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Functions of finite Dirichlet sums and compactifications of infinite graphs

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Abstract.

We introduce the p-resister for an infinite network and show a comparison theorem on the resisters for two infinite graphs of bounded degrees which are quasi isometric. Some geometric projections of the Royden p-compactifications of infinite networks are investigated and several observations are made in relation to geometric boundaries of hyperbolic networks in the sense of Gromov. In addition, a Riemannian manifold which is quasi isometric to the hyperbolic space form is constructed to illustrate a role of the bounded local geometry in studying points at infinity.

§1. Introduction

Let $G = (V, E)$ be a connected, infinite, and locally finite graph with vertex set V and edge set E. (Loops and multiple edges are not admitted in this paper.) It is known that given a family Φ of bounded functions $B(V)$ on V, there exists a compact Hausdorff space $\mathcal{C}_\Phi(G)$, unique up to homeomorphisms, satisfying the following properties: (1) V is embedded in $\mathcal{C}_\Phi(G)$ as an open and dense subset; (2) every function of Φ extends to a continuous function on $\mathcal{C}_\Phi(G)$; (3) the extended functions separate the points of the boundary $\partial\mathcal{C}_\Phi(G) = \mathcal{C}_\Phi(G) \setminus V$. For instance, we have the end compactification $\mathcal{E}(G)$ of G associated to the space of bounded functions on V that are locally constant outside a compact subset of V. We note that for two families Φ and Ψ in $B(V)$ with $\Phi \subset \Psi$, the identity map induces a continuous map from $\partial\mathcal{C}_\Psi(G)$ *onto* $\partial\mathcal{C}_\Phi(G)$.

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In this paper, we are concerned with compactifications of a graph relative to some spaces of functions with finite Dirichlet sums.

Given a weight, or a positive function r on the set of edges of a connected, infinite, and locally finite graph $G = (V, E)$, we have a weighted graph, or a network (G, r). Let $L^{1,p}(G, r)$ be the space of all functions f on the set of vertices whose Dirichlet sums of order $p(> 1)$ with weight r are finite, that is,

$$D_{r;p}(f) = \sum_{e \in E} \frac{|df(e)|^p}{r(e)^{p-1}} < \infty,$$

where, for an edge $e = |xy|$ with end points x and y, we set $|df(e)| = |df(|xy|)| = |f(x) - f(y)|$. Then $L^{1,p}(G, r)$ is a Banach space with respect to the norm $\|f\|_{r;p} = D_{r;p}(f)^{1/p} + |f(o)|$, where $o \in V$ is a fixed point. Let us denote by $L_0^{1,p}(G, r)$ the closure of the set of functions on V with finite support. A function on V that is locally minimizing with respect to $D_{r;p}$ is said $D_{r;p}$-harmonic, and we denote by $HL^{1,p}(G, r)$ the set of all $D_{r;p}$-harmonic functions h on V with $D_{r;p}(h) < +\infty$. Let $BL^{1,p}(G, r) = L^{1,p}(G, r) \cap B(V)$. This is a Banach algebra with unit element 1 with respect to the norm $\|f\|_{r;p,\infty} = D_{r;p}(f)^{1/p} + \|f\|_\infty$. Associated to $BL^{1,p}(G, r)$, we have a compactification $\mathcal{R}_p(G, r)$ called the Royden p-compactification of the network (G, r) and the boundary $\partial\mathcal{R}_p(G, r) = \mathcal{R}_p(G, r) \setminus V$ is called the Royden p-boundary of (G, r). There is an important part of the Royden p-boundary, called the harmonic p-boundary of the network (G, r), which is defined by

$$\Delta_p(G, r) := \{x \in \partial\mathcal{R}_p(G, r) \mid f(x) = 0, \quad \forall f \in BL_0^{1,p}(G, r)\}.$$

We recall further that for any function $f \in L^{1,p}(G, r)$, there exist uniquely $h \in HL^{1,p}(G, r)$ and $g \in L_0^{1,p}(G, r)$ such that $f = h + g$. (See e.g. [10], [12], [15] and [16] for the details on Royden compactifications.)

On a network (G, r), each edge e is taken to be of length $r(e)$ and also every finite path has its length, so that we can define a distance $d_{(G,r)}(x, y)$ between two vertices x and y by taking the infimum of the lengths of paths joining x and y. In this way, the network (G, r) can be thought as a metric space. Considering also each edge e as a continuous curve of length $r(e)$, we get a locally compact, geodesic space called the metric graph associated to the network.

