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On the middle Betti number of certain singularities with critical locus a hyperplane

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Abstract.

We study holomorphic germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with the following properties:

(i) the critical set H of the germ f is a hyperplane $H = \{x = 0\}$;
(ii) the transversal singularity of the germ f in points of the set $H \setminus \{0\}$ has type A_{k-1}.

We will investigate the topological structure of the Milnor fibre for f and give explicit formula for the middle Betti number of the Milnor fibre and for the quasihomogeneous case we express it in terms of weights and degrees.

§1. Introduction

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function for which the critical set is a hyperplane $H = \{(x, y_1, ..., y_n) \in \mathbb{C} \times \mathbb{C}^n | x = 0\}$. We consider as in [8] the group D_H of local analytic isomorphisms $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ with $\varphi(H) = H$. Let $\mathcal{O}_{n+1}$ be the ring of germs of holomorphic functions $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and consider two ideals of the ring $\mathcal{O}_{n+1}$: $J(f)$, the Jacobian ideal of f, namely

$$J(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n} \right)$$

and, for a natural number k the ideal $(x)^k$, the k-th power of the ideal

$$(x) = \left\{ h \in \mathcal{O}_{n+1} | h(0, y_1, \ldots, y_n) = 0 \text{ for any } (y_1, \ldots, y_n) \in \mathbb{C}^n \right\}.$$

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The group D_H acts on $(x)^k$ and the extended codimension of (the orbit of) f with respect to this action is defined by

$$c(f) = c_{I,e}(f) = \dim_{\mathbb{C}}((x)^k/\tau(f)),$$

where $\tau(f)$ is the tangent space to the orbit of f at the point f (see 1 for details), and $I = (x)^{k-1}$.

We will study germs $f \in (x^k)$ with $c(f) < \infty$ and prove

Theorem 1. *Milnor fibre of a hyperplane singularity $f = x^k g$, with $c(f) < \infty$ and an isolated singularity $g(0, y_1, \ldots, y_n)$ is homotopy equivalent to the bouquet of the circle S^1 and $k\mu+\sigma$ copies of the n-dimensional sphere, where $\mu = \mu(g(0, y_1, \ldots, y_n))$ is the Milnor number of the isolated singularity $g(0, y_1, \ldots, y_n)$, and σ is the number of Morse points in a special unfolding of f.*

We will give a proof of this theorem which is a generalization of our previous proof for the case of non-isolated hyperplane singularities when $k = 2$, see [10].

Moreover for a complex hyperplane singularity f with finite codimension the number σ of Morse points is given by

$$\sigma = \dim_{\mathbb{C}} \, (x)^k / \, (xf_x, f_{y_1}, \ldots, f_{y_n}).$$

In case $f = x^k g$ with g a quasihomogeneous function of degree d with weights $w_0, w_1, ..., w_n$,

$$\sigma = \left(\frac{d - w_0}{w_0} + 1 \right) \prod_{i=1}^{n} \frac{d - w_i}{w_i}.$$

Quasihomogeneous isolated singularities have been studied by Arnold [1], see also [2]. A formula for the Milnor number has been obtained using topological methods by Milnor and Orlik [7].

In detail, a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is called quasihomogeneous of degree d with weights $w_0, w_1, ..., w_n$ if for any $\lambda > 0$ one has

$$f(\lambda^{w_0} x, \lambda^{w_1} y_1, \ldots, \lambda^{w_n} y_n) = \lambda^d f(x, y_1, \ldots, y_n).$$

For quasihomogeneous isolated singularities the Milnor number is given by the formula

$$\mu(f) = \dim_{\mathbb{C}} \, (\mathcal{O}_{n+1}/J(f)) = \prod_{i=0}^{n} \frac{d - w_i}{w_i}.$$

One-dimensional quasihomogeneous singularities have been thoroughly investigated by D. Siersma [11, 12]. In this case $f = \sum_{i,j=1}^{n} h_{ij} g_i g_j$ with $h_{ij} \in \mathcal{O}_{n+1}$ and homogeneous g_i, whereas the Milnor number is given by the formula

$$\mu(f) = \prod_{i=0}^{n} \frac{d - w_i}{w_i} - [(2n+1)d - w - 3a] - \frac{\prod_{i=1}^{n} a_i}{\prod_{i=1}^{n} w_i},$$

where d is the degree of f, $w = \sum_{i=1}^{n} w_i$, $a = \sum_{i=1}^{n} a_i$, where w_i are the weights of f, and the a_i are the degrees of the homogeneous germs g_i.

§2. A characterization result

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function having H as its singular set. Then the following holds.

Lemma 1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic complex-valued function having H as its singular set. Then it can be represented in the form $f = x^k g(x, y_1, \ldots, y_n)$, where $k \geqslant 2$ and g is any germ from $\mathcal{O}_{n+1}$ with $g \notin (x)$.*

Proof. Since, by the assumption, $f(0, y_1, \ldots, y_n) = 0$, the initial function can be represented in the form

$$f(x, y_1, \ldots, y_n) = \int_0^1 \frac{df(tx, y_1, \ldots, y_n)}{dt} dt = \int_0^1 \frac{\partial f}{\partial x}(tx, y_1, \ldots, y_n) x \, dt.$$

Assume that

$$\int_0^1 \frac{\partial f}{\partial x}(tx, y_1, \ldots, y_n) dt = g_1(x, y_1, \ldots, y_n),$$

and, since f has a singularity on H, we have

$$g_1(0, y_1, \ldots, y_n) = \frac{\partial f}{\partial x}(0, y_1, \ldots, y_n) = 0.$$

Hence, applying this argument to $g_1(x, y_1, \ldots, y_n)$, we obtain a holomorphic function $g_2(x, y_1, \ldots, y_n)$ such that

$$g_1(x, y_1, \ldots, y_n) = x g_2(x, y_1, \ldots, y_n)$$

and, therefore,

$$f(x, y_1, \ldots, y_n) = x^2 g_2(x, y_1, \ldots, y_n).$$

One can repeat this process to obtain

$$f(x, y_1, \ldots, y_n) = x^k g_k(x, y_1, \ldots, y_n)$$

until $g_k(0, y_1, \ldots, y_n) \neq 0$. Then, taking $g = g_k$ we obtain the statement of the lemma. Q.E.D.

