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Stochastic flows and geometric analysis on path spaces

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Abstract.

Some aspects of geometric analysis on path spaces are reviewed. Special emphasis is given to the relevance of stochastic flows to this analysis, and to the role of Ricci and higher order Weitzenböck curvatures. Path spaces of diffeomorphism groups and of compact symmetric spaces are considered.

§1. Introduction

This is not a general review of path space analysis. It is rather a description of the role stochastic flows can play in such analysis, and a discussion of the special case of paths on compact Riemannian symmetric spaces, for example spheres. For a general introduction there is E. Hsu's monograph [31], also a Sugaku exposition with emphasis on loop spaces by S. Aida, [1], Leandre's survey [34] and, closer to this article, [25].

The structure is:

(1) Generalities about analysis on path spaces $C_a\mathcal{M}$;
(2) The case $\mathcal{M}$ =Diff(M), differential forms on C_{id}Diff(M), stochastic flows, generalised (raw) Bismut formulae;
(3) Paths on Lie groups, $C_{id}K$, and symmetric spaces, $C_{x_0}(K/G)$;
(4) Comments on the Markov uniqueness problem and vanishing of L^2 harmonic one-forms on $C_{x_0}(K/G)$.

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It draws heavily on the joint work with Xue-Mei Li, and with Yves LeJan in [17], [19], [20], [21], [15], and [18], and on the the work of Shigekawa [41] and Fang & Franchi [28].

Ricci and higher order Weitzenböck curvatures play an important role in this analysis, and one way they arise is via stochastic flows, which here will be considered as processes on diffeomorphism groups. There are various other reasons for considering the case $\mathcal{M}=\mathrm{Diff}(M)$. One is that many formulae, such as the raw Bismut formulae, naturally fit in that context. Another is that by using $C_{id}\mathrm{Diff}(M)$ as the "model" space the transformation to paths on M is via the smooth deterministic map obtained by evaluation at our base point x_0 of M, rather than by the solution map of an SDE as is the case if $C_0\mathbf{R}^m$ is used. This is especially striking when M is a symmetric space and our measure on $C_{id}\mathrm{Diff}(M)$ is supported on the path space of a compact Lie group.

§2. Generalities

Let $\mathcal{M}$ denote a separable metrisable C^∞ Banach manifold. In the sequel it will be either a compact manifold or a space of diffeomorphisms of a compact manifold. Take a base point $a \in \mathcal{M}$ and a fixed positive time T, and define the based path space $C_a\mathcal{M}$ to consist of those continuous $\sigma : [0,T] \to \mathcal{M}$ with $\sigma(0) = a$. This inherits a natural C^∞ Banach manifold structure from that of $\mathcal{M}$, see [13]. Recall that the tangent space at σ is given by

$$T_\sigma C_a\mathcal{M} = \left\{ v \in C_0 T\mathcal{M} : v(t) \in T_{\sigma(t)}, 0 \le t \le T \right\}.$$

Let μ_a be a Borel probability measure on $C_a\mathcal{M}$; we are thinking especially of the case when it is the law of some diffusion process on $\mathcal{M}$.

2.1. Bismut and L^2 tangent spaces

From the work of L. Gross in the 1960's on Abstract Wiener spaces, reviewed in [30], that of Daletskii & Shnaiderman for infinite dimensional Lie groups, [8], and the development of Malliavin Calculus [36], we expect that to get a useful version of differential analysis with a potential theory, Sobolev spaces etc., we need to restrict our differentiation to be along spaces of "admissible" directions given by Hilbert spaces $\mathcal{H}_\sigma, \langle -, - \rangle_\sigma$ defined for almost all $\sigma \in \mathcal{M}$, with continuous linear injections into the tangent space to $\mathcal{M}$ at σ:

$$i_\sigma : \mathcal{H}_\sigma \to T_\sigma C_a\mathcal{M}.$$

The images of these maps are sometimes called the *Bismut tangent spaces* and their disjoint union, $\mathcal{H} := \bigcup_\sigma \mathcal{H}_\sigma$, with its projection onto $C_a\mathcal{M}$, the *Bismut tangent bundle.*

Usually these spaces are constructed by taking a connection ∇ on $T\mathcal{M}$, or, in the case of the law of a degenerate diffusion on $\mathcal{M}$, on some bundle E, say, with an injection into $T\mathcal{M}$. If $\mathcal{M}$ is finite dimensional E would be a subundle of TM. From this one obtains a linear transport:

$$W_t^\sigma : T_a\mathcal{M} \to T_{\sigma(t)}\mathcal{M}$$

and a linear operator $\frac{\mathbb{D}}{dt}$ from

$$\left\{ v \in T_\sigma C_a\mathcal{M} : v(t) = W_t^\sigma(h(t)), h \in L_0^{2,1}([0,T]; T_a\mathcal{M}) \right\}$$

to $L^2 T_\sigma C_a\mathcal{M}$ defined by

$$\frac{\mathbb{D}}{dt} v = W_t^\sigma \frac{d}{dt} (W_t^\sigma)^{-1} v(t).$$

Here $L_0^{2,1}([0,T]; T_a\mathcal{M})$ refers to the finite energy paths in the tangent space to $\mathcal{M}$ at the base point and $L^2 T_\sigma C_a\mathcal{M}$ denotes the L^2 tangent space to our path space consisting of "tangent vectors" which are not necessarily continuous but are measurable and lie in L^2 for some fixed Riemannian metric on $\mathcal{M}$. In fact in the case of degenerate diffusions it is the bundle E which has a Riemannian metric but it is convenient to give $T\mathcal{M}$ a possibly weaker inner product or Finsler metric to make sense of the above. We then consider the restricted L^2 tangent bundle $L^2\mathcal{E}$ consisting of (equivalence classes of) paths which take values almost surely in E and are in L^2. These will be identified with elements of $L^2 T C_a\mathcal{M}$ via the inclusion of E in $T\mathcal{M}$.