Relative to the set of functions f in $L^{1,p}(G, r)$ such that for some finite subset K, $D_{r;p}(f) \le D_{r;p}(u)$ if $u \in L^{1,p}(G, r)$ coincides with f on K, we have a compactification of V called the Kuramochi p-compactification of (G, r) and denoted by $\mathcal{K}_p(G, r)$. The boundary $\partial\mathcal{K}_p(G, r)$ is called the Kuramochi p-boundary of (G, r). The identity map of V induces a continuous map from $\partial\mathcal{R}_p(G, r)$ onto $\partial\mathcal{K}_p(G, r)$. We note that $\mathcal{K}_p(G, r)$ is metrizable, although $\partial\mathcal{R}_p(G, r)$ is not in general. For example, the Kuramochi p-compactification of an infinite tree T with weight r is identified with the end compactification $\mathcal{E}(T)$.

Now in the case where $r = 1$, we simply write G for $(G, 1)$ and the distance $d_G (= d_{(G,1)})$ is the graph distance of G.

We recall here two results in [7].

Theorem 1. *Let $G = (V, E)$ and $G' = (V', E')$ be connected, locally finite and infinite graphs with bounded degrees. Suppose that G and G' are quasi isometric, that is, there exists a quasi isometry $\phi : (V, d_G) \longrightarrow (V', d_{G'})$. Then ϕ extends to a continuous map $\bar{\phi}$ of $\mathcal{R}_p(G)$ to $\mathcal{R}_p(G')$ whose restriction to $\partial\mathcal{R}_p(G)$ induces a homeomorphism $tr(\phi)$ between $\partial\mathcal{R}_p(G)$ and $\partial\mathcal{R}_p(G')$ such that $tr(\phi)(\Delta_p(G)) = \Delta_p(G')$. Moreover assigning a function h of $BHL^{1,p}(G')$ the unique function $\eta(h)$ of $BHL^{1,p}(G)$ whose trace $tr(\eta(h))$ on $\Delta_p(G)$ coincides with $tr(h) \circ tr(\phi)$ is bijective, and there exists a positive constant C such that*

$$C^{-1}D_p(h) \le D_p(\eta(h)) \le CD_p(h)$$

for all $h \in BHL^{1,p}(G')$. When $p = 2$, η induces a linear isomorphism between $BHL^{1,2}(G)$ and $BHL^{1,2}(G')$.

For a connected, complete, noncompact Riemannian manifold M of dimension n, we can define the function spaces $L^{1,p}(M)$, $L_0^{1,p}(M)$ and $HL^{1,p}(M)$, and also the Royden p-boundary $\partial\mathcal{R}_p(M)$ and the harmonic p-boundary $\partial\Delta_p(M)$ of M, which respectively correspond to those of an infinite network (G, r), $L^{1,p}(G, r)$, $L_0^{1,p}(G, r)$, $HL^{1,p}(G, r)$, $\partial\mathcal{R}_p(G, r)$ and $\partial\Delta_p(G, r)$ (cf. [10], [11]).

Theorem 2. *Let $G = (V, E)$ be a connected, locally finite and infinite graphs with bounded degrees, and let M be a connected, complete Riemannian manifold of dimension n such that the Ricci curvature is bounded from below and the volume of any ball of radius one is bounded below by a positive constant. Suppose that (V, d_G) and (M, d_M) are quasi isometric. Then there exists a bijective correspondence σ between $HL^{1,p}(G)$ and $HL^{1,p}(M)$ such that*

$$C^{-1}D_p(h) \le D_p(\sigma(h)) \le CD_p(h)$$

for some positive constant C and all $h \in HL^{1,p}(G)$. In the case of $p = 2$, σ is a linear isomorphism between $HL^{1,2}(G)$ and $HL^{1,2}(M)$. Moreover for $p > n$, a quasi isometry $\phi : (V, d_G) \to (M, d_M)$ induces *a homeomorphism between the Royden p-boundaries of G and M which sends the harmonic p-boundary of G to that of M.*

The outline of the paper is as follows. In Section 2, we introduce two functions related to the Green functions of a network and give a comparison theorem on these functions under quasi isometries. In Section 3, some projections of Royden compactifications are discussed and Section 4 is devoted to providing some examples of Kuramochi compactifications. The observations made in these two sections are found in [9] where the case of $p = 2$ is discussed. In Section 5, we illustlate a role of the condition of manifolds on volume of unit balls in Theorem 2 with an example of Riemannian manifolds which are quasi isometric to hyperbolic space forms.