As in [9], the following holds:

Proposition 1. *Let $f \in (x^k)$. Then the tangent space $\tau(f)$ to* $\mathrm{Orb}(f)$ *is a finite-dimensional subspace of the Euclidean space $\mathcal{O}_{n+1}$ which has the form*

$$\tau(f) = \mathfrak{m}\left(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}\right) + (x)\left(\frac{\partial f}{\partial x}\right)$$

at the point f.

Now we can formulate the characterization theorem

Theorem 2. *Let $f \in (x)^k$ such that $f = x^k g(x, y_1, \ldots, y_n)$ and $g(0, y_1, \ldots, y_n)$ has an isolated singularity. Then the following statements are equivalent:*

(a) *$c(f)$ is finite;*

(b) *the functions $g(x, y_1, \ldots, y_n)$ and $g(0, y_1, \ldots, y_n)$ have both an isolated singularity;*

(c) *outside the points on H where $g(0, y_1, \ldots, y_n) = 0$, the germ f is equivalent to x^k; at the points where $g(0, y_1, \ldots, y_n) = 0$, except for the origin, f is equivalent to $x^k y_1$.*

Proof. (a)$\Rightarrow$(b). Let codim $f < \infty$. Then

$$\dim_{\mathbb{C}} (x)^k / \tau(f) < \infty,$$

where $\tau(f)$ as in 1 has the form

$$(\xi x^{k-1} g + \xi x^k g_x, \eta_1 x^k g_{y_1}, \ldots, \eta_n x^k g_{y_n}),$$

with $\xi \in (x)$ and $\eta_i \in \mathfrak{m}$, $i = 1, \ldots, n$.

For the function g, we consider the Jacobian ideal $(g_x, g_{y_1}, \ldots, g_{y_n})$. By the Briançon-Skoda theorem [3], we have $g^{n+1} \in (g_x, g_{y_1}, \ldots, g_{y_n})$. Hence we see that $\tau^{n+1}(f)$, which already contains the terms expressed through g_x and g_{y_i}, $i = 1, \ldots, n$, is contained in the ideal $(x^k g_x, x^k g_{y_1}, \ldots, x^k g_{y_n})$. Since

$$\dim_{\mathbb{C}} (x)^k / \tau^{n+1}(f) = \dim_{\mathbb{C}} (x)^k / \tau(f) + \dim_{\mathbb{C}} \tau(f) / \tau^{n+1}(f) < \infty,$$

we obtain

$$\begin{aligned}&\dim_{\mathbb{C}} \mathcal{O}_{n+1}/(g_x, g_{y_1}, \dots, g_{y_n})\\ &= \dim_{\mathbb{C}}(x^k)/(x^k g_x, x^k g_{y_1}, \dots, x^k g_{y_n}) < \dim_{\mathbb{C}}(x)^k/\tau^{n+1}(f) < \infty.\end{aligned}$$

Therefore, $g(x, y_1, \dots, y_n)$ has an isolated singularity.

(b)⇒(c). Let g be an isolated singularity and $g(0, y_1, \dots, y_n)$ have an isolated singularity at zero, i. e. grad $g(0, y_1, \dots, y_n) \neq 0$ except for the origin. Assuming, for definiteness, that $\frac{\partial g}{\partial y_1}(0, y_1, \dots, y_n)$ is nonzero at the point z, we consider the transformation

$$\widetilde{x} = x, \widetilde{y}_1 = g(x, y_1, \dots, y_n), \widetilde{y}_i = y_i, i = 2, \dots, n$$

of class D_H. The germ f is reduced to the form $\widetilde{x}^k \widetilde{y}_1$.

At the points from the singular plane $\{x = 0\}$ which lie outside the space $\{g(0, y_1, \dots, y_n) = 0\}$, we consider the transformation

$$\widetilde{x} = x\sqrt{g(x, y_1, \dots, y_n)}, \widetilde{y}_i = y_i, i = 1, \dots, n$$

of class D_H. At the points where $x = 0$, the Jacobian of this transformation is equal to $\sqrt{g(0, y_1, \dots, y_n)}$ since $g(0, y_1, \dots, y_n) \neq 0$. Therefore, at these points the germ f is right-equivalent to the germ x^k.

(c)⇒(a). Let f be a representative of the germ of a given hyperplane singularity. In the domain where it is given, we define a sheaf $\mathcal{F}$ of $\mathcal{O}_{n+1}$-modules as follows:

$$\mathcal{F}(U) = (x)^{k+1}/(\tau(f) \cap (x)^{k+1})$$

where $(x)^{k+1}$ and $\tau(f)$ are considered as modules over the ring of holomorphic functions on $U \subset \mathbb{C}^{n+1}$ and $\mathcal{O}_{n+1}$ as a sheaf of germs of holomorphic functions on $\mathbb{C}^{n+1}$. The sheaf $\mathcal{F}$ is coherent [5].

