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Topology of abelian pencils of curves

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Abstract.

We study the geometry of a linear system of plane curves $C(\tau)$ ($\tau \in \mathbb{C}$) spanned by two irreducible curves C, C' of degree d such that $\pi_1(\mathbb{P}^2 - C \cup C')$ is abelian. We will show that the fundamental group $\pi_1(\mathbb{C}^2 - C(\vec{\tau}))$ is isomorphic to $\mathbb{Z} \times F(r-1)$ for a generic $\vec{\tau}$ where $\vec{\tau} = (\tau_1, \dots, \tau_r)$ and $C(\vec{\tau}) = C(\tau_1) \cup \cdots \cup C(\tau_r)$.

§1. Introduction

Consider two irreducible plane curves $C : F(X,Y,Z) = 0$ and $C' : G(X,Y,Z) = 0$ of the same degree d. Let $C \cap C' = \{P_1, \dots, P_s\}$ and let m_i be the local intersection number of C and C' at P_i. We consider a linear family of curves (which we call simply a *pencil of curves* in this paper) :

$$C(\tau): \quad \tau F(X,Y,Z) + (1-\tau) G(X,Y,Z) = 0, \ \tau \in \mathbb{C}.$$

Choose a generic line at infinity $L_\infty : Z = 0$ such that the affine space $\mathbb{C}^2 := \mathbb{P}^2 - L_\infty$ contains all the base points $\{P_1, \dots, P_s\}$. Our pencil is defined in this affine space by

$$C(\tau): \quad \tau f(x,y) + (1-\tau) g(x,y) = 0, \ \tau \in \mathbb{C}$$

where $x = X/Z$, $y = Y/Z$ and $f(x,y) = F(x,y,1)$, $g(x,y) = G(x,y,1)$. We consider a union of generic r curves in this pencil:

$$C(\vec{\tau}) := C(\tau_1) \cup \cdots \cup C(\tau_r), \ \vec{\tau} = (\tau_1, \dots, \tau_r).$$

The parameters $\tau_1, \dots, \tau_r$ are generic so that the topology of the pair $(\mathbb{P}^2, C(\vec{\tau}))$ does not depend on the choice of $\tau_1, \dots, \tau_r$ (see §3.1). We call

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$C(\vec{\tau})$ *a generic r-fold pencil curve* of the pencil $C(\tau)$, $\tau \in \mathbb{C}$. The pencil $C(\tau)$, $\tau \in \mathbb{C}$ is called *abelian* if $\pi_1(\mathbb{P}^2 - C(\tau_1) \cup C(\tau_2))$ is abelian for any generic τ_1, τ_2. Then $\pi_1(\mathbb{C}^2 - C(\tau_1) \cup C(\tau_2)))$ is also abelian, because L_∞ is assumed to be generic and thus we can use the central extension of fundamental groups:

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - C(\tau_1) \cup C(\tau_2)) \to \pi_1(\mathbb{P}^2 - C(\tau_1) \cup C(\tau_2)) \to 1$$

by [8]. We consider the d^2-fold branched covering:

$$\varphi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \varphi(x,y) = (f(x,y), g(x,y))$$

and we consider the pencil of lines: $L(\tau) = \{\tau x + (1-\tau)y = 0\}$, $\tau \in \mathbb{C}$. Put

$$L(\vec{\tau}) := L(\tau_1) \cup L(\tau_2) \cup \cdots \cup L(\tau_r), \ \vec{\tau} = (\tau_1, \dots, \tau_r).$$

Then we see that $\varphi^{-1}(L(\vec{\tau})) = C(\vec{\tau})$. Our main result is:

$$\varphi_\sharp : \pi_1(\mathbb{C}^2 - C(\vec{\tau})) \to \pi_1(\mathbb{C}^2 - L(\vec{\tau}))$$

is an isomorphism, if $C(\tau)$, $\tau \in \mathbb{C}$ *is an abelian pencil. In particular,* $\pi_1(\mathbb{C}^2 - C(\vec{\tau})) \cong \mathbb{Z} \times F(r-1)$ *and* $C(\vec{\tau})$ $(\tau \in \mathbb{C})$ *is of a non-torus type and the Alexander polynomial of* $C(\vec{\tau})$ *is given by* $(t^r-1)^{r-2}(t-1)$ (Main Theorem 7).

§2. Preliminaries

2.1. Taylor expansions and intersection numbers

Suppose that we have two smooth germs of curves at the origin $o = (0,0)$: $C : f(x,y) = 0$ and $C' : g(x,y) = 0$ and let $y = \varphi(x)$ be the solution of $f(x,y) = 0$ in y, assuming that $\frac{\partial f}{\partial y}(0,0) \neq 0$. Then the local intersection number, denoted as $I(C, C'; o)$, is defined to be $\operatorname{ord}_x g(x, \varphi(x))$. Suppose that $y = \psi(x)$ is the solution of $g(x,y) = 0$ and consider their Taylor expansions:

$$\varphi(x) = t_1 x + t_2 x^2 + \dots, \quad \psi(x) = s_1 x + s_2 x^2 + \dots$$

and put $m = \min\{j \mid t_j \neq s_j\}$. Then we have $I(C, C'; o) = m$.

2.2. Alexander polynomial

Let $C(\tau)$ be a pencil of smooth curves of degree d with base points $P_1, \dots, P_s$. Let m_i be the intersection multiplicity of two pencil lines $C(\tau)$ and $C(\tau')$ at P_i. We consider a generic r-fold pencil curve $C(\vec{\tau})$. Take local coordinates (u_i, w_i) at P_i and let $w_i = \phi_{i,0}(u_i)$ be the solution of $f(x,y,\tau)=0$ in w_i. Change the local coordinates at P_i as (u_i,v_i), where $v_i := w_i - \varphi_{i,0}(u_i)$. Then the local defining equation of $C(\vec{\tau})$ is given by

$$\prod_{i=1}^{r}(v_i - c_i(\tau_i)u_i^{m_i}) + (\text{higher terms}) = 0$$

and the Newton boundary is non-degenerate and the weight vector is given by ${}^t(1,m_i)$, which is independent of r. The local adjunction ideal $\mathcal{I}_k(P_i) \subset \mathcal{O}_{P_i}$ is given by the following equality ([1, 10, 11]):

$$\mathcal{I}_k(P_i) = \langle u_i^a v_i^b \mid a + m_i\, b \geq \alpha_{i,k}\rangle,\ \alpha_{i,k} := \lceil \frac{km_i}{d} \rceil - m_i.$$

An important observation here is that $\alpha_{i,k}$ does not depend on r. Put

$$\mathcal{O}(1,s) = \oplus_{i=1}^{s}\mathcal{O}_{P_i},\quad V_k(P_i) = \mathcal{O}_{P_i}/\mathcal{I}_k(P_i),\quad V_k(1,s) = \oplus_{i=1}^{s} V_k(P_i).$$

Let $O(j)$ be the set of polynomials in x, y whose degree is less than or equal to j. We consider the following canonical mappings:

$$\sigma_{k,i},: O(k-3) \to \mathcal{O}_{P_i},\ \sigma_k := \oplus\sigma_{k,i} : O(k-3) \to \oplus_{i=1}^{s}\mathcal{O}_{P_i}.$$

We denote the composites $O(k-3) \to \mathcal{O}_{P_i} \to V_k(P_i)$ by $\bar{\sigma}_{k,i}$ and their direct sum $O(k-3) \to \mathcal{O}(1,s) \to V_k(1,s)$ by $\bar{\sigma}_k$ respectively:

$$\bar{\sigma}_{k,i},: O(k-3) \to V_k(P_i),\ \bar{\sigma}_k := \oplus\sigma_{k,i} : O(k-3) \to V_k(1,s).$$

For positive integers n, j with $0 < j < n$, we put

$$\Delta_{n,j} = (t - \exp(\frac{2\pi j\sqrt{-1}}{n}))(t - \exp(\frac{2\pi(n-j)\sqrt{-1}}{n})).$$

Then the Alexander polynomial of $C(\vec{\tau})$ is given as follows (see [1, 6]).