Then we define $\mathcal{H}_\sigma$ to be the set of v with $\frac{\mathbb{D}}{dt} v \in L^2\mathcal{E}$.

The space $L^2\mathcal{E}$ forms a smooth vector bundle over $C_a\mathcal{M}$, with a Riemannian metric, and we can use our operator $\frac{\mathbb{D}}{dt}$ to transfer this to the Bismut tangent bundle, at least over some set of full μ_a-measure. See Sections 2.3 and 8.1 of [20].

2.2. H-differentiation, divergence operator, Clark–Ocone formulae

To proceed take a space of smooth (in the Fréchet sense) functions f on $C_a\mathcal{M}$, dense in $L^2(C_a\mathcal{M}, \mu_a; \mathbf{R})$, such as the space $Cyl^\infty\mathcal{M}$ of C^∞ cylindrical functions, and define their H-derivative $d^H f_\sigma : \mathcal{H}_\sigma \to \mathbf{R}$ at σ to be the composition of the usual derivative at σ with i_σ. In the situations we are interested in this will give a densely defined closable operator $f \mapsto d^H f$ from $L^2(C_a\mathcal{M}, \mu_a; \mathbf{R})$ to the space of L^2-sections of the dual "bundle" to $\mathcal{H}$, the L^2 H-one-forms on $C_a\mathcal{M}$. We will let d denote its closure and $\mathbb{D}^{2,1}$ the domain of d with graph norm. There is a discussion of such analysis in this generality in [26] with some emphasis on the resulting Dirichlet forms.

As usual we get the closed operator ∇, the gradient operator, from the domain of d to the space $L^2\Gamma\mathcal{H}$ of L^2 sections of $\mathcal{H}$, the L^2 H-vector fields. We shall use $\phi^\sharp$ to denote the H-vector field corresponding to an H-one-form ϕ and the other way round, so $(\phi^\sharp)^\sharp = \phi$. Then $\nabla f = (df)^\sharp$. There is also the adjoint, div, of $-\nabla$ defined from its domain in $L^2\Gamma\mathcal{H}$ to $L^2(C_a\mathcal{M}, \mu_a; \mathbf{R})$, so

$$\int_{C_a\mathcal{M}} df_\sigma(V(\sigma))d\mu_a(\sigma) = -\int_{C_a\mathcal{M}} f(\sigma)\operatorname{div}(V)(\sigma)d\mu_a(\sigma) \tag{1}$$

$$= \int_{C_a\mathcal{M}} \langle \nabla f(\sigma), V(\sigma)\rangle_\sigma d\mu_a(\sigma). \tag{2}$$

On $C_a\mathcal{M}$ there is the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ of sigma-sub-algebras of the Borel sigma-algebra where $\mathcal{F}_t$ is the sigma-algebra generated by the evaluation maps, $ev_s : C_a\mathcal{M} \to \mathcal{M}$ for $0 \le s \le t$. For a function defined on $C_a\mathcal{M}$ to be measurable with respect to $\mathcal{F}_t$ essentially means that it depends only on the restrictions of each path to the interval $[0,t]$. A vector field V on $C_a\mathcal{M}$ is said to be *adapted* if $\sigma \mapsto V(\sigma)(t)$ is $\mathcal{F}_t$-measurable for each t. In our situations adapted vector fields which are L^2-sections of $\mathcal{H}$ are in the domain of the divergence operator. Then the divergence is given by an Itô integral. In the notation of [15]

$$\operatorname{div} V(\sigma) = -\int_0^T \langle \frac{\mathbb{D}}{dt}V(\sigma), d\{\sigma\}(t)\rangle_{\sigma(t)}.$$

In our situations there will be a special class of solutions to the equation $\operatorname{div} V = f$ for given square integrable $f : C_a\mathcal{M} \to \mathbf{R}$ with $\int_{C_a\mathcal{M}} f d\mu_a = 0$. These are the solutions given by generalised Clark–Ocone formulae. For this, given an H-one-form ϕ in L^2, define $CO(\phi) \in L^2(C_a\mathcal{M}; \mathbf{R})$ by

$$CO(\phi) = -\operatorname{div}(\mathcal{P}\phi^\sharp) \tag{3}$$

where for any H-vector field V in L^2 we let $\mathcal{P}V$ be the orthogonal projection onto the L^2 adapted H-vector fields. This is determined by taking conditional expectations and is given by

$$\frac{\mathbb{D}}{dt}\mathcal{P}V = \mathbf{E}\Big\{\frac{\mathbb{D}}{dt}V|\mathcal{F}_t\Big\} = W_t\mathbf{E}\Big\{\frac{d}{dt}(W_t^{-1}V_t)|\mathcal{F}_t\Big\}. \tag{4}$$

Then if $f : C_a\mathcal{M} \to \mathbf{R}$ is in $\mathbb{D}^{2,1}$ the Clark–Ocone formula states that for almost all σ

$$(5) \qquad f(\sigma) = \int_{C_a\mathcal{M}} f d\mu_a + CO(df).$$

The first version of the Clark–Ocone formula for path spaces of manifolds was by S. Fang, [27]. He pointed out that it implies that the space of exact H-one-forms is closed in the space of H-one-forms, and so there is a spectral gap for the generalised Laplacian, or Ornstein–Uhlenbeck operator, $\triangle = \operatorname{div} \nabla$, with a Poincaré inequality. Later Capitaine, Hsu, & Ledoux [6] showed that it implies a logarithmic Sobolev inequality, see also [17]. For extensions to more general f on Abstract Wiener spaces, see [42].