For $x \neq 0$ the function f is regular at the point $(x, y_1, \dots, y_n)$ and since $(x)^{k+1} \cong (\mathcal{O}_{n+1})_p$, we have $\dim \mathcal{F}_p = 0$ and $\tau(f) \cong (\mathcal{O}_{n+1})_p$ at the point $p = (x, y_1, \dots, y_n)$. If $x = 0$ and $g(0, y_1, \dots, y_n) = 0$, then f is right-equivalent to the germ of the function x^k and $\dim_{\mathbb{C}} \mathcal{F}_p = 0$ since $(x)^k \cong \tau(f)$ at this point. Now assume that $x = 0$ and $g(0, y_1, \dots, y_n) = 0$; we see that f is right-equivalent to the germ of the function $x^k y_1$ provided that $(x, y_1, \dots, y_n) \neq (0, 0, \dots, 0)$. Therefore, outside the origin, we have $\tau(f) \cong (x)^{k+1}$, hence $\dim_{\mathbb{C}} \mathcal{F}_p = 0$. Since the support of $\mathcal{F}$ is zero, $\dim_{\mathbb{C}}(x)^{k+1}/((x)^{k+1} \cap \tau(f))$ is finite. Q.E.D.

§3. Unfoldings

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ have singular set H, $c(f) < \infty$ and in the representation $f = x^k g(x, y_1, \ldots, y_n)$, $g(0, y_1, \ldots, y_n)$ have an isolated singularity. As in [8] let us introduce

Definition 3.1. *An unfolding(with q parameters) of an analytic germ* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *is an analytic germ* $F : (\mathbb{C}^{n+1} \times \mathbb{C}^q, 0) \to (\mathbb{C}, 0)$ *such that* $F(z, 0) = f(z)$.

Definition 3.2. *A q-parametric unfolding of a germ of the analytic isomorphism* $h : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ *is a germ of an analytic mapping* $H : (\mathbb{C}^m \times \mathbb{C}^q, 0) \to (\mathbb{C}^m, 0)$ *such that* $H(z, 0) = h(z)$ *and* $h_t(z) = H(z, t)$ *belongs to the group D of germs of local diffeomorphisms for any sufficiently small* $t \in \mathbb{C}^q$.

Definition 3.3. *An unfolding* $H(z, t)$ *is called a q-parametric* D_H*-unfolding if* $H(z, 0) = h(z) \in D_H$ *and* $h_t \in D_H$ *for any sufficiently small* $t \in \mathbb{C}^q$.

Definition 3.4. *A q-parametric unfolding of the hyperplane singularity* $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *is a germ of the analytic function* $F : (\mathbb{C}^m \times \mathbb{C}^q, 0) \to (\mathbb{C}, 0)$ *such that* $F(z, 0) = f(z)$ *and* $f_t(z) = F(z, t)$ *belongs to the ideal* (x^k) *for any sufficiently small* $t \in \mathbb{C}^q$.

Definition 3.5. *Let F and G be q-parametric unfoldings of a hyperplane singularity* $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$. *A morphism* $\varphi : F \to G$ *between unfoldings is a pair* (H, λ), *where* $H \in D_H$ *is a q-parametric* D_H*-unfolding of the unit in* D_H *and* $\lambda : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ *is defined as a germ of the analytic mapping such that the diagram*

$$\begin{tikzcd} \mathbb{C}^m \times \mathbb{C}^q \arrow[r, "F"] \arrow[d, "{(H,\lambda)}"'] & \mathbb{C} \arrow[d, equal] \\ \mathbb{C}^m \times \mathbb{C}^r \arrow[r] & \mathbb{C} \end{tikzcd}$$

commutes, where

$$(H, \lambda)(z, t) = (H(z, \lambda(t)), \lambda(t)).$$

Definition 3.6. *A q-parametric unfolding F of a hyperplane singularity f is called versal if for each r-parametric unfolding of the G-hyperplane singularity f, there exists a morphism* $\varphi : G \to F$.

Definition 3.7. *Two q-parametric unfoldings are called equivalent if there exist two morphisms* $\varphi : F \to G$ *and* $\psi : G \to F$ *such that*

$$\varphi\psi = \mathrm{Id} \ \ \textit{and} \ \ \psi\varphi = \mathrm{Id}\,.$$

Definition 3.8. *A q-parametric unfolding of an F-hyperplane singularity induces an r-parametric unfolding G of the hyperplane singularity f if there exists an analytic germ* $i : (\mathbb{C}^r, 0) \to (\mathbb{C}^q, 0)$ *which is an embedding satisfying the condition* $G = i^*(F)$, *where* $i^*F(z,t) = F(z, i(t))$ *for* $t \in \mathbb{C}^r$.

Let us introduce some notation. Let F be a q-parametric unfolding and $t_1, t_2, \ldots, t_q$ be local coordinates in the space of parameters $\mathbb{C}^q$. For $i = 1, 2, \ldots, q$ we introduce notation

$$\partial_i F := \left.\frac{\partial F}{\partial t_i}\right|_{t=0}.$$

Proposition 2 (unfolding theorem). *Let F be a q-parametric unfolding of a hyperplane singularity f. Then the following statements are equivalent:*

(a) $\tau(f) + (\partial_1 F, \partial_2 F, \ldots, \partial_q F)) = (x)^k$;

(b) *F is a versal unfolding of the hyperplane singularity f.*

Proof. We consider a versal unfolding with respect to the group D_H. A more general pseudogroup is considered in [4] and a versal unfolding with respect to this group is constructed in the same paper. Taking into account the above definitions, our statements immediately follow from those results of [4]. Q.E.D.