Lemma 1.

$$\Delta_{C(\vec{\tau})}(t) = (t-1)^{r-1}\prod_{j=1}^{rd-1}\Delta_{rd,j}(t)^{\ell_j} \tag{1}$$

where the multiplicity ℓ_j is given by the dimension of the cokernel of $\bar{\sigma}_j$.

2.3. Fox calculus.

Let X be a path-connected topological space. Suppose that $\varphi : \pi_1(X) \to \mathbf{Z}$ is a given surjective homomorphism. Assume that $\pi_1(X)$ has a finite presentation:

$$\pi_1(X) \cong \langle x_1,\dots,x_n \mid R_1,\dots,R_m\rangle$$

where R_i is a word of $x_1, \dots, x_n$. This implies that we have a surjective homomorphism $\psi : F(n) \to \pi_1(X)$ where $F(n)$ is a free group of rank n, generated by $x_1, \dots, x_n$ and the kernel of ψ is normally generated by $R_1, \dots, R_m$. Consider the group ring of $F(n)$ with $\mathbb{C}$-coefficients $\mathbb{C}[F(n)]$. The Fox differential

$$\frac{\partial}{\partial x_j} : \mathbb{C}[F(n)] \to \mathbb{C}[F(n)]$$

is a $\mathbb{C}$-linear map which is characterized by the following property on words:

$$\frac{\partial}{\partial x_j} x_i = \delta_{i,j}, \quad \frac{\partial}{\partial x_j}(uv) = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j}, \ u, v \in F(n) \subset \mathbb{C}[F(n)].$$

The composite $\varphi \circ \psi : F(n) \to \mathbf{Z}$ gives a ring homomorphism $\gamma : \mathbb{C}[F(n)] \to \mathbb{C}[t, t^{-1}]$. *The Alexander matrix* A is the $m \times n$ matrix with coefficients in $\mathbb{C}[t, t^{-1}]$ whose (i, j)-component is given by $\gamma(\frac{\partial R_i}{\partial x_j})$. Then it is known that the Alexander polynomial $\Delta(t)$ is given by the greatest common divisor of the $(n-1)$-minors of A ([2]).

2.4. A smooth pencil of curves

Let $C(\tau) : F(X, Y, Z, \tau) = 0, \tau \in \mathbb{C}$ be a linear family of curves of degree d. The linear family $C(\tau)$, $\tau \in \mathbb{C}$ is called *a smooth pencil of curves of degree* d with the base point $P_1, \dots, P_s$ and the respective intersection multiplicities $m_1, \dots, m_s$, if it satisfies the following conditions.

(1) For a generic τ, $C(\tau)$ is an irreducible smooth plane curve of degree d. (This means $C(\tau)$ is smooth except for a finite number of parameters.)
(2) Every curve $C(\tau)$ passes through the base points $P_1, \dots, P_s$ and the local intersection number $I(C(\tau), C(\tau'); P_i)$ is equal to m_i for any $\tau \neq \tau'$ and $i = 1, \dots, s$ and the equality $\sum_{i=1}^{s} m_i = d^2$ holds.

The following is an immediate result of the definition.

Proposition 2. *Assume that* $C_1 : F(X, Y, Z) = 0$ *and* $C_2 : G(X, Y, Z) = 0$ *are two smooth irreducible curves of degree* d *and let* $C_1 \cap C_2 = \{P_1, \dots, P_s\}$ *and put* $m_i = I(C_1, C_2; P_i)$, $i = 1, \dots, s$. *Then the pencil of curves*

$$C(\tau) : \tau F(X, Y, Z) + (1-\tau) G(X, Y, Z) = 0, \tau \in \mathbb{C}$$

is a smooth pencil with the base points $P_1, \ldots, P_s$ and the respective intersection numbers are $m_1, \ldots, m_s$. Conversely any smooth pencil $C(\tau)$ is given in this way by choosing two smooth curves in the pencil.

A smooth pencil $C(\tau)$, $\tau \in \mathbb{C}$ is called *of a strict non-torus type* if $\bar{\sigma}_j : O(j-3) \to V_j(1,s)$ is surjective for $j < 2d$ and $\dim \mathrm{Coker}\, \bar{\sigma}_{2d} = 1$. In [12], we used terminology "*a pencil of a non-torus type*" for a pencil of a strict non-torus type. As it might be confused with a curve of a non-torus type, we change the terminology to "*pencil of a strict non-torus type*". In [12] we have shown the following. For $\vec{\tau} = (\tau_1, \ldots, \tau_r)$ with a positive integer r, put $C(\vec{\tau}) := C(\tau_1) \cup \cdots \cup C(\tau_r)$ as before.

Theorem 3. ([12]) *Assume that $C(\tau)$, $\tau \in \mathbb{C}$ be a smooth pencil of curves of a strict non-torus type. Then the generic Alexander polynomial of $C(\vec{\tau})$ is given by $(t^r - 1)^{r-2}(t-1)$.*

Remark 4. For a pencil $C(\tau), \tau \in \mathbb{C}$ to be of a strict non-torus type, it is necessary that each curve $C(\tau)$ is a curve of a non-torus type. However the sufficiency is not clear. Recall that $C(\tau)$ is a curve of a torus type if it can be defined by a polynomial of the type $F_n^p(X,Y,Z) + F_m^q(X,Y,Z) = 0$ with $np = mq = \deg\, C(\tau)$, $1 < p < q$ and F_n, F_m are homogeneous polynomials of degree n, m respectively.

2.5. A central line arrangement

Consider a central line arrangement of r lines $L^{(r)} = \bigcup_{i=1}^r L_i$ in $\mathbb{C}^2$. The topology of the complement does not depend on the choice of the lines $L_1, \ldots, L_r$. We may assume that

$$L^{(r)} : \quad y^r - x^r = 0.$$

The fundamental group $G^{(r)} = \pi_1(\mathbb{C}^2 - L^{(r)})$ can be easily computed using the van Kampen–Zariski pencil method ([15]) as follows.

$$G^{(r)} := \langle g_1, \ldots, g_r, \omega \,|\, \omega = g_r g_{r-1} \cdots g_1, \omega g_i = g_i \omega,\ i = 1, \ldots, r \rangle.$$

Eliminating either g_r or ω, we have two equivalent presentations

$$\begin{aligned} &(2) \quad G^{(r)} = \langle g_1, \ldots, g_{r-1}, \omega \,|\, g_i \omega = \omega g_i,\ i = 1, \ldots, r-1 \rangle \\ &(3) \quad \phantom{G^{(r)}} = \langle g_1, \ldots, g_r \,|\, g_r g_{r-1} \cdots g_1 = g_{r-1} \cdots g_1 g_2 = \cdots = g_1 g_r \cdots g_2 \rangle \end{aligned}$$

where the generators are taken in the line $x = 1$ as in Figure 1, where the bullets denote small loops which are oriented counterclockwise around the center. We call such loops *lassos* for $L^{(r)}$. The Alexander polynomial of $L^{(r)}$ is given by

$$\Delta_{L^{(r)}}(t) = (t^r - 1)^{r-2}(t-1).$$

This can be obtained either by the Fox calculus from the above presentation, or by Lemma 1. This also follows from a result of Randell [13].