§3. Paths on diffeomorphism groups

3.1. Wiener processes on Diff(M) and their Itô maps

Let M be a smooth compact and connected manifold. Let $\mathcal{D}^s(M)$ denote the space of diffeomorphisms of M in the Sobolev space H^s, for sufficiently large $s > 0$. By the Sobolev embedding theorem $\mathcal{D}^s(M)$ will consist of C^r diffeomorphisms if $s > \frac{1}{2}\dim M + r$. It is a C^∞ Hilbert manifold, and a topological group. Moreover each right translation $R_\theta : \mathcal{D}^s(M) \to \mathcal{D}^s(M)$ given by $R_\theta(h) = h \circ \theta$, by an element $\theta \in \mathcal{D}^s(M)$, is C^∞. Let Diff(M) denote the space of C^∞ diffeomorphisms. This is the intersection, $\cap_s \mathcal{D}^s(M)$, and our processes etc. will actually be on this space. For our purposes it is easier to have a Hilbert manifold structure so we will usually work on $\mathcal{D}^s(M)$ for high s. There are approaches to working directly on Diff(M), see [37] or [40], but stochastic differential equations theory in Fréchet, or even general Banach spaces, is not well developed.

We consider a Wiener process on Diff(M) in the sense of Baxendale [2]. This is a sample continuous stochastic process $\{\xi_t : t \geq 0\}$ with values in Diff(M), starting at the identity element id, with independent increments on the left which are identically distributed i.e. for $h > 0$ and $0 \leq s < t$ the increments $\xi_{t+h}\xi_{s+h}^{-1}$ and $\xi_t\xi_s^{-1}$ are independent and have the same law. For simplicity we shall make the symmetry assumption that $\{\xi_t : t \geq 0\}$ and $\{\xi_t^{-1} : t \geq 0\}$ have the same law. We will restrict to times $t \in [0, T]$. We will discuss analysis on the path space $C_{id}\mathcal{D}^s$ with measure μ_{id} the law of $\xi_\cdot$.

By Baxendale's theorem, [2], such a process can be considered as the solution flow to a Stratonovich stochastic differential equation on M driven by a possibly infinite dimensional Brownian motion.

More precisely there is a Hilbert space $\mathfrak{k}$, (a gothic k), of smooth vector fields on M continuously included in each of the spaces $\mathfrak{D}^s$ of H^s vector fields on M and inducing a Gaussian measure on each of them. In turn this determines a Gaussian measure $\mathbb{P}$ on $C_0\mathfrak{D}^s$ for which the canonical process $B_t : C_0\mathfrak{D}^s \to \mathfrak{D}^s$, $t \in [0,T]$, given by $B_t(\sigma) = \sigma(t)$ is a Wiener process. Then, treating $\mathfrak{D}^s$ as the tangent space at the identity to $\mathcal{D}^s$, our process $\xi_.$ can be taken, up to law, to be the solution to the right invariant Stratonovich stochastic differential equation:

$$d\xi_t = TR_{\xi_t} \circ dB_t \tag{6}$$

starting at the identity. Alternatively it is the solution flow to the stochastic differential equation on M:

$$dx_t = ev_{x_t} \circ dB_t \tag{7}$$

where $ev_y : \mathfrak{D}^s \to T_yM$ denotes evaluation at $y \in M$.

For us the main point is that we obtain a measurable map, the Itô map of the stochastic differential equation (6)

$$\mathcal{I}^{\mathfrak{D}^s} : C_0\mathfrak{D}^s \to C_{id}\mathcal{D}^s$$

given by $\mathcal{I}^{\mathfrak{D}^s}(\omega)_t = \xi_t(\omega)$. This sends $\mathbb{P}$ to μ_{x_0}. It will play the role of a chart for our non-linear path space.

3.2. The Bismut tangent spaces for Diff(M)

We have a Hilbert bundle E^s over $\mathcal{D}^s$ obtained by right translating our reproducing Hilbert space $\mathfrak{k}$. This is the image of the principal symbol of the possibly infinite-dimensional diffusion generator of $\xi_.$. Although left translation is not smooth on $\mathcal{D}^s$ the map

$$\mathcal{D}^{s+r} \times \mathcal{D}^s \to \mathcal{D}^s$$

given by $(f,h) \mapsto f \circ h$ is C^r, [12], so E^s is a smooth bundle with a natural smooth right invariant metric and flat connection. It gives rise to a smooth Hilbert bundle $L^2\mathcal{E}$ over $C_{id}\mathcal{D}^s$ as described above.

Our transport operators will be taken to be left translations:

$$W_t^\sigma = TL_{\sigma(t)} : T_{id}\mathcal{D}^s \to T_{\sigma(t)}\mathcal{D}^s$$

so

$$\frac{\mathbb{D}}{dt}v = TL_{\sigma(t)}\frac{d}{dt}(TL_{\sigma(t)})^{-1}v. \tag{8}$$

Then as above we obtain the Bismut tangent space, at a path σ. It will be denoted by $\mathcal{H}_\sigma^K$. It is defined for all $\sigma \in C_{id}\mathcal{D}^s$.

At any fixed time $t \in [0,T]$ our Itô map $\mathcal{I}_t^{\mathfrak{D}^s}$ has an H-derivative in the sense of Malliavin calculus, $T_\omega \mathcal{I}_t^{\mathfrak{D}^s} \in \mathbb{L}_2(H; T_{\mathcal{I}^{\mathfrak{D}^s}(\omega)_t}\mathcal{D}^s)$, for almost all $\omega \in C_0\mathfrak{D}^s$. Here $H = L_0^{2,1}([0,T];\mathfrak{k})$ and $\mathbb{L}_2$ refers to Hilbert-Schmidt maps. It is given by

$$\begin{aligned} T_\omega \mathcal{I}_t^{\mathfrak{D}^s}(h) &= TL_{\sigma(t)} \int_0^t (TL_{\sigma(r)})^{-1} TR_{\sigma(r)} \frac{d}{dr} h(r)dr \\ &= \left(\frac{\mathbb{D}}{dt}\right)^{-1} \left(TR_{\sigma(-)}\dot{h}\right) \end{aligned}$$

where $\sigma = \mathcal{I}^{\mathfrak{D}^s}(\omega)$.