Definition 3.9. *Let F be a q-parametric unfolding of a hyperplane singularity f. The unfolding F is called a miniversal unfolding of f if F is a versal unfolding and* $q = \dim_{\mathbb{C}} (x)^k / \tau(f)$.

Corollary 1. (a) *A hyperplane singularity f has a versal unfolding if and only if*

$$\dim_{\mathbb{C}} (x)^k / \tau(f) < \infty.$$

(b) *Any two versal unfoldings F and G of the singularity f are equivalent.*

Hence it follows that a miniversal unfolding F of a hyperplane singularity f can be written in the form

$$F(x, y_1, \ldots, y_n, \lambda) = f(x, y_1, \ldots, y_n) + \sum_{i=1}^{q} \lambda_i l_i(x, y_1, \ldots, y_n),$$

where

$$q = \operatorname{codim} f = \dim_{\mathbb{C}} (x)^k / \tau(f)$$

and $l_1, l_2, \ldots, l_q$ represent a $\mathbb{C}$-basis of the space $(x)^k / \tau(f)$.

Similarly to [10], the following is valid

Proposition 3. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a complex-valued analytic germ with singularities on H and $f \in (x^k)$. Then there exists an unfolding*

$$\widetilde{f} = x^k(g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1}),$$

where

$$\begin{aligned}
&\lambda_{n+1} \in \operatorname{Reg}(g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n),\\
&\lambda_i \in \operatorname{Reg}(\operatorname{grad} g(0, y_1, \dots, y_n)),
\end{aligned}$$

satisfying the following condition: $\widetilde{f}$ is equivalent to x^k or $x^k y_1$ on H and has only singular points of Morse type outside H.

Proof. Assume that on the hyperplane H, we have $g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1} \neq 0$. Consider the transformation

$$\begin{aligned}
&\widetilde{x} = x\sqrt[k]{g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1}},\\
&\widetilde{y}_i = y_i, \quad i = 1, \dots, n
\end{aligned}$$

of class D_H. Since

$$\left.\frac{d\widetilde{x}}{dx}\right|_{x=0} = \sqrt[k]{g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1}} \neq 0,$$

the Jacobian of this transformation is nonzero and $\widetilde{f}$ is equivalent to x^k in these coordinates.

Now let $g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1} = 0$; choose

$$\begin{aligned}
&\lambda_{n+1} \in \operatorname{Reg}(g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1}),\\
&\lambda_i \in \operatorname{Reg}(\operatorname{grad}(g(0, y_1, \dots, y_n)).
\end{aligned}$$

Now, assuming that $\frac{\partial g}{\partial y_1}(0, y_1, \dots, y_n) + \lambda_1 \neq 0$, we consider the transformation

$$\widetilde{x} = x,\ \widetilde{y}_1 = g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1},\ \widetilde{y}_i = y_i,\ i = 2, \dots, n.$$

This transformation belongs to the class D_H and has the Jacobian of the form

$$\frac{\partial g}{\partial y_1} + \lambda_1 \neq 0.$$

In the new coordinates, $\widetilde{f}$ is equivalent to $x^k y_1$ at the points of smooth $(2n-2)$-dimensional submanifolds which are the Milnor fibres of $g|_H$.

Outside the singular hyperplane, the unfolding $\widetilde{f}$ has finite number of Morse points. This is proved similarly as in [10]. Q.E.D.

In what follows, $\widetilde{f}$ will denote unfolding of indicated type, for some fixed value of the parameters λ.

§4. Topology of the Milnor fibre

The main idea in the study of topology of hyperplane singularities is to use special unfoldings of f to get a generic approximation $f_\lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ with properties

1. the induced fibration above a small circle around the origin of f_λ and f must be equivalent; and have therefore the same fibre.
2. the singular locus of f_λ and the local singularity type of f_λ at the points of the critical locus should be as easy as possible.

For the isolated singularities this idea was used by Le Dung Trang to determine the homotopy type of the Milnor fibre [6]; in the one-dimensional non-isolated case, this idea has been used by Siersma in [11].

Let $f : (\mathbb{C}_{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ having an isolated singularity on the set $H = \{x = 0\}$, $f \in (x^k)$, and f_λ be the unfolding of the singularity f obtained by Proposition 3. Choose $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, the fiber $f^{-1}(0) \pitchfork \partial B_\varepsilon$ is transversal to the ball. For such $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$f^{-1}(t) \pitchfork \partial B_\varepsilon \text{ for all } 0 < |t| < \eta(\varepsilon).$$

After fixing ε and η, consider the restriction

$$f : X_D = f^{-1}(D_\eta) \cap B_\varepsilon \to D_\eta,$$

where D_η is a disk of radius η in $\mathbb{C}^1$.

Lemma 2. *Consider the restriction*

$$f_\lambda : X_{D,\lambda} = f_\lambda^{-1}(D_\eta) \cap B_\varepsilon \to D_\eta$$

for any $0 < |\lambda| < \delta$ and $0 < |t| < \eta$, where δ and η are sufficiently small numbers. Then the following statements hold:

(1) $f_\lambda^{-1}(t) \pitchfork \partial B_\varepsilon$;
(2) *over the boundary of the disk ∂D_η, the induced fibrations f and f_λ are equivalent;*
(3) *X_D and $X_{D,\lambda}$ are homeomorphic.*

Proof. Proof of (1) is as in [11] and [10] (see also [13]). At the points of $H \cap \partial B_\varepsilon$, we have singular points where f_λ equivalent either to x^k or $x^k y_1$. If at $z \in H \cap \partial B_\varepsilon$ f is equivalent to x^k, i. e. there exist coordinates

$(x, y_1, \dots, y_n)$ such that $f_{\lambda(s)}(x, y_1, \dots, y_n) \sim x^k$, where $f_{\lambda(s)}$ is a one-parameter unfolding of the singularity f and the coordinate x smoothly depends on λ. For $t \neq 0$ the tangent plane $f^{-1}_{\lambda(s)}(t)$ is obtained from the equation

$$kx_0^{k-1}(x - x_0) = 0,$$

i. e. $x = x_0$ is a hyperplane and, thus, transversally intersects the boundary ∂B_ε.

