Φ^1 under $\mathbb{P}_\varepsilon\times\delta_{a(\varepsilon)\lambda}$ is the same as the law of y^ε.

Ledoux–Qian–Zhang [16] proved that $(\mathbb{P}_\varepsilon)_{\varepsilon>0}$ satisfies the Schilder type large deviation principle. Hence, $(\mathbb{P}_\varepsilon\times\delta_{a(\varepsilon)\lambda})_{\varepsilon>0}$ satisfies the type large deviation principle on $G\Omega_p(\mathbf{R}^n)\times A$ with a good rate function $\mathcal{I}$, where, for $(h,a')\in G\Omega_p(\mathbf{R}^n)\times\mathbf{R}$,

$$I(h,a'\lambda)=\begin{cases}\|h\|^2_{\mathcal{H}^n}/2, & \text{if } h\in\mathcal{H}^n\text{ and }a'=a(0),\\ \infty, & \text{otherwise}\end{cases}$$

Now we may use Lemma 2.3 to complete the proof. Q.E.D.

2.4. Laplace's method

In this subsection we consider Laplace's method for the SDE (4). This is the precise asymptotics for the large deviation studied in the previous subsection. As in the case of the large deviation (Proposoition 2.2), there is no condition on the growth of the derivatives of order 2,3,... of σ and b.

This kind of problem for SDEs has been extensively studied. From [2, 3] for finete dimensional SDEs to deeper results in the context of the Malliavin calculus and in the infinite dimensional setting. The first proof via the rough path theory was done in [1].

We impose the following conditions on the functions F and G. In what follows, we especially denote by D the Fréchet derivatives on $\mathrm{BV}(\mathbf{R}^n)$ and $\mathcal{P}^d$. In the following, for $h \in \mathcal{H}^n$, we denote by $\Theta : \mathcal{H}^n \to \mathcal{H}^d$ the mapping defined by $h \mapsto Y^h$ as in (10).

(H1): F and G are real-valued bounded continuous functions defined on $\mathcal{P}^d$.

(H2): The function $F_\Lambda := F \circ \Theta + \| \cdot \|^2_{\mathcal{H}^n}/2$ defined on $\mathcal{H}^n$ attains its minimum at a unique point $\gamma \in \mathcal{H}^n$. We will write $\phi := \Theta(\gamma) = Y^\gamma$.

(H3): The functions F and G are $m+3$ and $m+1$ times Fréchet differentiable on a neighborhood $B(\phi)$ of $\phi \in \mathcal{P}^d$, respectively. Moreover there exist positive constants $M_1, \ldots, M_{n+3}$ such that

$$\begin{aligned} \big|D^k F(\eta)\big[y, \ldots, y\big]\big| &\le M_k \|y\|^k_{\mathcal{P}^d}, \quad k = 1, \ldots, m+3, \\ \big|D^k G(\eta)\big[y, \ldots, y\big]\big| &\le M_k \|y\|^k_{\mathcal{P}^d}, \quad k = 1, \ldots, m+1, \end{aligned}$$

hold for any $\eta \in B(\phi)$ and $y \in \mathcal{P}^d$.

(H4): At the point $\gamma \in \mathcal{H}^n$, we consider the Hessian $A := D^2(F \circ \Theta)(\gamma)|_{\mathcal{H}^n \times \mathcal{H}^n}$. As a bounded self-adjoint operator on $\mathcal{H}^n$, the operator A is strictly larger than $-\mathrm{Id}_{\mathcal{H}^n}$ in the form sense.

Now we are in a position to state our main theorem. The explicit values of the constants $\{\alpha_m\}_{m=0}^\infty$ are the same as in the case of an SDE with bounded coefficients.

Theorem 2.4. *Assume that a, σ and b in (9) are C^∞ and that $\sigma, b, \hat{b}$ are of at most linear growth. Under conditions* **(H1)**–**(H4)** *we have the following asymptotic expansion:*

(12)
$$\begin{aligned} \mathbb{E}\Big[G(y^\varepsilon)\exp\big(-F(y^\varepsilon)/\varepsilon^2\big)\Big] &= \exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big)\exp\big(-c(\gamma)/\varepsilon\big) \\ &\qquad \cdot \big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_m\varepsilon^m + O(\varepsilon^{n+1})\big), \end{aligned}$$

where the constant $c(\gamma)$ in (12) is given by $c(\gamma) := DF(\phi)[\Xi_1(\gamma)]$. Here $\Xi_1(\gamma) \in \mathcal{P}^d$ is the unique solution of the differential equation

$$\begin{aligned} d\Xi_t - \nabla\sigma(\phi_t)[\Xi_t, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[\Xi_t]dt \\ = a'(0)\cdot b(\phi_t)dt \quad \text{with} \quad \Xi_0 = 0. \end{aligned} \tag{13}$$