Thus we obtain $T\mathcal{I}^{\mathfrak{D}^s} : C_0\mathfrak{D}^s \times H \to TC_{id}\mathcal{D}^s$ which maps the trivial H-bundle to our Bismut tangent bundle. It is only defined up to sets of measure zero but is isometric on fibres.

3.3. Analysis on Diff(M)

We can take the closure of our H-differentiation on the space of cylindrical functions $F : \mathcal{C}_{id}\mathcal{D}^s \to \mathbf{R}$ of the form $F(\sigma) = g\Big(\sigma(t_1)(x_1), \ldots, \sigma(t_k)(x_k)\Big)$ for g a smooth function on k copies of M and $t_1, \ldots, t_k$ in $[0,T]$ with $x_1, \ldots, x_k$ in M, to get the analysis described in Section 3.2. A Clark-Ocone formula with consequent log Sobolev inequality can be found in [17].

In the sense of [20] this is a situation where the Itô map has no "redundant noise". It is invertible and essentially fits into the framework of Fang & Franchi [28] though $\mathcal{D}^s$ is infinite dimensional. Either way we can deduce that our Itô map does give an "isomorphism" as in [28]. In particular an L^2 function $F : C_0\mathcal{D}^s \to \mathbf{R}$ is in $\mathbb{D}^{2,1}$ if and only if its composition with the Itô map is in the $\mathbb{D}^{2,1}$ space of the linear space $C_0\mathfrak{D}^s$ and then there is the chain rule:

$$d(F \circ \mathcal{I}^{\mathfrak{D}^s}) = \mathcal{I}^{\mathfrak{D}^s *}(dF) \tag{9}$$

where $\mathcal{I}^{\mathfrak{D}^s *}$ denotes the pull back operation on forms, i.e. composition with $T\mathcal{I}^{\mathfrak{D}^s}$ for H-one-forms or with its exterior powers for higher order forms. In fact we can define H-q-forms on $\mathcal{C}_0\mathcal{D}^s$ to be measurable sections of the exterior power $\wedge^q\mathcal{H}^K$ of our Bismut tangent bundle, obtain a closed exterior derivative operator on the corresponding spaces of L^2 forms and we see, as in [28], that the Itô map gives an isomorphism of the resulting deRham complex with that of the Abstract Wiener space corresponding to the inclusion of $\mathfrak{k}$ into $C_0\mathfrak{D}^s$. For the latter see [41]. In particular the L^2 cohomology is trivial.

3.4. Some special vector fields on Diff(M)

Assume in this section that $ev_y : \mathfrak{k} \to T_yM$ is surjective for all $y \in M$. This corresponds to non-degeneracy of the SDE (7). Give T_yM the quotient inner product, $\langle -,-\rangle_y$ say. Then ev_y has a right inverse $Y_y \in \mathbb{L}(T_yM;\mathfrak{k})$ defined by $\langle Y_y(v),\alpha\rangle_{\mathfrak{k}} = \langle v, ev_y\alpha\rangle_y$ for all $\alpha \in \mathfrak{k}$ and all $y \in M$. It can be written in terms of the reproducing kernel of $\mathfrak{k}$, as in [17], [15]. As in [19] we follow what is now a rather standard procedure to lift certain processes taking values in tensor bundles of M to tensor fields on flat Wiener space. This goes back Malliavin's approach to Hörmander's theorem, and to Bismut, for example see [4], [3]. We can then transfer them to $C_{id}\mathcal{D}^s$. Take $b \in T_{x_0}M$ and define an H-vector field $h^b : C_0\mathfrak{D}^s \to L_0^{2,1}([0,T];\mathfrak{k})$ on $C_0\mathfrak{D}^s$ by

$$h_t^b = \int_0^t Y_{\xi_r(x_0)}\left(T\xi_r(b)\right) dr. \tag{10}$$

Using formula (9), this can be pushed forward by $T\mathcal{I}^{\mathfrak{D}^s}$ to give the H-vector field V^b on $C_{id}\mathcal{D}^s$:

$$V^b(\xi_\cdot)_t = T\xi_t \int_0^t (T\xi_r)^{-1}\left(\dot{h}_r^b((\mathcal{I}^{\mathfrak{D}^s})^{-1}\xi_\cdot)\circ\xi_r\right) dr. \tag{11}$$

Note that since $ev_y(Y_yv) = v$ for all $v \in T_yM$, evaluation at x_0 yields

$$V^b(\xi_\cdot)_t(x_0) = tT_{x_0}\xi_t b. \tag{12}$$

These vector fields are in the domain of the divergence operator and their divergences can be expressed as Itô integrals, [17]:

$$\operatorname{div} V^b(\xi_\cdot) = -\int_0^T \langle \frac{d}{dr}h_r^b, dB_r\rangle_{\mathfrak{k}} \tag{13}$$

$$= -\int_0^T \langle T\xi_r(b), ev_{\xi_r(x_0)}dB_r\rangle_{\xi_r(x_0)} \tag{14}$$

where we are abusing notation by using $\mathcal{I}^{\mathfrak{D}^s}$ and its inverse as an identification. From this, if we set $V^i = V^{b^i}$ for tangent vectors $b^i, i = 1,\dots$ in $T_{x_0}M$, the divergence of the H-$(q+1)$-vector field $V^i \wedge V^2 \wedge \cdots \wedge V^{q+1}$ can be given by the usual formula, as in [41],