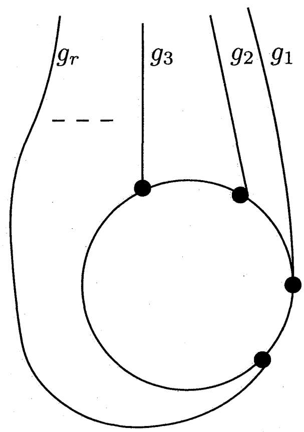

Fig. 1. Generators in $x = 1$

By the above presentation (2), we get

Proposition 5. *$G^{(r)}$ is isomorphic to $\mathbb{Z} \times F(r-1)$, where $F(r-1)$ is a free group generated by $g_1, \dots, g_{r-1}$ and $\omega := g_r \cdots g_1$ is in the center and ω generates the first factor $\mathbb{Z}$.*

§3. Abelian pencil and the main result

Let $C(\tau)$, $\tau \in \mathbb{C}$ be a pencil of curves of degree d, defined by

$$(4) \quad C(\tau): \quad \tau F(X,Y,Z) + (1-\tau) G(X,Y,Z) = 0, \quad \tau \in \mathbb{C}$$

with the base points $P_1, \dots, P_s$ and assume that $I(C(\tau), C(\tau'); P_i) = m_i$ for $i = 1, \dots, s$ and $\tau \neq \tau'$. This implies that $\sum_{i=1}^{s} m_i = d^2$. We assume that two generating curves $C(0)$, $C(1)$ are irreducible.

3.1. Generic pencil curves

Let C be a reduced curve in $\mathbb{P}^2$. The sum of the local Milnor numbers $\mu(C, P)$ for all the singular points P of C is called *the total Milnor number of C* and we denote it by $\mu(C)$. Let $C(\tau)$, $\tau \in \mathbb{C}$ be the above pencil of curves which is not necessarily smooth. Note that if the generating curves $C(0)$ and $C(1)$ have singularities simultaneously at some base point P_i, every pencil curve $C(\tau)$ has also a singularity at P_i. A curve in the pencil $C(\tau)$ is called *generic* if the total Milnor number $\mu(C(\tau))$ is minimal among all pencil curves. If $C(\tau)$ is generic, it is irreducible. Furthermore by the upper-semicontinuity of the local Milnor number, a generic curve $C(\tau)$ is smooth outside of the base points $\{P_1, \dots, P_s\}$ and $\mu(C(\tau), P_i) \le \mu(C(t), P_i)$ for any $t \in \mathbb{C}$ and $i = 1, \dots, s$. Let $U_1 \subset \mathbb{C}$ be the set of parameters τ such that $C(\tau)$ is generic. Then $\mathbb{C} - U_1$ contains only finite points. This follows from Bertini's theorem (see for example [4]). Now consider the set:

$$U_r := \{(\tau_1, \dots, \tau_r) \in \mathbb{C}^r \mid \tau_i \in U_1 \ (i = 1, \dots, r),\ \tau_i \ne \tau_j, \text{ if } i \ne j\}$$

An r-fold pencil curve $C(\vec{\tau})$ is called *generic* if $\vec{\tau} \in U_r$.

For the affine case, we fix a line at infinity L_∞ and we ask further the transversality condition. Thus we say that $C(\tau)$ is *affine generic* if $\tau \in U_1$ and $C(\tau)$ intersects L_∞ transversely. An r-fold curve $C(\vec{\tau})$ is *affine generic* if $\vec{\tau} \in U_r$ and $C(\vec{\tau})$ intersects L_∞ transversely.

Assertion 6. *The topology of $(\mathbb{P}^2, C(\vec{\tau}))$ is independent of $\vec{\tau} \in U_r$. Similarly the topology of $(\mathbb{C}^2, C(\vec{\tau}) \cap \mathbb{C}^2)$ is independent of an affine generic curve $C(\vec{\tau})$.*

Proof. First by the definition of U_1, the local Milnor numbers $\mu(C(\tau), P_i)$, $\tau \in U_1$ is constant. Thus by a result of D.T. Lê ([5]), the local embedded topology of $(C(\tau), P_i)$ (and also the global embedded topology in $\mathbb{P}^2$) is independent of $\tau \in U_1$. Suppose that $\mu(C(\tau), P_i) > 0$. Assume that $C(\tau)_{i,1}, \dots, C(\tau)_{i,r_i}$ be the local irreducible components of the germ $(C(\tau), P_i)$. Note that r_i is independent of $\tau \in U_1$ by the topological equivalence of the germs $(C(\tau), P_i)$ for $\tau \in U_1$ and the germs $(C(\tau)_{i,j}, P_i)$, $\tau \in U_1$ are topologically equivalent for any fixed j, $1 \le j \le r_i$.

Now we consider the germs of a curve $(C(\vec{\tau}), P_i)$ for $\vec{\tau} \in U_r$. Irreducible components are $\{C(\tau_a)_{a,j} \mid 1 \le a \le r,\ 1 \le j \le r_a\}$. First observe that the local intersection number $I(C(\tau_a), C(\tau_b); P_i)$ is constant and equal to m_i by the assumption. By the equality

$$I(C(\tau_a), C(\tau_b); P_i) = \sum_{j,k=1}^{r_i} I(C(\tau_a)_{i,j}, C(\tau_b)_{i,k}; P_i),\ 1 \le a < b \le r$$

and by the upper-semicontinuity of the local intersection numbers, the local intersection numbers $I(C(\tau_a)_{i,j}, C(\tau_b)_{i,k}; P_i)$, $\vec{\tau} \in U_r$ are constant for $a \ne b$ and fixed j, k. Certainly $I(C(\tau_a)_{i,j}, C(\tau_a)_{i,k}; P_i)$ are independent of $\tau_a \in U_1$. Thus the local intersection numbers of irreducible components of $(C(\vec{\tau}), P_i)$ are independent of $\vec{\tau} \in U_r$ and therefore $(C(\vec{\tau}), P_i)$

($\vec{\tau} \in U_r$) is a topologically equivalent family with a constant local Milnor numbers $\mu(C(\vec{\tau}), P_i)$ by Theorem 3.1 ([5]). Thus the family of curves $C(\vec{\tau})$, $\vec{\tau} \in U_r$ is a topologically equivalent family of curves in $\mathbb{P}^2$. The second assertion is proved similarly. Q.E.D.

3.2. Main result

Recall that the pencil $C(\tau)$ ($\tau \in \mathbb{C}$) is called *abelian* if $\pi_1(\mathbb{P}^2 - C(\tau_1) \cup C(\tau_2))$ is abelian for any pair of generic pencil curves $C(\tau_1)$, $C(\tau_2)$. Take a generic line at infinity $L_\infty : Z = 0$ for $C(\tau_1) \cup C(\tau_2)$ and put $f(x, y) = F(x, y, 1)$ and $g(x, y) = G(x, y, 1)$ so that $C(\tau)$ is defined by $\tau f(x, y) + (1 - \tau) g(x, y) = 0$ in $\mathbb{C}^2$. Now we can state our main result of this paper. We consider the d^2-fold branched covering:

$$\varphi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \varphi(x, y) = (f(x, y), g(x, y)).$$

We also consider the following central line arrangement

$$L(\vec{\tau}) = L(\tau_1) \cup L(\tau_2) \cup \cdots \cup L(\tau_r), \ \vec{\tau} = (\tau_1, \ldots, \tau_r)$$

where $L(\tau)$ is defined by $\tau x + (1 - \tau) y = 0$.