Proof. We give a sketch of proof. First, note that

(14)

$$\begin{aligned} &\mathbb{E}\Big[G(y^\varepsilon)\exp\big(-F(y^\varepsilon)/\varepsilon^2\big)\Big] \\ &= \int_{G\Omega_p(\mathbf{R}^n)\times \mathrm{BV}(\mathbf{R})} G(\Phi(X,\lambda')^1)\exp\big(-F(\Phi(X,\lambda')^1)/\varepsilon^2\big) \\ &\qquad\qquad \times d(\mathbb{P}_\varepsilon \times \delta_{a(\varepsilon)\lambda})(X,\lambda') \end{aligned}$$

Let U and $\tilde{U}$ be open neighborhoods of $\gamma \in G\Omega_p(\mathbf{R}^n)$ and $a(0)\lambda \in \mathrm{BV}(\mathbf{R})$, respectively. By the large deviation for y^ε (see (2.2)), the integration (14) above outside $U \times U'$ is dominated by $\exp(-(k+F_\Lambda(\gamma))/\varepsilon^2)$ for some positive constant k. Therefore, it does not contribute. So, it is sufficient to consider the integration (14) above on $U \times U'$. If we choose a sufficiently small U, U', then there exists $r > 0$ such that $\Phi = \Phi_r$ on the neighborhood $U \times U'$. Therefore, the problem is reduced to the case of bounded σ and b, which is done (in the rough path context) by Aida [1], Inahama [11], and Inahama and Kawabi [13, 14]. Q.E.D.

§3. An application for the large deviation for a diffusion process on loop goups

In this section we apply the arguments in the previous sections to the case of infinite dimensional diffusions. The example we treat here is the Brownian motion on loop groups over a compact Lie groups. Large deviation for such a process with respect to the usual sup-norm was first shown by Fang and Zhang [8], then by Inahama and Kawabi [12] by the method of [16]. On the other hand, Brzeźniak and Elworthy [6] constructed this process as a solution of an SDE on a M-type 2 Banach space. We prove in this section that the large deviation of Fang and Zhang holds with respect to the stronger topology, too.

Let $q \in [1,\infty)$ and $\theta \in (0,1)$. For an $\mathbf{R}^n$-valued function f defined on $[0,1]$, set

$$\|f\|_{\theta,q}^q := \int_0^1 |f(x)|^q dx + \int_0^1\int_0^1 \frac{|f(x_1)-f(x_2)|^q}{|x_1-x_2|^{1+\theta q}} dx_1 dx_2$$

and $W^{\theta,q}([0,1];\mathbf{R}^n) := \{f \in L^q([0,1];\mathbf{R}^n) \mid \|f\|_{\theta,q} < \infty\}$. If $q \in [2,\infty)$, $W^{\theta,q}([0,1];\mathbf{R}^n)$ is M-type 2 Banach space (see p. 71, [6]). If $1/q < \theta$, $W^{\theta,q}([0,1];\mathbf{R}^n)$ is continuously imbedded in $C([0,1];\mathbf{R}^n)$ (see p. 66, [6]). In this case we may set $\mathcal{X}^n_{\theta,q} = \{f \in W^{\theta,q}([0,1];\mathbf{R}^n) \mid f(0) = f(1) = 0\}$.

If $q > 2, 1/q < \theta < 1/2$, then the triplet $(\mathcal{X}^n_{\theta,q}, \mathcal{H}^n_0, \mu_0)$ becomes an abstract Wiener space, where $\mathcal{H}^n_0 = \{h \in \mathcal{H}^n \mid h(0) = h(1) = 0\}$ and μ_0 is the d-dimensional pinned Wiener measure (see [4] or p. 73, [6]). Indeed, we can easily see that $E^{\mu_0}[\| \cdot \|^q_{\theta,q}] < \infty$ since $E^{\mu_0}[|f(x_1) - f(x_2)|^q] \le c|x_1 - x_2|^{q/2}$ for some constant $c = c_q$.

Let $a \in C^4_b(\mathbf{R}^d; M(d,n))$ and $v \in C^4_b(\mathbf{R}^d;\mathbf{R}^d)$. Set

$$\sigma : \mathcal{X}^d_{\theta,q} \to L(\mathcal{X}^n_{\theta,q}, \mathcal{X}^d_{\theta,q}) \qquad \text{by} \qquad (\sigma(f)g)(\tau) = a(f(\tau))g(\tau),$$

for $f \in \mathcal{X}^d_{\theta,q}$, $g \in \mathcal{X}^n_{\theta,q}$, $\tau \in [0,1]$. Similarly, $b : \mathcal{X}^d_{\theta,q} \to \mathcal{X}^d_{\theta,q}$ is defined from v. Then, σ, b and $\hat{b}$ are of linear growth, C^3 in the Fréchet sense, and thier derivatives are bounded on any bounded set (see [6] for the definition of Tr in the definition of $\hat{b}$).