(15)

$$
\begin{aligned}
&q \operatorname{div} V^1 \wedge V^2 \wedge \cdots \wedge V^{q+1} \\
&= \sum_{j=1}^{q+1} (-1)^j (\operatorname{div} V^j) V^1 \wedge \cdots \wedge \hat{V^j} \wedge \cdots \wedge V^{q+1} \\
&- \sum_{1 \le i \le j \le q+1} (-1)^{i+j} [V^i, V^j] \wedge V^1 \wedge \ldots \hat{V^i} \wedge \cdots \wedge \hat{V^j} \wedge \cdots \wedge V^{q+1}.
\end{aligned}
$$

In the special case of *gradient flows*, when all the elements of $\mathfrak{k}$ are gradient vector fields, it is shown in [19] that the vector fields $h^b, b \in T_{x_0}M$ commute; so therefore will $\{V^b : b \in T_{x_0}M\}$ and the Lie brackets above vanish. In this situation, if we take an orthonormal base $E^1, E^2, \ldots$ for $\mathfrak{k}$ and choose $f^j : M \to \mathbf{R}$ with $f^j(x_0) = 0$ such that $E^j = \nabla f^j$ for each j, we have an isometric immersion $\Psi : M \to \mathfrak{k}$ given by $y \mapsto \sum_j f^j(y) E^j$ with derivative Y. Commutativity holds after a short calculation because this integrability of Y implies that the exterior derivative of Y vanishes.

§4. Descent to M

4.1. Induced semi-groups on functions and forms on M

For bounded measurable $f : M \to \mathbf{R}$ and $t \ge 0$ define $P_t f : M \to \mathbf{R}$ by

$$P_t f(y) = \mathbf{E} f(\xi_t(y)),$$

the expectation being with respect to $\mathbb{P}$. This gives a diffusion semigroup with generator restricting to a smooth diffusion operator, $\mathcal{A}$ say, on smooth functions. It will have a Hörmander form representation obtained by taking an orthonormal basis of $\mathfrak{k}$; so there may be an infinite sum of squares of vector fields in this representation.

There are also extensions of this to differential forms ϕ on M by defining $P_t \phi = \mathbf{E} \xi_t^*(\phi)$. The generator of this semi-group, $\mathcal{A}^q$ say for q-forms, is obtained by applying the Hörmander form representation just mentioned for $\mathcal{A}$ and interpreting it as a sum of squares of Lie derivatives acting on forms; see [14] and [17]. Note that these commute with exterior differentiation: $dP_t^q = P_t^{q+1} d$.

4.2. The diffeomorphism bundle, related connections and decomposition of ξ.

Let $p = ev_{x_0} : \mathcal{D}^s \to M$. This is a C^∞ submersion giving $\mathcal{D}^s$ the structure of a principal bundle with group $\mathcal{D}^s_{x_0}$, those diffeomorphisms in $\mathcal{D}^s$ which fix the base point x_0, acting on the right, though we need to remain careful about the differentiability.

Set $x_t = p(\xi_t) = \xi_t(x_0)$ and let μ_{x_0} denote its law as a measure on $C_{x_0}M$. Suppose the generator $\mathcal{A}$ is elliptic. Then a principal connection is induced on $p : \mathcal{D}^s \to M$. The horizontal lift map, $\mathbf{h}_\theta : T_yM \to T_\theta\mathcal{D}^s$ for $\theta \in \mathcal{D}^s$ with $\theta(x_0) = y$, is determined by the reproducing kernel of our space $\mathfrak{k}$ of vector fields, see [16] or [15]:

$$\mathbf{h}_\theta(v)(x) = Y_y(v)(\theta(x)). \tag{16}$$

In fact ellipticity of $\mathcal{A}$ can be replaced by *cohesiveness* at the expense of dealing with semi-connections, [15].

From this connection we obtain a connection, $\hat{\nabla}$ say, on the tangent bundle TM to M. If σ is a piecewise C^1 path in M starting at x_0 let $\tilde{\sigma}$ be its horizontal lift to $\mathcal{D}^s$ starting from the identity; it will lie in Diff(M). Then, [15], the $\hat{\nabla}$ parallel translation along σ will be given by

$$\hat{/\!/}_t^\sigma = T_{x_0}\tilde{\sigma}(t).$$

Stochastic calculus gives a lift of $\{x_t : 0 \le t \le T\}$ to a horizontal process $\tilde{x}_t : C_0\mathfrak{D}^s \to \mathcal{D}^s$, $0 \le t \le T$, starting at the identity, and from this we obtain a decomposition:

$$\xi_t = \tilde{x}_t \circ g_t^{x.} \tag{17}$$

where for μ_{x_0}- almost all paths σ the process $g^\sigma_.$ is a $\mathcal{D}^s_{x_0}$-valued diffusion independent of $\{x_t; 0 \le t \le T\}$. See [15] where such decompositions are discussed in some generality.

4.3. Raw generalised Bismut formulae

We continue with the previous notation, but assume that the process $\{x_t := ev_{x_0}\xi_t : 0 \le t \le T\}$, i.e. the solution to the SDE (7) from a point $x_0 \in M$ is a non-degenerate diffusion with generator $\mathcal{A}$ and that the connection $\hat{\nabla}$ induced on TM as above (or equivalently that obtained by projection using the evaluation map, $ev_- : M \times \mathfrak{k} \to TM$: the LJW-connection of the SDE (7) in the sense of [17]), is the Levi–Civita connection, ∇, for the Riemannian metric on M determined by $\mathcal{A}$. Then with respect to that Riemannian structure, $\mathcal{A} = \frac{1}{2}\triangle$ and the solutions of equation (7) are Brownian motions on M. This will hold, for example, if $\mathfrak{k}$ consists of gradient vector fields as described above. It also holds in the symmetric space situation discussed below.