Main Theorem 7. *Let $C(\tau)$, $\tau \in \mathbb{C}$ be an abelian pencil of curves of degree d. Let r be a positive integer and $C(\vec{\tau})$ be an affine generic r-fold pencil curve. Then $\varphi : (\mathbb{C}^2, C(\vec{\tau})) \to (\mathbb{C}^2, L(\vec{\tau}))$ induces an isomorphism of the fundamental groups*

$$\varphi_\sharp : \pi_1(\mathbb{C}^2 - C(\vec{\tau})) \to \pi_1(\mathbb{C}^2 - L(\vec{\tau})) = G^{(r)}.$$

In particular, the Alexander polynomial of a generic curve $C(\vec{\tau})$ is given by $(t^r - 1)^{r-2}(t - 1)$ and $C(\vec{\tau})$ is a curve of a non-torus type.

Proposition-Remark 8. *Assume that $C(\tau)$, $\tau \in \mathbb{C}$ is a smooth abelian pencil. The second assertion $\Delta_{C(\vec{\tau})} = (t^r - 1)^{r-2}(t - 1)$ of Main Theorem 7 follows from the surjectivity of $\varphi_\sharp$.*

Proof. We only use the surjectivity of $\varphi_\sharp$ which is easy to be proved (see Appendix 4.3). In fact, assume that $r = 3$. The surjectivity of $\varphi_\sharp$ implies the divisibility of $\Delta_{C(\vec{\tau})}(t)$ by $\Delta_{L(\vec{\tau})}(t) = (t^3 - 1)(t - 1)$. On the other hand, we will show show in the next section that $\pi_1(\mathbb{C}^2 - C(\vec{\tau}))$ is generated by three lasso generators g, h, k and they satisfy the relations $[g, kh] = [gk, h] = e$ where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. On the other hand, the abstract group

$$G = \langle g, k, h \mid [g, kh] = [gk, h] = e \rangle$$

has the Alexander polynomial $(t^3-1)(t-1)$ (with respect to the canonical surjection $\phi : G \to \mathbb{Z}$, $g, k, h \mapsto 1$) which can be proved by a Fox calculus. As the presentation of $\pi_1(\mathbb{C}^2 - C(\vec{\tau}))$ is obtained from G by adding some more relations, $\Delta_{C(\vec{\tau})}(t)$ divides $\Delta_G(t) = (t^3 - 1)(t - 1)$. Thus $\Delta_{C(\vec{\tau})}(t) = (t^3 - 1)(t - 1)$ and the assertion is true for $r = 3$. Now this implies that $\bar{\sigma}_j : O(j - 3) \to V_j(1, s)$ is surjective for $j < 2d$ and $\bar{\sigma}_{2d}$ has 1 dimensional cokernel by Lemma 1 and therefore $C(\tau), \tau \in \mathbb{C}$ is a smooth pencil of a strict non-torus type. Now the assertion follows from Theorem 3. Q.E.D.

Remark 9. Shimada has considered curves which are pull backs of weighted homogeneous hypersurfaces by polynomial mappings from $\mathbb{P}^2$ to $\mathbb{P}^n$ and also their fundamental groups ([14]). See also Oka [9].

§4. Proof of the main theorem

In this section, we prove Main Theorem 7.

4.1. Computation of the monodromy relations.

Consider a pencil of curves

$$C(\tau) : (1 - \tau)G(X, Y, Z) + \tau F(X, Y, Z) = 0, \ \tau \in \mathbb{C}$$

as before which is generated by two irreducible curves $C(1) = \{F(X, Y, Z) = 0\}$ and $C(0) = \{G(X, Y, Z) = 0\}$ of degree d. We may assume that $C(\tau)$ is generic for any τ, $0 \leq \tau \leq 1$, changing $C(0)$, $C(1)$ if necessary. This can be done as follows. Take a generic curve $C(t_1)$ and choose a generic line at infinity L_∞ for $C(t_1)$ and fix it. Then choose $\varepsilon > 0$ so that $C(\tau)$ is affine generic for any τ, $|\tau - t_1| \leq \varepsilon$. Replace $C(0)$ by $C(t_1)$ and $C(1)$ by $C(t_1 + \varepsilon)$ and reparametrize the pencil. Let $C^{(2)} := C(0) \cup C(1)$ and $C^{(3)}(\tau) = C^{(2)} \cup C(\tau)$. Thus the base points P_j $(j = 1, \ldots, s)$ are in the affine part $C(\tau) \cap \mathbb{C}^2$. We consider the pencil of lines $L_\eta := \{x = \eta\}$ to compute the fundamental group. Take a generic member of the pencil line L_{η_0} and we fix η_0. Put

$$C(0) \cap L_{\eta_0} = \{\rho_1, \ldots, \rho_d\}, \quad C(1) \cap L_{\eta_0} = \{\xi_1, \ldots, \xi_d\}.$$

Note that $\rho_j \neq \xi_i$ for any $1 \leq i, j \leq d$. We take generators $\{g_j, h_j \mid j = 1, \ldots, d\}$ of $\pi_1(L_{\eta_0} - L_{\eta_0} \cap C^{(2)})$ where $g_1, \ldots, g_d$ (respectively $h_1, \ldots, h_d$) are lassos of $C(0)$ (resp. of $C(1)$) and g_j, h_j are given as in Figure 2. The oriented arc ℓ_j and the disk D_j are explained in Observation 11. Hereafter every lasso has the counterclockwise orientation.

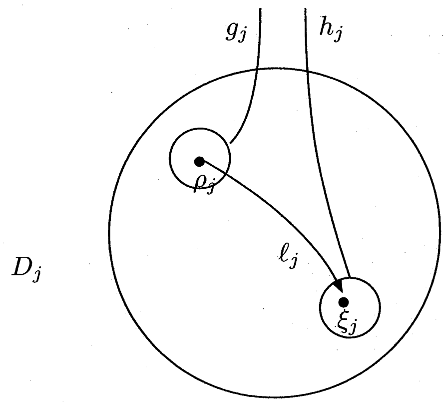

Fig. 2. Generators g_j, h_j in L_{η_0}

As we have assumed that $\pi_1(\mathbb{P}^2 - C^{(2)})$ is abelian and the line at infinity L_∞ is generic, $\pi_1(\mathbb{C}^2 - C^{(2)})$ is abelian and isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (see [8]). Thus we may assume that our generators satisfy the following relations:

$$(Ab) \quad g_1 = \cdots = g_d, \quad h_1 = \cdots = h_d, \quad [g_1, h_1] = e$$

where $[g_1, h_1] = g_1 h_1 g_1^{-1} h_1^{-1}$. Let $\Sigma_2 = \{\eta \mid L_\eta \text{ singular}\}$ and put $\Sigma_2 = \{\alpha_1, \dots, \alpha_\nu\}$. (We say that L_η is called *a singular line* if $\sharp\,(C^{(2)} \cap L_\eta) < 2d$ ([3]).) We may assume, for simplicity, that the tangent cones of $C(0)$, $C(1)$ at P_i are transversal to the vertical line passing through P_i, $i = 1, \dots, s$. Thus we can write $\Sigma_2 = \Sigma_c \amalg \Sigma_t$ such that $\Sigma_c = \{\alpha_1, \dots, \alpha_s\}$ and $\Sigma_t = \{\alpha_{s+1}, \dots, \alpha_\nu\}$. The line L_{α_j} passes through P_j for $j = 1, \dots, s$. (So α_i is the x-coordinate of P_i for $1 \le i \le s$.) For $\alpha_j \in \Sigma_t$, L_{α_j} is tangent to either $C(0)$ or $C(1)$. We take generators $\beta_1, \dots, \beta_\nu$ of $\pi_1(\mathbb{C} - \Sigma_2, \eta_0)$. We assume that these generators are presented by disjoint lassos around α_j, $1 \le j \le \nu$. We recall how *the monodromy action* of $\pi_1(\mathbb{C} - \Sigma_2, \eta_0)$ on $\pi_1(L_{\eta_0} - C^{(2)})$ is defined. Take a base point $b_0 = (\eta_0, y_0) \in L_{\eta_0} - C^{(2)}$ for the fundamental group $\pi_1(\mathbb{C}^2 - C^{(2)})$. We assume that y_0 is purely imaginary and the imaginary part of y_0 is positive and sufficiently large so that b_0 is near enough to the base point of the pencil $[0 : 1 : 0]$. We take y as the coordinate function of each pencil line L_η. Fix a cross section $\gamma : \mathbb{C} \to \mathbb{C}^2 - C^{(2)}$ of the projection $p : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x$, so that $\gamma(\eta_0) = b_0$. We assume that the generator β_i (considered as a lasso) is presented as $\beta_i = B_i \cdot S_i \cdot B_i^{-1}$, where S_i is an oriented circle of the radius $\delta > 0$ centered at α_i and a simple path