§2. Royden compactifications

Let $G = (V, E)$ and r be respectively a connected, locally finite, infinite graph and a weight on E. Let

$$M_{G,r}^{(p)}(x) = \sup\left\{ \frac{|g(x)|^p}{D_{r;p}(g)} \mid g \in L_0^{1,p}(G,r) \right\}, \quad x \in V.$$

We say that the network (G, r) is p-parabolic if $M_{G,r}^{(p)}(x)$ is infinite for some (and hence for all) $x \in V$, and (G, r) is p-nonparabolic otherwise.

To begin with, we state some results of Yamasaki [14], [16], [17].

Theorem 3. (1) *The following three conditions are mutually equivalent:* (i) (G, r) *is p-parabolic,* (ii) $\Delta_p(G,r) = \emptyset$, (iii) $L_0^{1,p}(G,r) = L^{1,p}(G,r)$.

(2) *Suppose that* (G, r) *is p-nonparabolic. Then* $f \in L_0^{1,p}(G,r)$ *if* $f = 0$ *on* $\Delta_p(G,r)$.

(3) *Suppose that* $\partial\mathcal{R}_p(G,r) \setminus \Delta_p(G,r)$ *is not empty. Then any point of* $\partial\mathcal{R}_p(G,r) \setminus \Delta_p(G,r)$ *is not a* G_δ *set; in particular,* $\partial\mathcal{R}_p(G,r)$ *is not metrizable. In addition, for any closed subset* $F \subset \partial\mathcal{R}_p(G,r) \setminus \Delta_p(G,r)$, *there exists a function* $g \in L_0^{1,p}(G,r)$ *such that* $g(x)$ *tends to infinity as* $x \in V \to F$.

Now we have the following

Proposition 4. *The following three conditions are mutually equivalent:* (i) $M_{G,r}^{(p)}$ *is bounded on* V, (ii) *all* $g \in L_0^{1,p}(G,r)$ *are bounded,* (iii) $\partial\mathcal{R}_p(G,r) = \Delta_p(G,r)$, *that is, for any bounded* $g \in L_0^{1,p}(G,r)$, $g(x)$ *tends to zero as* $x \to \infty$.

Proof. Obviously (i) or (iii) implies (ii), and in view of Theorem 3, (iii) follows (ii). Now we suppose that (ii) holds. Then the bounded inverse theorem shows that the two norms $\|\cdot\|_{r;p}$ and $\|\cdot\|_{r;p,\infty}$ are equivalent in $L_0^{1,p}(G,r)(= BL_0^{1,p}(G,r))$. Therefore we have $\|f\|_\infty \leq MD_{r;p}(f)$ for some positive constant M and all $f \in L_0^{1,p}(G,r)$. This proves that $\sup_{x\in V} M_{G,r}^{(p)}(x) \leq M$. Q.E.D.

We assign, to any pair of vertices x and y, a nonnegative number $R_{G,r}^{(p)}(x,y)$ defined by

$$R_{G,r}^{(p)}(x,y) = \sup\left\{ \frac{|f(x)-f(y)|^p}{D_{r;p}(f)} \,\middle|\, f \in L^{1,p}(G,r),\ D_{r;p}^{(p)}(f) \neq 0 \right\}.$$

We note that ${R_{G,r}^{(p)}}^{1/p}$ is a distance on V and $R_{G,r}^{(p)}(x,y)^{1/p} \leq d_{(G,r)}(x,y)^{1-1/p}$ for all $x, y \in V$. When $p=2$, $R_{G,r}^{(2)}(x,y)$ is called the effective resistance between two vertices x and y of the network (G,r), and it is known that $R_{G,r}^{(2)}$ itself induces a distance on V called the resistance metric of (G,r). For instance, $R_{T,r}^{(p)} = {d_{(T,r)}}^{p-1}$ for an infinite tree T with weight r.