In this situation the semi-groups induced by $\xi_.$ on forms are the usual heat semi-groups with generators $\mathcal{A}^q = \frac{1}{2}\triangle$ for $\triangle$ the Hodge Laplacian, [17]. To demonstrate one use of this let ϕ be a smooth one-form on M

and consider b^1, b^2 in $T_{x_0}M$ as above. Fix $\tau \in (0, T]$. Pull ϕ back to get the one form Φ on $\mathcal{C}_{id}\mathcal{D}^s$:

$$\Phi_\sigma(v) = \phi_{\sigma(\tau)(x_0)}(v(\tau)(x_0)) \tag{18}$$

for $\sigma \in C_{id}\mathcal{D}^s$ and $v \in T_\sigma C_{id}\mathcal{D}^s$.

Since exterior differentiation commutes with pull-backs, we have by integration by parts on $C_{id}\mathcal{D}^s$:

$$\begin{aligned}
d(P^1_\tau(\phi))(b^1 \wedge b^2) &= P^2_\tau(d\phi)(b^1 \wedge b^2) \\
&= \mathbf{E}\Big\{ d\phi_{\xi_\tau(x_0)}\left(T\xi_\tau b^1 \wedge T\xi_\tau b^2\right) \Big\} \\
&= \frac{1}{\tau^2}\int_{C_{id}\mathcal{D}^s} d\Phi_\sigma\left((V^1 \wedge V^2)(\sigma)\right) d\mu_{id}(\sigma) \\
&= -\frac{1}{\tau^2}\int_{C_{id}\mathcal{D}^s} \Phi_\sigma\left(\operatorname{div}(V^1 \wedge V^2)(\sigma)\right) d\mu_{id}(\sigma) \\
&= \frac{1}{2\tau}\mathbf{E}\Big\{ -\int_0^T \langle T\xi_r(b^1), ev_{\xi_r(x_0)}dB_r\rangle_{\xi_r(x_0)}\phi_{\xi_\tau(x_0)}(b^2) \\
&\quad + \frac{1}{2\tau}\int_0^T \langle T\xi_r(b^2), ev_{\xi_r(x_0)}dB_r\rangle_{\xi_r(x_0)}\phi_{\xi_\tau(x_0)}(b^1)\Big\}
\end{aligned}$$

using equations (13), (15) and the fact that in our situation the bracket term of equation (15) vanishes on evaluation, [19].

By the martingale property of stochastic integrals this yields Xue-Mei Li's formula, [35]

$$\begin{aligned}
d(P^1_\tau\phi)(b^1 \wedge b^2) = \frac{1}{2\tau}\mathbf{E}\Big\{ &-\int_0^\tau \langle T\xi_r(b^1), ev_{\xi_r(x_0)}dB_r\rangle_{\xi_r(x_0)}\phi_{\xi_\tau(x_0)}(b^2) \\
&+\frac{1}{2\tau}\int_0^\tau \langle T\xi_r(b^2), ev_{\xi_r(x_0)}dB_r\rangle_{\xi_r(x_0)}\phi_{\xi_\tau(x_0)}(b^1)\Big\}.
\end{aligned}$$

(The factor of $\frac{1}{2\tau}$ rather than $\frac{1}{\tau}$ comes from using the conventions of [21] rather than [19].)

This gives a formula for the exterior derivative of $P_t\phi$ in terms of ϕ and so extends to bounded measurable forms. The same argument works for any $q \in \{0, 1, 2, \dots\}$, with $q = 0$ being the better known, and most useful, case. As it stands it is not intrinsic, since it depends on the choice of stochastic flow $\xi_\cdot$. However by "integrating out the redundant noise", i.e. conditioning with respect to evaluation at x_0, an intrinsic version can be found, [24]. It involves Weitzenböck curvatures. These arise because the conditional expectation, $\overline{\wedge^q T_{x_0}\xi_t}(V_0) := \mathbf{E}\{\wedge^q T_{x_0}\xi_t(V_0)|p(\xi_.) = \sigma\}$ is given by

$$\frac{D}{dt}\overline{\wedge^q T_{x_0}\xi_t}(V_0) = -\frac{1}{2}\mathcal{R}^q_{\sigma(t)}\overline{\wedge^q T_{x_0}\xi_t}(V_0) \tag{19}$$

for almost all $\sigma \in C_{x_0}M$ where $V_0 \in T_{x_0}M$ and $\mathcal{R}^q_y : \wedge^q T_y M \to \wedge^q T_y$ is the q-th Weitzenböck curvature, corresponding to the Ricci curvature when $q = 1$; see [23] for gradient systems and [17] in general.

This conditioning is essentially a push forward operation by integration over the fibres of $ev_{x_0} : C_{id}\mathcal{D}^s \to C_0M$. This could be achieved by using the decomposition (17) but the approach in [17] following [23] is simpler. There are other versions and generalisations of these formulae in [10], see also [38].

For more geometric applications of this sort of analysis of stochastic flows see Kusuoka [32] and Elworthy & Rosenberg [22].

4.4. Analysis on $C_{x_0}M$

We continue with the previous notation and assumptions, so that M is Riemannian, compact, and furnished with its Levi–Civita connection. We will also let $p : C_{id}\mathcal{D}^s \to C_{x_0}M$ denote the induced map on the path spaces. Again it is a C^∞ submersion. It also maps μ_{id} to the Brownian motion measure μ_{x_0}, and the composition of it with $\mathcal{I}^{\mathfrak{D}^s}$ is the Itô map $\mathcal{I} : C_0\mathfrak{D}^s \to C_{x_0}M$, of the SDE (7), so $x_t(\omega) = \mathcal{I}(\omega)_t$, $0 \le t \le T$.