Consider the SDE (9) for these σ and b, which is well-defined as an SDE over an M-type 2 Banach space. Set

$$\mathcal{K}^n = \{g : [0,1] \to \mathcal{H}^n_0 \mid \text{absolutely continuous and } \textstyle\int_0^1 \|g'_t\|^2_{\mathcal{H}^n_0} dt < \infty\}.$$

As before, for $z \in \mathcal{P}(\mathcal{X}^d_{\theta,q})$, we set

$$\hat{I}(z) = \begin{cases} \inf\{\|h\|^2_{\mathcal{K}^n}/2 \mid z = Y^h\}, & \text{if } z = Y^h \text{ for some } h \in \mathcal{K}^n, \\ \infty, & \textit{otherwise}. \end{cases}$$

Proposition 3.1. *Assume $q > 3$ and $1/q < \theta < 1/2$, and $\hat{\sigma}$ and $\hat{b}$ are C^4_b. Define σ and b as above and let y^ε be as in the SDE (9) and let $\hat{I}$ be as above. Then, the law of y^ε satisfies the large deviation principle as $\varepsilon \searrow 0$ with a good rate function $\hat{I}$.*

Proof. Basically, the proof goes in the same way as in the finite dimensional case. So, we give a sketch of proof.
(i) The Wong–Zakai approximation theorem for SDEs on M-type 2 Banach spaces is given by [5].
(ii) Existence of a cutoff function does not seem very obvious in this Banach setting, because the norm function is not always smooth. As we will see, however, under this assumption we can prove that (a power of) the norm function is C^3. Therefore, a C^3 cutoff function exists. Note that the assumption $q > 3$ is used here.

First, we consider the case $d = 1$. Let $G(f) = \|f\|_{\theta,q}^q$ for $f \in \mathcal{X}_{\theta,q}^d$. By a straight forward computation,

$$\nabla G(f)\langle h\rangle/q = \int_0^1 |f(x)|^{q-1}\mathrm{sgn}(f(x))h(x)dx$$

$$+\int_0^1\int_0^1 \frac{|f(x_1)-f(x_2)|^{q-1}\mathrm{sgn}(f(x_1)-f(x_2))(h(x_1)-h(x_2))}{|x_1-x_2|^{1+\theta q}}dx_1dx_2,$$

$$\nabla^2 G(f)\langle h_1,h_2\rangle/q(q-1) = \int_0^1 |f(x)|^{q-2}h_1(x)h_2(x)dx$$

$$+\int_0^1\int_0^1 \frac{|f(x_1)-f(x_2)|^{q-2}\prod_{i=1}^2(h_i(x_1)-h_i(x_2))}{|x_1-x_2|^{1+\theta q}}dx_1dx_2,$$

$$\frac{\nabla^3 G(f)\langle h_1,h_2,h_3\rangle}{q(q-1)(q-2)} = \int_0^1 |f(x)|^{q-3}\mathrm{sgn}(f(x))h_1(x)h_2(x)h_3(x)dx$$

$$+\int_0^1\int_0^1 \frac{|f(x_1)-f(x_2)|^{q-3}\mathrm{sgn}(f(x_1)-f(x_2))}{|x_1-x_2|^{1+\theta q}}$$

$$\times\prod_{i=1}^3(h_i(x_1)-h_i(x_2))dx_1dx_2.$$

From these, there exists $c > 0$ such that $\|\nabla^i G(f)\| \le c\|f\|^{q-i}$ for $i = 1,2,3$. Thus, we have shown that G is of C^3 and derivatives are bounded on every bounded set.

The case $d \ge 2$ can be done in a similar way with

$$\hat{G}(f) = \sum_{i=1}^d\Big(\int_0^1 |f_i(x)|^q dx + \int_0^1\int_0^1 \frac{|f_i(x_1)-f_i(x_2)|^q}{|x_1-x_2|^{1+\theta q}}dx_1dx_2\Big)$$

for $f = (f_1,\dots,f_d)$, since $c^{-1}\|f\|_{\theta,q}^q \le \hat{G}(f) \le c\|f\|_{\theta,q}^q$ for some constant $c > 0$.