The same argument as in the proof of Proposition 4 is valid to prove the following

Proposition 5. *$R_{G,r}^{(p)}$ is bounded if and only if any $u \in L^{1,p}(G,r)$ is bounded, that is, $L^{1,p}(G,r) = BL^{1,p}(G,r)$. Moreover in these cases, the conditions of Proposition 4 are satisfied and $\partial\mathcal{R}_p(G,r)(=\Delta_p(G,r))$ is metrizable.*

We remark that if $p=2$ and $\sup_{x,y\in V} R_{G,r}^{(2)}(x,y)$ is finite, then $\partial\mathcal{R}_2(G,r)$ coincides with $\partial\mathcal{K}_2(G,r)$ (cf. [9]); however it is in question whether this would be true for any p other than 2.

In what follows, we study the case where $r=1$.

We first notice that for $p<q$, $D_q(f)^{1/q} \leq D_p(f)^{1/p}$ for all $f \in L^{1,p}(G)$. In fact, we have

$$D_q(f) = \sum_{e\in E} |df(e)|^p (|df(e)|^p)^{(q-p)/p} \leq D_p(f) D_p(f)^{(q-p)/p} = D_p(f)^{q/p}.$$

As consequences, we get $L_0^{1,p}(G) \subset L_0^{1,q}(G)$, $L^{1,p}(G) \subset L^{1,q}(G)$ and moreover

$$R_G^{(p)}(x,y) \leq R_G^{(q)}(x,y), \quad x,y \in V;\ M_G^{(p)}(x) \leq M_G^{(q)}(x)(\leq +\infty), \quad x \in V.$$

As a result of the last inequality, G is q-parabolic if it is p-parabolic, and the parabolic index of G can be introduced as the infimum of the indices p so that G is p-parabolic (cf. [14]).

Now we shall exhibit examples of graphs whose Royden p-boundaries consist of single points.

(I) Let $G = G_1 \times G_2$ be the cartesian product of connected, locally finite, and infinite graghs G_1 and G_2. If G_2 satisfies the conditions in Proposition 4 for an exponent $p(>1)$, then the Royden p-boundary $\partial\mathcal{R}_p(G)$ consists of a single point. For instance, $\partial\mathcal{R}_p(G_1 \times \mathbf{Z}^n) = \{\infty\}$ for $1 < p < n$.

(II) Let $G = (V, E)$ be a connected, locally finite, infinite graph. Given a subset A of V, we denote by $\partial_b A$, called the vertex boundary of A, the subset of A consisting of points adjacent to those outside of A. Let $\{V_n\}$ be an increasing family of finite subsets of V whose union coincides with V. Suppose that the finite subgraphs B_n of G generated by $\partial_b V_n$ are connected and satisfies the following Sobolev type inequality $(S_{p,q})$ $(1 < p < q)$:

$$\left(\sum_{x\in\partial_b V_n} |f(x) - \bar{f}|^{pq/(q-p)}\right)^{1-p/q} \le CD_p(f), \quad f \in L^{1,p}(B_n),$$

where we set $\bar{f} = \sum_{x\in\partial_b V_n} f(x)/|\partial_b V_n|$, $|\partial_b V_n|$ stands for the cardinality of $\partial_b V_n$, and C is a positive constant independent of n. Then we see that $\partial\mathcal{R}_p(G) = \{\infty\}$.

We are interested in properties invariant under quasi isometries as mentioned in the introduction. In this place, we prove the following

Theorem 6. *Let $G = (V, E)$ and $G' = (V', E')$ be two connected, locally finite, infinite graphs with bounded degrees. Suppose that they are quasi isometric and let $\phi : (V, d_G) \longrightarrow (V', d_{G'})$ be a quasi isometry. Then there exist constants $C \ge 1$ and $C' \ge 0$ such that*

$$(1) \quad \frac{1}{C}R_G^{(p)}(x,y) - C' \le R_{G'}^{(p)}(\phi(x),\phi(y)) \le CR_G^{(p)}(x,y), \; x,y \in V;$$

$$(2) \qquad \frac{1}{C}M_G^{(p)}(x) - C' \le M_{G'}^{(p)}(\phi(x)) \le CM_G^{(p)}(x), \; x \in V.$$

Proof. We recall first that $D_p(f \circ \phi) \le C_1 D_p(f)$ for some constant $C_1 > 0$ and every $f \in L^{1,p}(G')$, since the degrees of the graphs are uniformly bounded and $R_G^{(p)}(x,y) \le d_G(x,y)^{p-1}$ holds for all $x,\ y \in V$ (see [7] for details). Then it follows that for any pair of points x and y

of V,

$$\frac{|f(\phi(x))-f(\phi(y))|^p}{D_p(f)} \le C_1 \frac{|f(\phi(x))-f(\phi(y))|^p}{D_p(f\circ\phi)} \le C_1 R_G^{(p)}(x,y).$$

This shows that $R_{G'}^{(p)}(\phi(x),\phi(y)) \le C_1 R_G^{(p)}(x,y)$.