Define transport operators along almost all paths σ in M by

$$\frac{D}{dt}W_t^\sigma(v_0) = -\frac{1}{2}\mathrm{Ric}^\sharp(W_t^\sigma(v_0)) \tag{20}$$

for $v_0 \in T_{x_0}M$ where $\mathrm{Ric}^\sharp : TM \to TM$ is given by the Ricci tensor. This is the *damped* or *Dohrn–Guerra* parallel translation. It defines $\frac{\mathbb{D}}{dt}$ and the Bismut tangent spaces $\mathcal{H}_\sigma$ as before. As a subset of $T_\sigma C_0M$ we can characterise $\mathcal{H}_\sigma$ as the set of tangent vectors v such that $t \mapsto (/\!/_t^\sigma)^{-1}v(t)$ is in $L^{2,1}([0,T]; T_{x_0}M)$. This is a standard definition, but we give it the damped inner product :

$$\langle v^1, v^2\rangle_\sigma = \int_0^T \langle \frac{\mathbb{D}}{dt}v^1, \frac{\mathbb{D}}{dt}v^2\rangle_{\sigma(t)}dt.$$

This has various advantages, as will be seen, and is often used, e.g. [7], [39] since it simplifies many formulae.

From formula (12), if V^b is one of the H-vector fields on $C_{id}\mathcal{D}^s$ described above then $T_\xi p(V^b(\xi_.))_t = tT_{x_0}\xi_t(b)$. Since $T_{x_0}\xi_t$ is generally only continuous in t, even after parallel translation back to $T_{x_0}M$, we see that Tp *does not map Bismut tangent spaces to Bismut tangent spaces* in general. However if we condition as described above and let the result be denoted by $\overline{TpV^b}(\sigma) \in T_\sigma C_{x_0}M$, for almost all $\sigma \in C_{x_0}M$ we get precisely that

$$\overline{TpV^b}(\sigma)_t = W_t^\sigma(tb)$$

and so we obtain an H-vector field on $C_{x_0}M$. We can clearly replace $t \mapsto tb$ by any element of $L_0^{2,1}T_{x_0}M$. In fact from [20] we see that we can replace V^b by any L^2 H-vector field V on $C_{id}\mathcal{D}^s$ to get an L^2 H-vector field $\overline{TpV}$ on $C_{x_0}M$. Dually the usual pull back map of one-forms by p extends to a continuous linear map

$$p^* : L^2\Gamma\mathcal{H}^* \to L^2\Gamma\mathcal{H}^{K*}$$

from L^2 H-one-forms on $C_{x_0}M$ to those on $C_{id}\mathcal{D}^s$. This is given by stochastic integration, [20].

One of the main results is that if $f : C_{x_0}M \to \mathbf{R}$ is in $\mathbb{D}^{2,1}$ then so is $f \circ \mathcal{I}$, and hence $f \circ p$, with the chain rule, as in (9),

$$d(f \circ p) = p^*(df). \tag{21}$$

A major open problem is whether $f \circ p \in \mathbb{D}^{2,1}$ implies $f \in \mathbb{D}^{2,1}$. This is equivalent to *Markov uniqueness* for the "Laplace" or "Ornstein–Uhlenbeck" operator $\triangle$ on C_{x_0M} given by $\triangle = div\nabla$ acting on smooth cylindrical functions, as shown in [20] using Eberle's work [11]. Markov uniqueness is slightly weaker than essential self-adjointness. It corresponds to the existence of a unique "Brownian motion" on $C_{x_0}M$; that is a Markov process whose generator agrees with $\frac{1}{2}\triangle$ on smooth cylindrical functions, [11]. This will also follow if we can show that if g is in $\mathbb{D}^{2,1}$ on $C_{id}\mathcal{D}^s$ then its conditional expectation with respect to p is also in $\mathbb{D}^{2,1}$, [20].

The key to the relationship between Markov uniqueness and composition properties with p is the notion of *weak differentiability.* We say that an L^2 function f on M is *weakly differentiable* if it is in the domain of the adjoint of the restriction of the divergence operator to $\mathbb{D}^{2,1}$-H-vector fields. Here we use the damped Markovian connection described in Section 4.5 below. Let $\tilde{d}f \in \mathbb{L}^2\Gamma\mathcal{H}^*$ be the resulting weak derivative, i.e. the adjoint of this restriction of the divergence, and let $W^{2,1}$ denote the space of such functions. For flat Wiener space, $\mathbb{D}^{2,1}$-H-vector fields are dense in the domain of the divergence and so there is no difference between $\mathbb{D}^{2,1}$and $W^{2,1}$. Eberle showed that this equality is equivalent to Markov uniqueness, though he based his definition of weak differentiability on cylindrical one-forms rather than $\mathbb{D}^{2,1}$ vector fields. From

[20] we have

$$f \in W^{2,1} \Longleftrightarrow f \circ p \in \mathbb{D}^{2,1}. \tag{22}$$

4.5. L^2 H-forms and Hodge Theory

In the presence of curvature, such as when using the Levi–Civita connection for a non-flat manifold M, the Lie bracket of H-vector fields will not generally be an H-vector field, [9], and defining H-2-forms to be sections of the dual bundle $\wedge^2\mathcal{H}^*$ to the exterior product bundle, with Hilbert space completion, $\wedge^2\mathcal{H}$, does not lead to a closed exterior derivative operator on L^2 H-forms, [33]. The difficulty reveals itself in the formula (15) for the divergence. This problem was skirted by Leandre in [33] to get a cohomology theory for H-forms using such exterior powers, but this did not include an L^2 Hodge decomposition or self-adjoint Laplacian.

An alternative procedure given in detail in [21] is to modify the definition of H-form. As usual they will be sections of dual bundles to a bundle of exterior powers of tangent vectors $\mathcal{H}^q$, $q = 1, 2, \ldots$, with $\mathcal{H}^1 = \mathcal{H}$. However the bundles $\mathcal{H}^q$ are perturbations of the usual $\wedge^q\mathcal{H}$ by a transformation using the curvature of M. They were originally defined by the projection obtained by integrating out the noise from exterior powers of the H-derivative of the Itô map $\mathcal{I}$. In [18] they are shown to depend only on the Riemannian structure of M.