Now assume that at $z \in H \cap \partial B_\varepsilon$ f is equivalent to $x^k y_1$. Thus we have

$$f_{\lambda(s)}(x, y_1, \dots, y_n) \sim x^k y_1.$$

Hence the tangent space to $f^{-1}_{\lambda(s)}(t)$ at a point $(x_0, \mathbf{y}^0) = (x_0, y_1^0, \dots, y_n^0)$ with $\mathbf{y}^0$ of sufficiently small norm has the form

$$(x - x_0)x_0^{k-1}y_1^0 + (y_1 - y_1^0)x_0^2 = 0$$

and, therefore, contains the plane $\{x = x_0,\ y_1 = y_1^0\}$, which is transversal to ∂B_ε since the set $\{y_1 = y_1^0\}$ coincides with the set

$$\{g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} = y_1^0\},$$

which transversally intersects $\partial B_\varepsilon \cap H$.

We have established that the transversality condition is fulfilled at points $z \in H \cap \partial B_\varepsilon$ and the mapping $f_{\lambda(s)}\big|_{S_\varepsilon}$ is a submersion at the points $z \in \partial B_\varepsilon \setminus H$. Since $f^{-1}(0) \cap \partial B_\varepsilon$ is compact and the transversality is an open property, we obtain that $f^{-1}_{\lambda(s)}(t) \pitchfork \partial B_\varepsilon$ for $0 < \|\lambda\| < \delta$ and $0 < |t| < \eta$; this completes the proof of statement (1).

Statements (2) and (3) can be proved similarly to the proof of Lemma 2. Q.E.D.

Let us proceed with describing the main construction. Let b_1, b_2, ..., b_σ be Morse critical points for the unfolding $\widetilde{f}$ with critical values $\widetilde{f}(b_1), \dots, \widetilde{f}(b_\sigma)$. Let $B_1, \dots, B_\sigma$ be disjoint $(2n+2)$-dimensional balls in $\mathbb{C}^{n+1}$ centered at the points $b_1, \dots, b_\sigma$ and $D_1, \dots, D_\sigma$ be disjoint 2-dimensional disks centered at the points $\widetilde{f}(b_1), \dots, \widetilde{f}(b_\sigma)$. Choose them such that the mapping

$$\widetilde{f} : B_i \cap \widetilde{f}^{-1}(D_i) \to D_i,\ i = 1, \dots, n,$$

defines a locally trivial Milnor fibration and the transversality condition

$$f^{-1}(t) \pitchfork \partial B_i,$$

holds for all $t \in D_i$, $i = 1, \ldots, \sigma$. Choose a small cylinder B_0 around H and a 2-dimensional disk $D_0 \in \operatorname{int} \widetilde{f}(B_0)$ satisfying the condition

$$\partial B_0 \pitchfork \widetilde{f}^{-1}(t), \; t \in D_0.$$

We consider the restriction

$$\widetilde{f} : B_0 \cap \widetilde{f}^{-1}(D_0) \to D_0$$

of this fibration. A fiber $\widetilde{f}^{-1}(t) \cap B_0$ of this map can be stratified by using the projection π on $B_\varepsilon \cap (H \setminus U)$, where U is a tubular neighborhood of the smooth nonsingular manifold $g(0, y_1, \ldots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1} = 0$, and

$$\pi(x, y_1, \ldots, y_n) = (0, y_1, \ldots, y_n).$$

This projection may have singularities. To show this, consider the mapping

$$\Phi_{\widetilde{f}} : \widetilde{f}^{-1}(D_0) \cap B_0 \to \mathbb{C} \times \mathbb{C}^n,$$

which is defined by the equality

$$\Phi_{\widetilde{f}}(x, y_1, \ldots, y_n) = (\widetilde{f}(x, y_1, \ldots, y_n), y_1, \ldots, y_n).$$

The Jacobi matrix of this mapping is

$$\begin{pmatrix} \dfrac{\partial \widetilde{f}}{\partial x} & \dfrac{\partial \widetilde{f}}{\partial y_1} & \cdots & \dfrac{\partial \widetilde{f}}{\partial y_n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Therefore, the critical set Γ of $\Phi_{\widetilde{f}}$ is given by the equality

$$\frac{\partial \widetilde{f}}{\partial x} = 0.$$

We see that it consists of the hyperplane H and the hypersurface $\Gamma_{\widetilde{f}}$, i. e. $\Gamma = H \cup \Gamma_{\widetilde{f}}$ and the projection

$$\pi : \widetilde{f}^{-1}(t) \cap B_0 \to B_\varepsilon \cap (H \setminus U)$$

is smooth outside the points from $\Gamma_{\widetilde{f}}$.

Lemma 3. *At all points of $\Gamma_{\widetilde{f}} \cap H$, $\widetilde{f}$ is equivalent to $x^k y_1$.*

Proof. We show that at some point of $\Gamma_{\widetilde{f}} \cap H$, if $\widetilde{f}$ is equivalent to x^k, then $\Gamma_{\widetilde{f}}$ coincides with H.