Now we take a quasi isometry $\psi : (V', d_{G'}) \to (V, d_G)$ such that $d_G(x, \psi(\phi(x))) \le C_2$ for some constant $C_2 \ge 0$ and all $x \in V$. Then by the same reason as above, we see that $R_G^{(p)}(\psi(\phi(x)), \psi(\phi(y))) \le C_3 R_{G'}^{(p)}(\phi(x),\phi(y))$ for some constant C_3 and all $x, y \in V$. Then for any $u \in L^{1,p}(G)$ and every pair of points $x, y \in V$, we get

$$\begin{aligned}
&\frac{|u(x)-u(y)|^p}{D_p(u)} \\
&\le 4^{p-1}\left(\frac{|u(x)-u(\psi(\phi(x)))|^p}{D_p(u)} + \frac{|u(\psi(\phi(x)))-u(\psi(\phi(y)))|^p}{D_p(u)}\right. \\
&\quad \left. + \frac{|u(\psi(\phi(y)))-u(y)|^p}{D_p(u)}\right) \\
&\le 4^{p-1}\Big(d_G(x,\psi(\phi(x)))^{p-1} + R_G^{(p)}(\psi(\phi(x)),\psi(\phi(y))) \\
&\quad + d_G(y,\psi(\phi(y)))^{p-1}\Big) \\
&\le 4^p C_2^{p-1} + 4^{p-1} C_3 R_{G'}^{(p)}(\phi(x),\phi(y)).
\end{aligned}$$

Hence we have

$$R_G^{(p)}(x,y) \le 4^p C_2^{p-1} + 4^{p-1} C_3 R_{G'}^{(p)}(\phi(x),\phi(y)).$$

Thus (1) is verified, and (2) can be also derived from the same arguments as above. Q.E.D.

As a corollary of estimate (2), we see that the parabolic index is invariant under quasi isometries among the set of connected, locally finite, infinite graphs of bounded degrees. This is a result due to Soardi and Yamasaki [13].

§3. Projections of Royden boundaries

Since the Royden p-compactification of an infinite network is very complicated in general, it will be useful to investigate its projections onto some metric spaces.

Let $G = (V, E)$ be a connected, locally finite and infinite graph and r a weight on E. Let (X, d_X) be a metric space. Define the Dirichlet sum of order p with weight r of a map ϕ from V to X by

$$D_{r;p}(\phi) = \sum_{e=|xy|\in E} \frac{d_X(\phi(x), \phi(y))^p}{r(|xy|)^{p-1}}.$$

Observe that if $D_{r;p}(\phi)$ is finite, then for any Lipschitz continuous function f with Lipschitz constant L on X, the composition $f \circ \phi$ belongs to $L^{1,p}(G, r)$ and

$$D_{r;p}(f \circ \phi) \leq L^p D_{r;p}(\phi).$$

Let $\tilde{X}$ be a compactification of X satisfying $\liminf_{x\to\xi, y\to\eta} d_X(x, y) > 0$ for all $\xi, \eta \in \partial X (= \tilde{X} \setminus X)$ with $\xi \neq \eta$. Then it is easy to see that a map $\phi : V \to X$ with finite $D_{r;p}(\phi)$ extends to a continuous map from the Royden p-compactification of (G, r) onto the closure of the image $\phi(V)$ in $\tilde{X}$.

For instance, considering the identity map I of V as a map of the graph to the metric space $(V, d_{(G,r)})$, we see that

$$D_p(I) \leq \sum_{e\in E} r(e)^p,$$

so that if the right side is finite, that is, $r \in L^p(E)$, then I extends to a continuous map from $\partial\mathcal{R}_p(G)$ onto $\partial\mathcal{L}(V, d_{(G,r)})$, where $\mathcal{L}(V, d_{(G,r)})$ stands for the compactification of V associated to the space of bounded Lipschitz continuous functions with respect to the distance $d_{(G,r)}$.