B_i joins η_0 and $\alpha_i' \in S_i$. We assume that $B_i \cap B_j = \{\eta_0\}$ for $i \neq j$ and the circles $S_1, \dots, S_\nu$ are disjoint. The loop β_i can be lifted to a loop $\hat{\beta}_i$ in $\mathbb{C}^2 - C^{(2)}$ by $\hat{\beta}_i(t) = \gamma(\beta_i(t))$ and we have a family of diffeomorphisms $\varphi_{i,t} : (L_{\eta_0}, C^{(2)} \cap L_{\eta_0}, b_0) \to (L_{\beta_i(t)}, C^{(2)} \cap L_{\beta_i(t)}, \gamma(\beta_i(t)))$, $0 \le t \le 1$, by the local triviality of the projection $p : (\mathbb{C}^2, C^{(2)} \cup \gamma(\mathbb{C})) \to \mathbb{C}$ over $\mathbb{C} - \Sigma_2$. Then the actions $g_j^{\beta_i}$, $h_j^{\beta_i}$ of β_i on g_j, h_j are simply given by $\varphi_{i,1} \circ g_j$ and $\varphi_{i,1} \circ h_j$. The monodromy relations are given by

$$g_j^{\beta_i} = g_j,\ h_j^{\beta_i} = h_j,\ j = 1, \dots, d$$

which are the result of the homotopies

$$\begin{aligned} g_j &\underset{F}{\simeq} (\hat{\beta}_i)^{-1} \cdot g_j \cdot \hat{\beta}_i \underset{H}{\simeq} \varphi_{i,1} \circ g_j = g_j^{\beta_i}, \\ h_j &\underset{F}{\simeq} (\hat{\beta}_i)^{-1} \cdot h_j \cdot \hat{\beta}_i \underset{H}{\simeq} \varphi_{i,1} \circ h_j = h_j^{\beta_i}. \end{aligned}$$

(The multiplication of paths starts from the left side.) For the construction of the homotopy F, we only need to observe that $\hat{\beta}_i \simeq c_{b_0}$ on $\gamma(\mathbb{C})$ where c_{b_0} is the constant loop. For the construction of H, we use the composite of g_j by $\varphi_{i,t}$ as follows. Put J_s be the path from $\hat{\beta}_i(s)$ to $\hat{\beta}_i(1) = b_0$ along the lift $\hat{\beta}_i$. Then the deformation of $(\hat{\beta}_i)^{-1} \cdot g_j \cdot \hat{\beta}_i$ at level s ($=H|_{I \times \{s\}}$) is given by $J_s^{-1} \cdot (\varphi_{i,s} \circ g_j) \cdot J_s$. Thus we observe that

Observation 10. *The monodromy relations $g_j = g_j^{\beta_i}$, $h_j = h_j^{\beta_i}$, $1 \le i \le \nu$, $1 \le j \le d$ are obtained on the subspace $p^{-1}(K_i) \cup \gamma(\mathbb{C})$ where $K_i = B_i \cup S_i$ and $p : \mathbb{C}^2 \to \mathbb{C}$ is the projection $(x, y) \mapsto x$.*

Now we consider the compact set $K = \bigcup_{i=1}^{\nu}(B_i \cup S_i) \subset \mathbb{C} - \Sigma_2$ and $T = \bigcup_{i=1}^{\nu} B_i$. We have observed that the monodromy relations (Ab) are obtained in the subspace $p^{-1}(K) \cup \gamma(\mathbb{C})$, using the pencil lines L_η, $\eta \in K$. Note that T is contractible.

For the sake of later arguments, we introduce the following notion. Put $C^{(2)} \cap L_\eta = \{\rho_1(\eta), \xi_1(\eta), \dots, \rho_d(\eta), \xi_d(\eta)\}$ for $\eta \in T$. We can choose $\rho_j(\eta)$ and $\xi_j(\eta)$ to be continuous on T and $\rho_j(\eta_0) = \rho_j$, $\xi(\eta_0) = \xi_j$. Let $S_{j,0}(\eta)$, $S_{j,1}(\eta)$ be the circle of the radius ε_0 with the center $\rho_j(\eta)$, $\xi_j(\eta)$ in L_η respectively. We can choose a sufficiently small number ε_0 so that the circles $\{S_{j,0}(\eta), S_{j,1}(\eta) \mid j = 1, \dots, d\}$ are disjoint for any $\eta \in T$, as T is compact. Take a temporary base point $b_j'(\eta), b_j(\eta)$ on $S_{j,0}(\eta), S_{j,1}(\eta)$ respectively. Choose simple paths $M_j'(\eta_0)$, $M_j(\eta_0)$ which connect b_0 to $b_j'(\eta_0)$ and b_0 to $b_j(\eta_0)$ respectively. By the triviality of the projection $p : (\mathbb{C}^2, C^{(2)} \cup \gamma(\mathbb{C})) \to \mathbb{C}$ over T, we may choose a continuous family of $M_j'(\eta), M_j(\eta)$ for any $\eta \in T$. Consider the lassos $M_j'(\eta) \cdot S_{j,0}(\eta) \cdot M_j'(\eta)^{-1}$

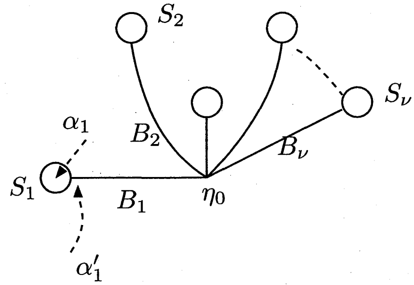

Fig. 3. The compact subsets K and T

and $M_j(\eta) \cdot S_{j,1}(\eta) \cdot M_j(\eta)^{-1}$. The temporary base point $\gamma(\eta)$ can be connected to the base point $b_0 = \gamma(\eta_0)$ in the obvious way by the path $\hat{\beta}_i$ for $\eta \in K_i$. We call such lassos *continuous families of lassos of the radius* ε_0 over T.

Now we are ready to consider $\pi_1(\mathbb{C}^2 - C^{(r)}(\tau))$ for $r \geq 3$.

4.1.1. *Case* $r = 3$ We first consider the case $r = 3$, as the argument for $r > 3$ is exactly the same with the case $r = 3$.