The elements of $\mathcal{H}^2$ can be described using the *damped Markovian connection* on $\mathcal{H}$. This is conjugate by $\frac{\mathbb{D}}{dt}$ to the Levi–Civita connection, or "pointwise connection ", on the L^2 tangent bundle, $L^2TC_{x_0}M$, inherited as described in [13] from the Levi–Civita connection of M. Let $\mathbb{R}$ denote its curvature operator. Then $u \in \mathcal{H}^2$ if and only if $u - \mathbb{R}(u) \in \wedge^2\mathcal{H}^1$, [21]. In [19] it was shown that exterior powers such as $V^1(x_0) \wedge V^2(x_0)$, for V^1 and V^2 as in Section 3.4, are sections of $\mathcal{H}^2$.

It seems not so easy to characterise elements of $\mathcal{H}^q$ for $q > 2$ and so far closability of exterior differentiation on the resulting forms has only been shown for one and for two forms, with consequent self-adjoint Hodge Laplacian, $\triangle = -(dd^* + d^*d)$ and Hodge decomposition for such forms, [21], [18]. It is reasonable to expect that the corresponding L^2 deRham cohomology groups should vanish since $C_{x_0}M$ is contractible and the measure involved is finite; see for example [5] and [29] for discussions of analogous finite dimensional situations. We consider this when M is a symmetric space next.

§5. Paths on symmetric spaces

We now specialise further to assume that M is a compact Riemannian symmetric space with associated compact Lie group K acting on its left, as isometries. Let $p : K \to M$ be $k \mapsto k.x_0$, giving an identification of M with the quotient K/G of K by $G := p^{-1}(x_0)$, the isotropy group of x_0. Under the right multiplication of G on K the map $p : K \to M$ is a principal G-bundle. This is naturally included into the diffeomorphism bundle discussed above, and we will use the notation used there.

The standard example is given by $M = S^2$, with x_0 the North Pole, $K = SO(3)$ acting as rotations, and $G = SO(1) = S^1$.

We suppose $\xi_.$ takes values in K and is a standard Brownian motion there for the bi-invariant metric on K which projects down to the metric on M. Then $\mathfrak{k}$ will be the Lie algebra of K with induced inner product. The Bismut tangent spaces $\mathcal{H}^K_{\xi_.}$ are as constructed on the full diffeomorphism group in Section 3.2 above.

The process $\{x_t : 0 \leq t \leq T\}$ is a Brownian motion on M, and the connection induced by $\xi_.$ on M is the Levi–Civita connection, [17], so that we are in the situation of Section 4.4 above. In particular if $f : C_{x_0}M \to \mathbf{R}$ is in $\mathbb{D}^{2,1}$ then $f \circ p$ is in $\mathbb{D}^{2,1}$.

As with the full diffeomorphism group we keep the L^2-deRham complex on $C_{id}K$ based on exterior powers $\wedge^q \mathcal{H}^K$. The fact that our stochastic flow $\xi_.$ is a flow of isometries allows us to pull back H-2-forms on $\mathcal{C}_{x_0}M$ to $\mathcal{C}_{id}K$ giving a continuous linear map, [18],

$$p^* : L^2\Gamma\mathcal{H}^{2*} \to L^2\Gamma \wedge^2 \mathcal{H}^{K*}.$$

This enables us to give a vanishing result for the first L^2 deRham cohomology of $C_{x_0}M$. We use d^1 for the exterior derivative as a closed operator from its domain in $L^2\Gamma\mathcal{H}^*$ to $L^2\Gamma\mathcal{H}^{2*}$:

Theorem 1. *Suppose M is a compact Riemannian symmetric space and $C_{x_0}M$ is furnished with Brownian motion measure.*
If $\phi \in L^2\Gamma\mathcal{H}^*$ with $d^1\phi = 0$ then $\phi = \tilde{d}f$ for some $f \in W^{2,1}$.

The details of the proof and related results will appear elsewhere. The main tool is the Clark–Ocone formula (5) and the main steps are as follows:

(1) $CO(\phi) \circ p = CO(p^*\phi)$
(2) $d^1\phi = 0 \Longrightarrow d^1(p^*\phi) = 0$
(3) Apply Fang & Franchi [28] to see

$$d\,(CO(p^*\phi)) = p^*\phi$$

(4) Therefore $CO(\phi) \in W^{2,1}$ and $\tilde{d}(CO(\phi)) = \phi$.

We used the equivalence (22).

An immediate corollary is:

Corollary 2. *If Markov uniqueness holds the first L^2 deRham cohomology group of $C_{x_0}M$ vanishes.*

Since an H-one-form ϕ is harmonic if and only if $d^1\phi = 0$ and $d^*\phi = 0$ we can use the definition of the weak derivative to obtain:

Corollary 3. *There are no non-zero $\mathbb{D}^{2,1}$ harmonic one-forms on $C_{x_0}M$.*

Recall that Markov uniqueness would follow if it were true that the conditional expectation of any $\mathbb{D}^{2,1}$ function f on $C_{id}K$ with respect to the sigma-algebra generated by p is also in $\mathbb{D}^{2,1}$. Intuitively this conditioning corresponds to integration over the fibres of p so in some sense the question is about the smoothness of the measure μ_{id} on $C_{id}K$ as it relates to the principal bundle structure of $p : C_{id}K \to C_{x_0}M$.

Added in proof. This approach has been extended in joint work with Yuxin Yang to prove Theorem 1 and its corollaries for the case of M an arbitrary compact Riemannian manifold with Levi–Civita connection, and in the symmetric space case to prove Theorem 1 with $f \in \mathbb{D}^{2,1}$ and $\phi = df$.

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