There are certain cases where the compactification $\mathcal{L}(V, d_{(G,r)})$ as above play important roles in geometries of the graphs.

Let

$$S(n) = \{x \in V \mid d_G(x, o) = n\},$$
$$E(n) = \{|xy| \in E \mid x,\ y \in S(n) \text{ or } x \in S(n),\ y \in S(n-1)\},$$

where $n = 1, 2, \dots$ and o is a fixed vertex of G. For a positive constant ε, we define a weight r_ε on G by letting $r_\varepsilon(e) = \exp(-\varepsilon n)$ if $e \in E(n)$. Then we have

$$D_p(I) \leq \sum_{n=1}^{\infty} |E(n)|\ \exp(-\varepsilon p n)(\leq +\infty),$$

where $|A|$ denote the cardinality of a finite set A. Let

$$e(G) = \limsup_{n\to\infty} \frac{1}{n} \log |E(n)| (\in [0, +\infty]).$$

Then if $e(G)$ is finite and $p\varepsilon > e(G)$, then $D_p(I)$ is finite and hence the identity map I of V as a map of the graph G to the metric space $(V, d_{(G,r)})$ extends to a continuous map from $\partial\mathcal{R}_p(G)$ onto $\partial\mathcal{L}(G, r_\varepsilon)$.

We shall recall here a basic fact on proper geodesic spaces which are hyperbolic in the sense of Gromov: if the metric graph $(|G|, d_{|G|})$ associated with the graph G is hyperbolic, then there exists $\varepsilon_0(G) \in (0, +\infty]$ such that $\partial\mathcal{L}(G, r_\varepsilon)$ coincides with the Gromov boundary $\partial_H(|G|, d_{|G|})$ for any positive $\varepsilon < \varepsilon_0(G)$ (cf. [6], [4]).

In addition, we can deduce from simple calculations that the identity map induces a homeomorphism between $\partial\mathcal{R}_p(G, r_\delta)$ and $\partial\mathcal{L}(G, r_\varepsilon)$ for a positive number δ less than $(\varepsilon p - e(G))/(p-1)$.

Now we consider a finitely generated, infinite, properly discontinuous subgroup Γ of isometries of a proper geodesic space (X, d_X). Fix a point o of X and let $\psi : \Gamma \to X$ be a map from Γ into X defined by $\psi(g) = g^{-1}(o)$, $g \in \Gamma$. We take a finite generating set S of Γ with $S = S^{-1}$ and consider the Cayley graph $G_\Gamma = (\Gamma, E_S)$. First we carry out "conformal" changes of the metric of X as in the case of graphs. For a positive continuous function w on X, the w-length $L_w(c)$ of an arc-length parametrized curve $c : [a, b] \to (X, d_X)$ is given by $L_w(c) = \int_a^b w(c(t))dt$. Then we define a distance $d_w(x, y)$ between two points x and y of X by the infimum of $L_w(c)$ where c ranges over all arc-length parametrized curves joining x and y. As in the case of graphs, we are interested in the case when $w_\varepsilon = \exp(-\varepsilon d_X(*, o))$, where ε is a positive constant and o is a fixed point of X. The Dirichlet sum of order p of the map ψ from G_Γ into (X, d_{w_ε}) is given by $D_p(\psi) = \sum_{g\in\Gamma, a\in S} d_{w_\varepsilon}(\psi(g), \psi(ag))^p$. If the completion $\overline{X}^\varepsilon$ of the metric space (X, d_{w_ε}) is compact, the critical exponent of Γ defined by

$$\bar{e}(\Gamma) = \inf\{s > 0 \mid \sum_{g\in\Gamma} \exp(-s d_X(o, g(o))) < +\infty\}$$

is finite, and $\varepsilon p > \bar{e}(\Gamma)$, then we can conclude that ψ extends to a continuous map of the Royden p-boundary of the Cayley graph G_Γ onto the intersection of the closure of the orbit of $\Gamma(o)$ in $\overline{X}^\varepsilon$ and the boundary of $\overline{X}^\varepsilon$.

We refer the reader to [1], [2], [3] and [5] for related topics and results.