Let $\widetilde{f} = x^k \widetilde{g}$ where $\widetilde{g} = g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1}$ and $\widetilde{g}(0, \dots, 0) \neq 0$. Then

$$\frac{\partial \widetilde{f}}{\partial x} = k x^{k-1} \widetilde{g} + x^k \widetilde{g}_x$$

and, since $\widetilde{g}(0, \dots, 0) \neq 0$, we can express x modulo (x^k). Therefore,

$$(x^{k-1}) \subset \left(\frac{\partial \widetilde{f}}{\partial x} \right) + (x^k).$$

Hence, by the Nakayama lemma, we obtain

$$(x^{k-1}) = \left(\frac{\partial \widetilde{f}}{\partial x} \right).$$

Therefore, the set defined by the equation $\frac{\partial \widetilde{f}}{\partial x} = 0$ coincides with the set $\{x = 0\}$, i. e. $\Gamma_{\widetilde{f}} = H$. Q.E.D.

This lemma implies that the projection π is a locally trivial fibration outside points, where $\widetilde{f}$ is equivalent to $x^k y_1$, whose fiber is given by the equation $\{\widetilde{f} = t\}$. Since the set π^{-1} $(B_\varepsilon \cap (H \setminus U))$ is compact and consists of singular points, where $\widetilde{f}$ is equivalent to x^k, the fiber locally consists of k points. Due to the compactness of this set, we can choose the radius of B_0 such that π defines the k-sheeted covering over $B_\varepsilon \cap (H \setminus U)$.

Consider the space $B_\varepsilon \cap (H \setminus U)$. The space $B_\varepsilon \cap (H \setminus U)$ is homotopy equivalent to the bouquet of S^{-1} and μ copies of n-dimensional spheres S^n, where μ is the Milnor number of the isolated singularity $g(0, y_1, \dots, y_n)$ ([9]). By virtue of the preceding lemma, we obtain the following assertion.

Lemma 4. *The Milnor fiber $\widetilde{f}^{-1}(t) \cap B_0$ covers the bouquet of S^1 and μ copies of n-dimensional spheres S^n by k-sheets.*

Let $n \geqslant 2$. To find out the homotopy type of $\widetilde{f}^{-1}(t) \cap B_0$, we represent the set $B_\varepsilon \cap (H \setminus U)$, which is homotopy equivalent to the bouquet of the circle S^1 and μ copies of spheres S^n, as the union $U_1 \cup U_2$, where $U_1 \cap B_\varepsilon \cap (H \setminus U)$ is homotopy equivalent to the circle S^1, $U_2 \cap B_\varepsilon \cap (H \setminus U)$ is homotopy equivalent to the bouquet of μ copies of S^n, and $U_1 \cap U_2$ is contractible. Since π is the k-sheeted covering $\widetilde{f}^{-1}(t) \cap B_0$

on $B_\varepsilon \cap (H \setminus U)$, we conclude that $\pi^{-1}(U_1) \cap \widetilde{f}^{-1}(t) \cap B_0$ is homotopy equivalent to S^1 and, since $U_2 \cap B_\varepsilon \cap (H \setminus U)$ is simply connected, any covering on it is trivial and, therefore, $\pi^{-1}(U_2 \cap B_\varepsilon \cap (H \setminus U))$ is homotopy equivalent to the direct product of k points by the bouquet of μ copies of spheres S^n. It is obvious that $\widetilde{f}^{-1}(t) \cap B_0$ is homotopy equivalent to the bouquet of the circle S^1 and $k\mu$ copies of spheres S^n.

We have proved the following lemma.

Lemma 5. *For $n \geqslant 2$, the Milnor fiber $f^{-1}(t) \cap B_0$ has the homotopy type of a bouquet of the circle S^1 and $k\mu$ copies of an n-dimensional sphere S^n.*

Similarly to the case of a hyperplane singularity of transversal type A_1, we can calculate the homology groups of the Milnor fiber in the ball B_ε.

Let us consider the case where $n = 1$, i.e., where $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an analytic germ of two complex variables having the form $f = x^k g(x, y)$, where $g(x, y)$ and $g(0, y)$ are isolated singularities. Then we obtain the following assertion.

Lemma 6. *The Milnor fiber $f^{-1}(t) \cap B_0$ has the homotopy type of the bouquet of $k\mu + 1$ copies of S^1, where $\mu = \mu(g(0, y))$ is the Milnor number of the isolated singularity $g(0, y)$.*

Proof. The set $B_\varepsilon \cap (H \setminus U)$ has the homotopy type of a bouquet of $\mu + 1$ copies of the circle S^1, where $\mu = \mu(g(0, y))$ is the Milnor number of the isolated singularity $g(0, y)$ ([9]).

We have $\pi^{-1} : f^{-1}(t) \cap B_0 \to B_\varepsilon \cap (H \setminus U)$ and, therefore, the k-sheeted covering of the space $f^{-1}(t) \cap B_0$ is the Eilenberg-MacLane space whose fundamental group $\pi_1(f^{-1}(t) \cap B_0)$ is a subgroup of index k of a free group of $\mu + 1$ generators. This follows from the following exact sequence of the covering π:

$$\begin{aligned} \cdots \to \pi_1(k) \to \pi_1(f^{-1}(t) \cap B_0) \to \pi_1(B_\varepsilon \cap (H \setminus U)) \to \\ \to \pi_0(k) \to \pi_0(f^{-1}(t) \cap B_0) \to \pi_0(B_\varepsilon \cap (H \setminus U)). \end{aligned}$$

Since $\pi_1(k) = 0$, we have $\pi_0(f^{-1}(t) \cap B_0) = \pi_0(B_\varepsilon \cap (H \setminus U)) = 0$ since the subsets $f^{-1}(t) \cap B_0$ and $B_\varepsilon \cap (H \setminus U)$ are connected. Hence we obtain

$$0 \to \pi_1(f^{-1}(t) \cap B_0) \to \pi_1(B_\varepsilon \cap (H \setminus U)) \to \pi_0(k) \to 0;$$

this yields

$$\mathrm{Ind}\big[\pi_1(f^{-1}(t) \cap B_0) | \, \pi_1(B_\varepsilon \cap (H \setminus U))\big] = k.$$

Therefore $\pi_1(f^{-1}(t)\cap B_0)$ has the rank

$$1+(\mu+1-1)k=1+k\mu.$$

Since $f^{-1}(t)\cap B_0$ is the Eilenberg–MacLane space, we see that it is homotopy equivalent to the bouquet of $k\mu+1$ copies of the circle. Q.E.D.