We may assume that $C(0)$, $C(1)$ are near enough so that there exist small disjoint disks D_j, $j = 1, \dots, d$ of the radius ε_1 in the line L_{η_0} with ρ_j, $\xi_j \in D_j$. For this, we take $C(\varepsilon)$ with sufficiently small $\varepsilon > 0$ and replace $C(1)$ by $C(\varepsilon)$. Recall that $S_{j,0}(\eta_0), S_{j,1}(\eta_0)$ are ε_0 disks. Taking again new ε_0 small enough if necessary, we may assume that $\operatorname{Int} D_j \supset S_{j,0}(\eta_0) \cup S_{j,1}(\eta_0)$ where $\operatorname{Int} D_j$ is the interior of D_j. Put $C(\tau) \cap L_{\eta_0} = \{\theta_1(\tau), \dots, \theta_d(\tau)\}$ in a suitable order. We may assume that $\theta_j(\tau) \in \operatorname{Int} D_j$ for any $0 \leq \tau \leq 1$.

Now we consider the following path:

$$\ell_j : [0,1] \to D_j, t \mapsto \theta_j(t).$$

We identify ℓ_j with the image of ℓ_j in D_j. A key observation is :

Key Observation 11. ℓ_j is a simple path joining ρ_j and ξ_j in $\operatorname{Int} D_j$.

Proof. In fact, assume that there exists a multiple point Q_j of ℓ_j. This implies that there exist two different times $t = t_1, t_2$ with $0 \leq t_1 < t_2 \leq 1$ such that $Q_j \in C(t_1) \cap C(t_2)$. This gives an obvious contradiction by the Bézout theorem. In fact, as $C(t_1) \cdot C(t_2)$ contains

P_i with intersection multiplicity m_i for $i = 1, \ldots, s$, we get the contradiction:

$$d^2 = I(C(t_1), C(t_2)) \geq I(C(t_1), C(t_2); Q_j) + \sum_{i=1}^{s} I(C(t_1), C(t_2); P_i)$$
$$= I(C(t_1), C(t_2); Q_j) + \sum_{i=1}^{s} m_i > d^2.$$

This proves the assertion. Q.E.D.

Topological model. To simplify the figure, we take an oriented horizontal line $\hat{\ell}_j$ in each D_j as a topological model and let $\hat{L}_{\eta_0}$ be the generic fiber with this model in D_j. It is easy to see that there exists a diffeomorphism $\phi_{mod} : L_{\eta_0} \to \hat{L}_{\eta_0}$ which satisfies the next conditions.

- There exists a positive number δ_1 so that ϕ_{mod} is the identity map outside of δ_1 neighborhood of $\cup_{j=1}^{d} D_j$.
- $\phi_{mod}(D_j) = D_j$ and $\phi_{mod}(\ell_j) = \hat{\ell}_j$ as oriented paths.

In the case that L_{η_0} is as in Figure 4, (we may assume that) ϕ_{mod} is the composite of "straightening" and a rotation of angle $3\pi/2$ in clockwise orientation inside D_j. We assume hereafter that the topological situation

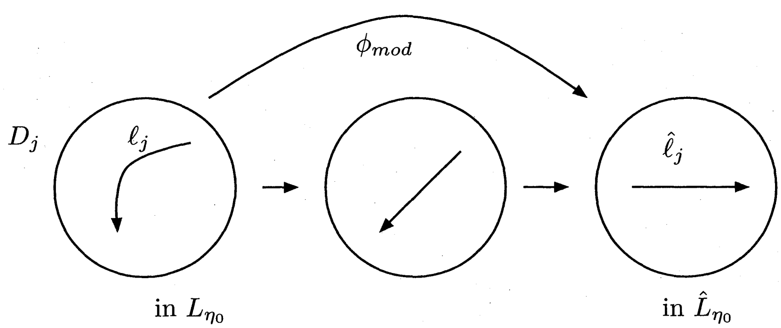

Fig. 4. Model path $\hat{\ell}_j$

is as in $\hat{L}_{\eta_0}$, using this identification. Put $\Delta(q; r) = \{(\eta, y) \mid |y - \xi| \leq r\}$ for $q = (\eta, \xi) \in L_\eta$ (the disk of the radius r with the center q in the line L_η).

For a fixed $\varepsilon_0 > 0$ as above, we can take $\delta_K > 0$ small enough so that
(1) for any τ with $|\tau - 1| < \delta_K$, $\theta_j(\tau) \in \Delta(\xi_j; \varepsilon_0/5)$ and
(2) for any τ with $|\tau| < \delta_K$, $\theta_j(\tau) \in \Delta(\rho_j; \varepsilon_0/5)$.
We fix such a δ_K hereafter.

Now we take lassos $g_1, \dots, g_d$ for $C(0) \cap L_{\eta_0}$, lassos $h_1, \dots, h_d$ for $C(1) \cap L_{\eta_0}$ and lassos $k_1(t), \dots, k_d(t)$ for $C(t) \cap L_{\eta_0}$ as in the left side of Figure 5. We assume that the lassos $g_j, k_j(t), h_j$ are connected to the base point b_0 by homotopically same paths outside of D_j in the left side figure. This means that the left side generators are homotopically the same as the right side generators in Figure 5. For the simplicity of drawing, we use hereafter the left side drawing style. (If ℓ_j is as is indicated in the left side of Figure 6 for example, the actual generators are the pull-back of $g_j, k_j(t), h_j$ by ϕ_{mod}.) They satisfy the following continuity property.

$$\text{(Continuity)} \qquad \begin{cases} k_j(t) \to h_j, & t \to 1 \\ k_j(t) \to g_j, & t \to 0. \end{cases}$$