§4. Kuramochi compactifications

In this section, we give two observations on the Kuramochi compactifications of infinite graphs.

Let $G = (V, E)$ be a connected, locally finite, infinite graph. For a positive integer n, we denote by $B_n = (V_n, E_n)$ the graph with the set of vertices $\{x, z_1, \ldots, z_n, y\}$ and the set of edges $\{|xz_i|, |z_iy|, i = 1, \ldots, n\}$. By assigning a positive integer $\nu(e)$ to each edge $e \in E$ and replacing each edge $e = |xy| \in E$ with the graph $B_{\nu(e)}$, we obtain a connected, locally finite, infinite graph $G_\nu = (V_\nu, E_\nu)$, where V is assumed to be a subset of V_ν. We remark that two metric spaces (V, d_G) and (V_ν, d_{G_ν}) are quasi isometric. For $p > 1$, we introduce a weight function $r_{\nu;p}$ on E defined by $r_{\nu,p}(e) = 2\nu(e)^{-1/(p-1)}$ for $e \in E$. Then we get a network $(G, r_{\nu;p})$ so that the identity map of V extends to a continuous map of $\partial\mathcal{R}_p(G)$ onto $\partial\mathcal{R}_p(G, r_{\nu;p})$. Moreover the inclusion map of V into V_ν extends to a homeomorphism between $\partial\mathcal{R}_p(G, r_{\nu;p})$ and $\partial\mathcal{R}_p(G_\nu)$ which induces also a homeomorphism between the Kuramochi p-boundaries $\partial\mathcal{K}(G, r_{\nu,p})$ and $\partial\mathcal{K}(G_\nu)$.

For instance, we assume that a given graph $G = (V, E)$ is hyperbolic in the sense of Gromov and further $p\varepsilon_0(G) - e(G)$ is positive. Then we choose a function $\nu : E \to \mathbf{Z}^+$ such that $C^{-1}\exp((p-1)\varepsilon n) \leq \nu(e) \leq C\exp((p-1)\varepsilon n)$ if $e \in E(n)$ $(n = 1, 2, \ldots)$, where C is a constant greater than 1 and ε is a positive number less than $(p\varepsilon_0(G) - e(G))/(p-1)$. Then it turns out that $\partial\mathcal{R}_p(G_\nu)$ is also identified with $\partial_H(|G|, d_{|G|})$; in addition, if $p = 2$, then $\partial\mathcal{R}_2(G_\nu) = \partial\mathcal{K}_2(G_\nu) = \partial_H(|G|, d_{|G|})$ (cf. [9]). The degree of G_ν is clearly unbounded.

Now we are given a connected, locally finite, and infinite graph $G = (V, E)$ such that $\sup_{x,y\in V} R_G^{(p)}(x, y)$ is finite (see Proposition 5). In what follows, we construct a connected, locally finite, infinite graph $G_* = (V_*, E_*)$ such that the Kuramochi boundary of G_* is homeomorphic to the union of the metric graph $|G|$ associated to G and a quotient space of the Royden p-boundary of G.

We first take a family of locally finite, infinite graphs indexed by the set of the vertices V of G, $\{G_\alpha = (V_\alpha, E_\alpha) \mid \alpha \in V\}$, such that the Royden p-boundary of G_α consists of a single point p_α for all $\alpha \in V$. Secondly we choose a sequence $\{p_{\alpha;n}\}$ of vertices in V_α tending to p_α for each $\alpha \in V$, and then for any edge $(\alpha, \beta) \in E$, we connect V_α with V_β by a family of paths $(C_{\alpha\beta}, E_{\alpha\beta}) = \{(C_{\alpha\beta}^{(n)}, E_{\alpha\beta}^{(n)}) \mid n = 1, 2, \ldots\}$ joining $\{p_{\alpha;n}\}$ to $\{p_{\beta;n}\}$, that is

$$
\begin{aligned}
C_{\alpha\beta}^{(n)} &= \{p_{\alpha;n} = x_{\alpha\beta;0}^{(n)}, x_{\alpha\beta;1}^{(n)}, \ldots, x_{\alpha\beta;r(\alpha,\beta;n)}^{(n)} = p_{\beta;n}\}; \\
E_{\alpha\beta}^{(n)} &= \{(x_{\alpha\beta;i}^{(n)}, x_{\alpha\beta;i+1}^{(n)}) \mid i = 0, \ldots, r(\alpha, \beta; n) - 1\},
\end{aligned}
$$