Using the standard technique from [11] we can obtain as in [10] that

$$\begin{cases} H_n(f^{-1}(t)\cap B_\varepsilon)=H_n(f^{-1}(t)\cap B_\varepsilon)\oplus\underbrace{\mathbb{Z}\cdots\oplus\mathbb{Z}}_{\sigma-times} \\ H_k(f^{-1}(t)\cap B_\varepsilon)=H_k(f^{-1}(t)\cap B_\varepsilon), \quad \text{if} \quad k\neq n. \end{cases}$$

Then combining the preceding lemmas, we obtain the following

Proposition 4. *In the case $n=1$, homology groups of the Milnor fiber $f^{-1}(t)\cap B_\varepsilon$, are given by the formulas*

$$\begin{aligned} &H_1(f^{-1}(t)\cap B_\varepsilon)=\mathbb{Z}^{\widetilde{\mu}_1+\sigma},\\ &H_0(f^{-1}(t)\cap B_\varepsilon)=\mathbb{Z},\\ &H_i(f^{-1}(t)\cap B_\varepsilon)=0, \quad i\neq 0,1, \end{aligned}$$

where $\widetilde{\mu}_1=k\mu+1$ and μ is the number of Morse point for the deformation f. In the case $n\geqslant 1$, we gave

$$\begin{aligned} &H_n(f^{-1}(t)\cap B_\varepsilon)=\mathbb{Z}^{\widetilde{\mu}_2+\sigma},\\ &H_1(f^{-1}(t)\cap B_\varepsilon)=\mathbb{Z},\\ &H_0(f^{-1}(t)\cap B_\varepsilon)=\mathbb{Z},\\ &H_i(f^{-1}(t)\cap B_\varepsilon)=0, \quad i\neq 0,1,n, \end{aligned}$$

where $\widetilde{\mu}_2=k\mu$.

Similarly to the case of an isolated hyperplane singularity of transversal type A_1, the above proposition implies the following theorem (see [9]).

Theorem 3. *In the case $n=1$, the Milnor fiber $f^{-1}(t)\cap B_\varepsilon$ has the homology type of a bouquet of $k\mu+1+\sigma$ copies of the circle, where $\mu=\mu(g(0,y))$ is the Milnor number of the isolated singularity $g(0,y)$. In the case $n\geqslant 2$, it has homotopy type of a bouquet of the circle and $k\mu+\sigma$ copies of an n-dimensional sphere, where σ is the number of Morse point in a deformation of f.*

§5. The Betti number

As we see the Morse points of the special unfoldings is an important topological invariant because it appears in the expression of the homotopy type of the Milnor fibre.

The number of Morse points can be calculated by an algebraic method similarly to the case of an isolated hyperplane singularity.

Theorem 4. *The number of Morse points for a unfolding of f is calculated by the formula*

$$\sigma = \dim_{\mathbb{C}} (x^k)/(xf_x, f_{y_1}, \dots, f_{y_n}).$$

Proof. Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^\lambda, 0) \to (\mathbb{C}, 0)$, $\lambda \in \mathbb{C}^\lambda$, be a versal unfolding of the singularity f, where $f = x^k g(x, y_1, \dots, y_n)$. Such a unfolding can be constructed as in Sec. 3. It has the form $F = x^k G(x, y_1, \dots, y_n)$, where $G \in \mathcal{O}_{x,y_1,\dots,y_n}$ satisfies the condition $G|_{\lambda=0} = g$.

The number of Morse point σ is obtained as the number of solutions of the following system of equations lying outside the singular plane $\{x = 0\}$ for a sufficiently small parameter λ:

$$F_x = 0,\ F_{y_1} = 0, \dots, F_{y_n} = 0,$$

and similarly to the case of the hyperplane singularity f, the number of Morse points coincides with the index of the intersection of the plane $\{\lambda = 0\}$ with a germ of the surface $S \subset \mathbb{C}^{n+1} \times \mathbb{C}^\lambda$ given as the closure of a germ of the set

$$S = \{F_x = 0,\ F_{y_1} = 0, \dots, F_{y_n} = 0,\ \ x \neq 0\}.$$

Since $x \neq 0$, we can cancel it and obtain

$$S = \{kG + xG_x = 0,\ G_{y_1} = 0, \dots, G_{y_n} = 0\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^\lambda.$$

Therefore

$$\begin{aligned}
\sigma &= \dim_{\mathbb{C}} \mathcal{O}_{x,y_1,\dots,y_n,\lambda}/(kG + xG_x, G_{y_1}, \dots, G_{y_n}) \\
&= \dim_{\mathbb{C}} \mathcal{O}_{x,y_1,\dots,y_n}/\mathcal{O}_{x,y_1,\dots,y_n}(kg + xg_x, g_{y_1}, \dots, g_{y_n}) \\
&= \dim_{\mathbb{C}} (x^k)/\mathcal{O}_{x,y_1,\dots,y_n}(kx^k g + x^{k+1} g_x, x^k g_{y_1}, \dots, x^k g_{y_n}) \\
&= \dim_{\mathbb{C}} (x^k)/(xf_x, f_{y_1}, \dots, f_{y_n}),
\end{aligned}$$

this completes the proof. Q.E.D.