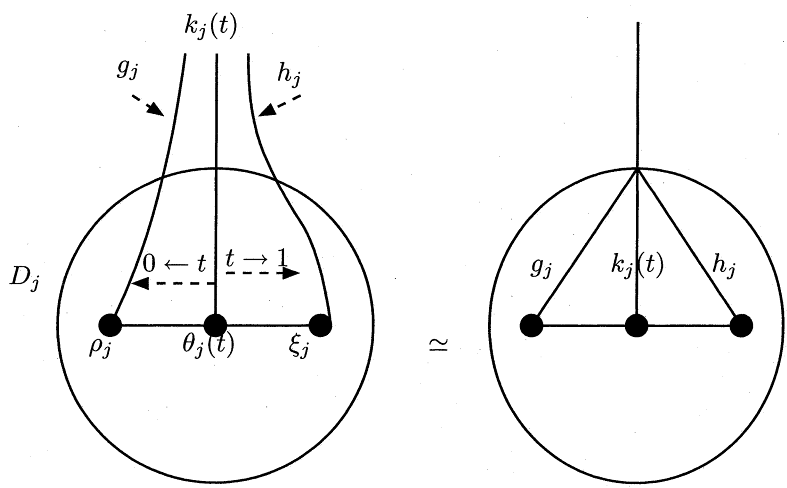

Fig. 5. $g_j, h_j, k_j(t)$

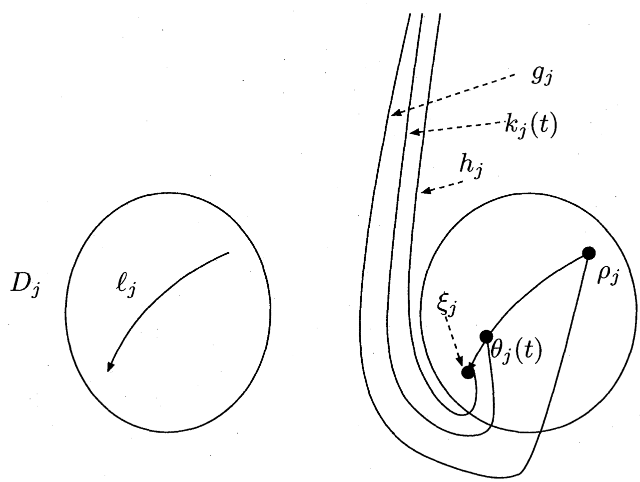

Fig. 6. $g_j, h_j, k_j(t)$-bis

Recall that a bullet denotes a lasso around the corresponding center which can be arbitrarily small. Consider two disjoint circles in D_j: $S_{j,0} = \partial\Delta(\rho_j;\varepsilon_0)$ and $S_{j,1} = \partial\Delta(\xi_j;\varepsilon_0)$ as above. They are oriented counterclockwise. For any $0 < t < t' < 1$, there is an isotopy $\psi_{t,t',s} : (\mathbb{C}^2, C^{(3)}(t)) \to (\mathbb{C}^2, C^{(3)}(s))$ for $t \le s \le t'$ (we write $\psi_s = \psi_{t,t',s}$ for simplicity) such that $L_{\eta_0} \cup \gamma(\mathbb{C})$ is stable by ψ_s and $\psi_t = \mathrm{id}$, and ψ_s induces an isotopy $(L_{\eta_0}, L_{\eta_0} \cap C^{(3)}(t)) \to (L_{\eta_0}, L_{\eta_0} \cap C^{(3)}(s))$ with $\psi_s(b_0) = b_0$. So $\psi_{t'}$ gives a diffeomorphism between $(L_{\eta_0}, L_{\eta_0} \cap C^{(3)}(t))$ and $(L_{\eta_0}, L_{\eta_0} \cap C^{(3)}(t'))$. (Strictly speaking, ψ_s is only a C^0-homeomorphism near $P_1, \dots, P_s$.)

(a) We first consider the extreme case: s_1, $|1 - s_1| < \delta_K$. Fix such an s_1 and consider the elements

$$\Omega_j(s_1) := k_j(s_1) h_j,\ j = 1, \dots, d.$$

See Figure 7. Note that the figures which follow hereafter are not on the exact scale, but they only show the topological situations. It is important to observe that Ω_j is presented by a lasso, $M_j \cdot S_{j,1} \cdot M_j^{-1}$ where M_j is a simple path joining b_0 and a point b_j on $S_{j,1}$. Strictly speaking, Ω_j

is a lasso for $C(1)$ if we ignore $C(s_1)$. Note that $\theta_j(s_1) \in \operatorname{Int} S_{j,1}$ where $\operatorname{Int} S_{j,1}$ is the interior of $S_{j,1}$. We can consider that $k_j(s_1)$, h_j are lassos first starting at b_j and then b_j is connected to b_0 by the same path M_j. See Figure 7.

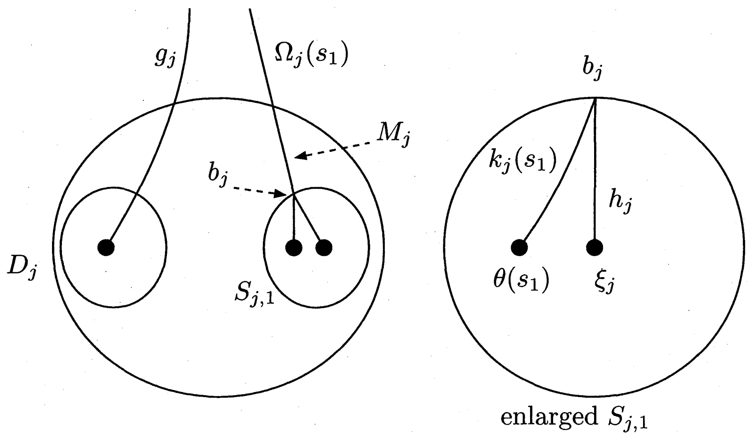

Fig. 7. $g_j, h_j, k_j(s_1)$ for $|s_1 - 1| < \delta_K$

We consider the same deformation of the loops g_j, Ω_j along K for $C^{(3)}(s_1)$ as that of $C^{(2)}$ which we used to obtain (Ab). During this deformation, we assume that the radius of the circle $S_{j,1}(\eta)$ is constant and equal to ε_0 so that the two points $\theta_j(s_1)$, ξ_j behave like one point inside $\operatorname{Int} S_{j,1}$. Thus the lassos,

$$\{g_1, \ldots, g_d, \Omega_1(s_1), \ldots, \Omega_d(s_1)\}$$

satisfy the same relations as (Ab). The point $\theta_j(s_1)$ stays always inside $S_{j,1}(\eta)$ and it does not come out of the circle $S_{j,1}(\eta)$ during the deformation over K by the choice of δ_K. Namely we have the following relations from (Ab):

$$(R_1) \quad g_1 = \cdots = g_d, \quad \Omega_1(s_1) = \cdots = \Omega_d(s_1), \quad [g_1, \Omega_1(s_1)] = e.$$

Using the isomorphism $\psi_{t,s_1,s_1\sharp} : \pi_1(\mathbb{C}^2 - C^{(3)}(t)) \to \pi_1(\mathbb{C}^2 - C^{(3)}(s_1))$, we obtain the following assertion.

Observation 12. *The lasso element $k_j(t)$ for $C^{(3)}(t)$ in an arbitrary t, $0 < t < 1$ in Figure 5 and the lasso element $k_j(s_1)$ for $C^{(3)}(s_1)$ in Figure 7 for $|1 - s_1| \leq \delta_K$ are identical under a canonical isotopy ψ_{t,s_1,s_1} by Observation 11.*

Using this observation, (R_1) can be translated in the following relations in $\pi_1(\mathbb{C}^2 - C^{(3)}(t))$:

$$(R_1)' \quad \begin{cases} g_1 = \cdots = g_d,\ \Omega_1(t) = \cdots = \Omega_d(t),\ [g_1, \Omega_1(t)] = e,\ 0 < t < 1 \\ \text{with } \Omega_j(t) = k_j(t)h_j,\ j = 1, \ldots, d. \end{cases}$$

(b) Similarly we consider the other extreme situation $t \to 0$. Fix an s_1', $0 < s_1' \leq \delta_K$. Recall that $|\theta_j(s_1') - \rho_j| \leq \varepsilon_0/5$. Thus $\theta_j(s_1') \in S_{j,0}$. In this case, we consider an element

$$\Omega_j'(s_1') = g_j k_j(s_1'), \quad j = 1, \ldots, d.$$

See Figure 8. Thus by the same argument as in the case (1), we get the relations:

$$(R_2) \quad \Omega_1'(s_1') = \cdots = \Omega_d'(s_1'),\ h_1 = \cdots = h_d,\ [\Omega_1'(s_1'), h_1] = e$$

and this implies

$$(R_2)' \quad \begin{cases} \Omega_1'(t) = \cdots = \Omega_d'(t),\ h_1 = \cdots = h_d,\ [\Omega_1'(t), h_1] = e,\ 0 < t < 1 \\ \text{with } \Omega_j(t)' = g_j k_j(t),\ j = 1, \ldots, d. \end{cases}$$

The relations $(R_1)'$, $(R_2)'$ imply also that $k_1(t) = \cdots = k_d(t)$. Thus using three generators $g = g_1, k = k_1(t)\,(t \neq 0, 1), h = h_1$, we have shown that $\pi_1(\mathbb{C}^2 - C^{(3)}(t))$ has the following presentation.

$$\pi_1(\mathbb{C}^2 - C^{(3)}(t)) = \langle g, k, h \mid [g, kh] = [gk, h] = e, R_1, \ldots, R_b \rangle,$$

where $R_1, \ldots, R_b$ are possible other relations to be added. We want to show that $b = 0$. Namely we do not need any other relations. We prove this assertion simultaneously for the general case $r \geq 3$ in the next section.