and the length $r(n)$ of the path $C_{\alpha\beta}^{(n)}$ is assumed to satisfy

$$\sum_{n=1}^{\infty} \frac{1}{r(n)^{p-1}} = 1$$

for all $(\alpha, \beta) \in E$. Set $V_* = (\cup_{\alpha \in V} V_\alpha) \cup (\cup_{(\alpha\beta) \in E} C_{\alpha\beta})$ and $E_* = (\cup_\alpha E_\alpha) \cup (\cup_{(\alpha,\beta) \in E} E_{\alpha\beta})$. Then we can deduce that the resulting graph $G_* = (V_*, E_*)$ has the desired properties mentioned above. It is in question whether there would be any need to take a quotient space of the Royden p-boundary of G. In fact, in case $p = 2$, the Kuramochi boundary of G_* is homeomorphic to the union of $|G|$ and $\partial\mathcal{R}_2(G)(= \partial\mathcal{K}_2(G))$ (see [9]). We remark that G_* is connected at infinity, and further that it is of bounded degrees if the degrees of G and all G_α are bounded uniformly.

§5. Riemannian manifolds quasi isometric to hyperbolic space forms

In this section, we construct a Riemannian manifold of dimension 4 which is quasi isometric to the hyperbolic space form $H^3(-1)$ of dimension 3 and admits infinite dimensional space of harmonic functions with finite Dirichlet integrals.

We first define a Riemannian metric on $\mathbf{R}^4$ by using the Hopf map from the unite 3-sphere S^3 onto the unit 2-sphere S^2, $\pi : S^3 \longrightarrow S^2$. Let g_0 be the standard metric of S^3. We choose a vector field ξ on S^3 such that $|\xi| = 1$ and $d\pi(\xi) = 0$. The tangent vector space of S^3, TS^3, is decomposed into the orthogonal sum $TS^3 = \{\xi\}_{\mathbf{R}} \oplus H$. Let ω_ξ be the 1-form on S^3 which satisfies $\omega_\xi(\xi) = 1$ and $\omega(H) = 0$. Let $\eta(r)$ be a smooth function such that $\eta(r) = \sinh r$ for $0 \leq r < 1$ and $\eta(r) = e^{-br}$ for $2 < r$, where b is a constant in $(0, 2)$. Then we define a Riemannian metric on $\mathbf{R}^4$ by

$$g_\eta = dr^2 + (\sinh r)^2(g_0 - \omega_\xi^2) + \eta(r)^2\omega_\xi^2.$$

Then the Ricci curvature of $(\mathbf{R}^4, g_\eta)$ is uniformly bounded from below; however the volume of unit balls in $(\mathbf{R}^4, g_\eta)$ tends to zero as $r \to \infty$. On the other hand, the space of all Dirichlet finite harmonic functions on $(\mathbf{R}^4, g_\eta)$ has infinite dimension. Indeed, Dirichlet finite harmonic functions on $(\mathbf{R}^4, g_\eta)$ are obtained as follows. Let g_1 be the standard metric on S^2. Given an eigenfunction ψ associated to an eigenvalue λ on (S^2, g_1), we let $\phi(\theta) = \psi(\pi(\theta))$. Let $u(r)$ be the solution of a differential equation $((\sinh r)^2\eta(r)u(r)')' - \lambda\eta(r)u(r) = 0$ subject to the initial conditions: $u(0) = u'(0) = 0$. Then $f(r,\theta) = u(r)\phi(\theta)$ turns out to be a Dirichlet finite harmonic function on $(\mathbf{R}^4, g_\eta)$ (cf. [8]).

Finally, we define a map $\tilde{\pi} : \mathbf{R}^4 \to H^3(-1)$ by

$$\tilde{\pi} : \boldsymbol{R}^4 = [0,\infty) \times S^3 \longrightarrow H^3(-1) = [0,\infty) \times S^2,\ \tilde{\pi}(r,\theta) = (r, \pi(\theta)).$$

Then $\tilde{\pi}$ induces a quasi isometry between $(\mathbf{R}^4, g_\eta)$ and $H^3(-1)$. Note that $H^3(-1)$ admits no nonconstant Dirichlet finite harmonic functions.

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