(iii) We must show that the Brownian rough paths exist. By [15], it suffices to see "the exactness condition" holds for μ_0 and $\mathcal{X}_{\theta,q}^n \otimes \mathcal{X}_{\theta,q}^n$ (the projective norm). Let $\theta < h < 1/2$. Then, h-Hölder norm is stronger than the norm of $\mathcal{X}_{\theta,q}^n$. For such a Hölder norm and a Gaussian measure satisfying $E^{\mu_0}[|f(x_1) - f(x_2)|^2] \le c|x_1 - x_2|$, the exactness holds (see p. 575, [15]). Therefore, $\mathcal{X}_{\theta,q}^n \otimes \mathcal{X}_{\theta,q}^n$ and μ_0 are exact, too. Q.E.D.

As a corollary of Proposition (3.1), we can slightly improve Fang and Zhang's large deviation for loop group-valued Brownian motion ([8]). We will show below that it also holds for the $W^{\theta,q}$-topology. Note that, in [6], this process is treated as the main example of the SDE theory on M type-2 Banach spaces.

Let G be a compact connected Lie group of dimension n. Its Lie algebra $\mathfrak{g}$ is equipped with an Ad_G-invariant inner product. Let A_i, $(1 \leq i \leq n)$ be an orthonormal basis of $\mathfrak{g}$, which are naturaly regarded as left-invariant vector fields on G. Imbed G into $\mathbf{R}^d$ for sufficiently large d so that the unit element $e \in G$ is mapped to the origin of $\mathbf{R}^d$. Extend A_i so that it is compactly supported. (Again we call this A_i.) Set, for $y \in \mathbf{R}^d$, $a(y) = [A_1(y), \ldots, A_n(y)] \in \mathrm{Mat}(d,n)$ (i.e., ith column vector of a is A_i) and set $v = 0$.

For this a and $v(=0)$, construct σ and $b(=0)$ as above and consider the SDE (9). Then, the law of y^ε is a probability measure on the path space over the loop group, i.e.,

$$(15) \quad \{z \in C([0,1], \mathcal{X}^d_{\theta,q}) \mid z(t)(\tau) \in G, z(0)(\tau) = e = z(t)(0) = z(t)(1)\}.$$

When $\varepsilon = 1$, $(y^1_t)_{0 \leq t \leq 1}$ is first appeared in Malliavin [20] and is called the Brownian motion over loop group, since its generater is the Gross Laplacian.

Let $\mathcal{L}^d = \{l \in C([0,1], \mathbf{R}^d) \mid l(0) = l(1) = 0\}$ with the usual sup-norm. Large deviation for y^ε on

$$\{z \in C([0,1], \mathcal{L}^d) \mid z(t)(\tau) \in G, z(0)(\tau) = e = z(t)(0) = z(t)(1)\}.$$

As $\varepsilon \searrow 0$ was first shown by Fang and Zhang [8], and then by Inahama and Kawabi [12] via the rough path theory. Using Proposition 3.1, we can improve slightly.

Corollary 3.2. *Let θ, q be as in Proposition 2.2 and let y^ε be the (scaled) Brownian motion over loop group as above. Then, the law of y^ε satisfies the large deviation principle on the space described in (15) as $\varepsilon \searrow 0$ with a good rate function.*

Proof. This is immediately shown from (3.1). Q.E.D.

Remark 3.3. In Proposition 3.1 and Corollary 3.2, the case $2 < q \leq 3$ is not mentioned. But, by the following argument, this case is also true. So, the large deviation holds for all $q > 2$.

Let $2 < q \leq 3$. Then by the expression of the derivatives of G and a simple inequality

$$\left| |x|^{q-2} - |y|^{q-2} \right| \leq |x-y|^{q-2}, \qquad x, y \in \mathbf{R},$$

we see that $\nabla^2 G$ is $(q-2)$-Hölder continuous.

Lyons' continuity theorem holds under the following $\mathrm{Lip}(2+\epsilon)$-type condition;

(A) the coefficient is in Lip$(2+\epsilon)$ $(0 < \epsilon \leq 1)$, that is, the coefficient and its derivatives of order 1,2 are bounded and the second derivative is ϵ-unformly Hölder continuous.

This extension is shown by Davie [7] in finite dimensional setting and cannnot be used in our context. But, it also holds in Banach-setting (see Section 5.3, [18] or pp. 343–344, [17]).

By using this extension and choosing p so that $2 < p < q =: 2+\epsilon$, we can prove Proposition 3.1 and Corollary 3.2 for the case $2 < q \leq 3$. (Here, p is the roughness.) Note also that, in Section 2, (local) Lipschitz continuity of the Itô map is not actually used. Only continuity is used. To the author's knowledge, this is the first application of Lip$(2+\epsilon)$-type extension of Lyons' continuity theorem to a concrete example.

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Graduate School of Mathematics
Nagoya University
Furocho, Chikusa-ku
Nagoya, 464-8602
Japan

E-mail address: `inahama@math.nagoya-u.ac.jp`