Now assume that $f = x^k g$, where g is a quasi-homogeneous function of degree d with weights $\omega_0, \omega_1, \dots, \omega_n$.

One has

Proposition 5. *Let $f \in (x^k)$, with $\omega(f) < +\infty$, and $g(0, y_1, \dots, y_n)$ be an isolated singularity, with the quasihomogeneous mapping $g(x, y_1, \dots, y_n)$, then*

$$\dim_{\mathbb{C}} \big((x^k)/(xf_x, f_{y_1}, \dots, f_{y_n})\big) = \dim_{\mathbb{C}} \big(\mathcal{O}_{n+1}/(xg_x, g_{y_1}, \dots, g_{y_n})\big).$$

Proof. Suppose that the mapping g is quasihomogeneous of degree d with weights $w_0, w_1, ..., w_n$. Then we have $g \in (g_x, g_{y_1}, \dots, g_{y_n})$, where $\ell g = xg_x + y_1 g_{y_1} + \dots + y_n g_{y_n}$ for some $\ell \in \mathbb{C}$. Thus,

$$\begin{aligned} xf_x &= x(kx^{k-1}g + x^k g_k) \\ &= x\big[kx^{k-1}m(xg_x + y_1 g_{y_1} + \dots + y_n g_{y_n}) + x^k g_x\big] \\ &= kx^{k+1}mg_x + kmx^k y_1 g_{y_1} + \dots + kmx^k y_n g_{y_n} + x^{k+1} g_x \\ &= (km+1)x^{k+1}mg_x + kmx^k y_1 g_{y_1} + \dots + kmx^k y_n g_{y_n}, \end{aligned}$$

where $m = \frac{1}{\ell}$, and

$$f_{y_i} = x^k g_{y_i}, \quad i = 1, \dots, n.$$

One has an isomorphism

$$\begin{aligned} &(xf_x, f_{y_1}, \dots, f_{y_n}) \\ &= \big((km+1)x^{k+1}g_x + kmx^k y_1 g_{y_1} + \dots + kmx^k y_n g_{y_n}, x^k g_{y_1}, ..., x^k g_{y_n})\big) \\ &\cong (x^k)((km+1)xg_x, g_{y_1}, \dots, g_{y_n}). \end{aligned}$$

As a result we obtain the needed equality

$$\dim_{\mathbb{C}} \big((x^k)/(xf_x, f_{y_1}, \dots, f_{y_n})\big) = \dim_{\mathbb{C}} \big(\mathcal{O}_{n+1}/(xg_x, g_{y_1}, \dots, g_{y_n})\big).$$

Q.E.D.

Remark. For a quasihomogeneous function g of degree d with weights $w_0, w_1, \dots, w_n$ the mapping

$$(x, y_1, \dots, y_n) \mapsto (g_x, g_{y_1}, \dots, g_{y_n})$$

is also quasihomogeneous, of multidegree $(d_0, d_1, \dots, d_n)$ with weights $w_0, w_1, \dots, w_n$, where $d_k = d - w_k$. Thus the mapping

$$(x, y_1, \dots, y_n) \mapsto (xg_x, g_{y_1}, \dots, g_{y_n})$$

is quasihomogeneous of multidegree $(d, d - w_1, \dots, d - w_n)$.

One has

Theorem 5. *$f \in (x^k)$, with $\omega(f) < +\infty$ and g is quasihomogeneous one has*

$$\sigma = \left(\frac{d - w_0}{w_0} + 1\right) \prod_{i=1}^{n} \frac{d - w_i}{w_i},$$

where d is degree of g, with respect to the weights $w_0, w_1, \ldots, w_n$.

Proof. From proposition we conclude

$$\sigma = \dim_{\mathbb{C}} \left((x^k)/(xf_x, f_{y_1}, \ldots, f_{y_n})\right) = \dim_{\mathbb{C}} \left(\mathcal{O}_{n+1}/(xg_x, g_{y_1}, \ldots, g_{y_n})\right).$$

According to the previous remark the map $(xg_x, g_{y_1}, \ldots, g_{y_n})$ is quasihomogeneous of multidegree $(d, d - w_1, \ldots, d - w_n)$, with weights w_0, w_1, ..., w_n. Using the result [2] about computation of the dimension of the local algebra of a quasihomogeneous map, we obtain

$$\sigma = \frac{d}{w_0} \prod_{i=1}^{n} \frac{d - w_i}{w_i} = \left(\frac{d - w_0}{w_0} + 1\right) \prod_{i=1}^{n} \frac{d - w_i}{w_i}.$$

Q.E.D.

Corollary 2. *The middle Betti number of the Milnor fible $F = f^{-1}(t) \cap B_\varepsilon$ is given by*

$$b_n(F) = \left(k + 1 + \frac{d - w_0}{w_0}\right) \prod_{i=1}^{n} \frac{d - w_i}{w_i}.$$

Proof. The middle Betti number $b_n(F) = \sigma + k\mu(g(0, y_1, \ldots, y_n))$, where σ is above and $\mu(g(0, y_1, \ldots, y_n))$ is Milnor number of the isolated singularity $g(0, y_1, \ldots, y_n)$, so $\mu(g(0, y_1, \ldots, y_n)) = \prod_{i=1}^{n} \frac{d-w_i}{w_i}$ this immediatelly implies the result. Q.E.D.

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