Remark 13. The locus of the singular lines Σ_3 for $C^{(3)}(s_1)$ with $|1 - s_1| \leq \delta_K$ can be written as $\Sigma_3 = \Sigma_2 \cup \Sigma'$ so that Σ_2 is that of $C^{(2)}$ and $\alpha \in \Sigma'$ corresponds to a line L_α which is tangent to $C(s_1)$. If $|1 - s_1|$ is sufficiently small, α is very near to some $\alpha_i \in \Sigma_t$ and therefore α is bifurcated from α_i and $\alpha, \alpha_i \in \operatorname{Int} S_i$ as in Figure 9. Define $\beta_i' \in \pi_1(\mathbb{C} -$

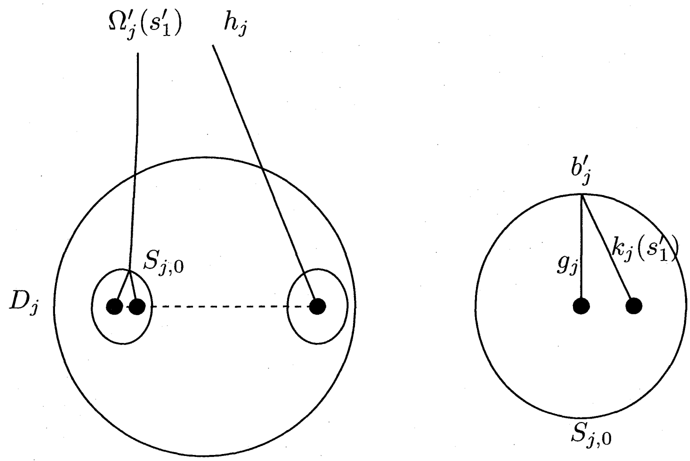

Fig. 8. $g_j, h_j, k_j(s'_1)$ for $|s'_1| < \delta_K$

Σ_3) by the product of two generators of $\pi_1(\mathbb{C} - \Sigma_3)$ corresponding to α_i and α. Thus the monodromy relation which we used for the generators $g_1, \Omega_1, \dots, g_d, \Omega_d$ under the generator $\beta_i \in \pi_1(\mathbb{C} - \Sigma_2)$ is in fact the action of $\beta'_i \in \pi_1(\mathbb{C} - \Sigma_3)$ on $g_j, \Omega_j \in \pi_1(L_{\eta_0} - C^{(3)}(s_1) \cap L_{\eta_0})$. Namely β'_i acts on g_j, Ω_j in the exact same way as the action of $\beta_i \in \pi_1(\mathbb{C} - \Sigma_2)$ on $g_j, h_j \in \pi_1(\mathbb{C}^2 - C^{(2)})$.

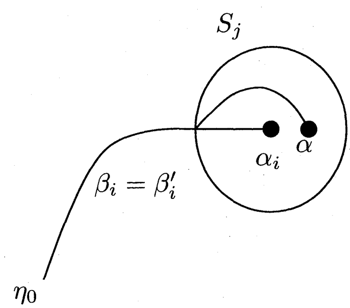

Fig. 9. Bifurcation of singular lines

4.1.2. *General case $r \geq 3$* For the general case $C(\vec{\tau})$, $\vec{\tau} = (\tau_1, \dots, \tau_r)$ with $\tau_1 = 0 < \tau_2 < \cdots < \tau_r = 1$, we do the same discussion as above. Take generators $k_j(\tau_i)$, $1 \le i \le r$, $1 \le j \le d$. It is important to choose $\tau_2, \dots, \tau_{r-1}$ on the open interval $(0,1)$ so that $\theta_j(\tau_1) = \rho_j, \theta_j(\tau_2), \dots, \theta_j(\tau_{r-1}), \theta_j(\tau_r) = \xi_j$ are on the path ℓ_j in this order. For simplicity, we write $k_{j,i}$ for $k_j(\tau_i)$ hereafter. Note that $k_{1,i}, \dots, k_{d,i}$ are generators for $\pi_1(L_{\eta_0} - C(\tau_i) \cap L_{\eta_0})$.

Fix an i, $1 \le i \le r-1$. Choose $\tau_2, \dots, \tau_{r-1}$ so that they satisfy

$$\tau_1 = 0 < \tau_2 < \cdots < \tau_i < \delta_K \ll 1 - \delta_K < \tau_{i+1} < \cdots < \tau_r = 1.$$

Note that $\theta_j(\tau_i)$ is on the path ℓ_j for each $i = 2, \dots, r-1$ so that

$$\{\rho_j, \theta_j(\tau_2), \dots, \theta_j(\tau_i)\} \subset \operatorname{Int} S_{j,0},\ \{\theta_j(\tau_{i+1}), \dots, \theta_j(\tau_{r-1}), \xi_j\} \subset \operatorname{Int} S_{j,1}.$$

See Figure 10. Two collections of the points $P = \{\rho_j, \theta_j(\tau_2), \dots, \theta_j(\tau_i)\}$ and $Q = \{\theta_j(\tau_{i+1}), \dots, \theta_j(\tau_{r-1}), \xi_j\}$ behave like two points $\{\rho_j, \xi_j\}$ under the monodromy actions of $\beta_1, \dots, \beta_\nu$, using the loops K_j, $j = 1, \dots, \nu$. Put

$$\Xi_{j,i} = k_{j,1} \cdots k_{j,i}, \quad \Omega_{j,i} = k_{j,i+1} \cdots k_{j,r}, \quad \text{where } k_{j,1} = g_j,\ k_{j,r} = h_j.$$

Two elememts $\Xi_{j,i}$, $\Omega_{j,i}$ are presented by lassos with circle $S_{j,0}$, $S_{j,1}$ of the radius ε_0 which contains $\{\rho_j, \theta_j(\tau_2), \dots, \theta_j(\tau_i)\}$ and $\{\theta_j(\tau_{i+1}), \dots, \theta_j(\tau_{r-1}), \xi_j\}$ respectively. Thus we see that the following relations are satisfied, which are derived from (Ab).

$$(R_i) \qquad \Omega_{1,i}\Xi_{1,i} = \Xi_{1,i}\Omega_{1,i},\ \Xi_{1,i} = \cdots = \Xi_{d,i},\ \Omega_{1,i} = \cdots = \Omega_{d,i},$$
$$\text{where} \quad \Xi_{j,i} = k_{j,1} \cdots k_{j,i},\ \Omega_{j,i} = k_{j,i+1} \cdots k_{j,r},\ i = 1, \dots, r-1.$$

Let $\vec{t} = (t_1, \dots, t_r)$, $t_1 = 1 < t_2 < \cdots < t_{r-1} < t_r = 1$ be an arbitrarily chosen r-fold vector of parameters. Thus by the same discussion as in the case $r = 3$, we have a homeomorphism

$$\phi_{\vec{t},\vec{\tau}} :\ (\mathbb{C}^2, C(\vec{t})) \to (\mathbb{C}^2, C(\vec{\tau}))$$

which induces a diffeomorphism of $(L_{\eta_0}, C(\vec{t}) \cap L_{\eta_0}) \to (L_{\eta_0}, C(\vec{\tau}) \cap L_{\eta_0})$ and $\phi_{\vec{t},\vec{\tau}}(\gamma(\eta)) = \gamma(\eta)$ for any η. Using this diffeomorphism, we can identify $k_j(\tau_i)$ and $k_j(t_i)$ for $i = 1, \dots, r$ as in the case $r = 3$. Thus the relations R_i can be understood as the relations for $k_{ji} = k_j(t_i)$ for $j = 1, \dots, d$, $i = 1, \dots, r$. Therefore using $(R_1), \dots, (R_{r-1})$ inductively, we get the relations:

$$k_{1,i} = \cdots = k_{d,i},\ i = 1, \dots, r.$$

Put $a_i := k_{1,i}$, $i = 1, \dots, r$. Thus we need only r generators a_i, $i =$

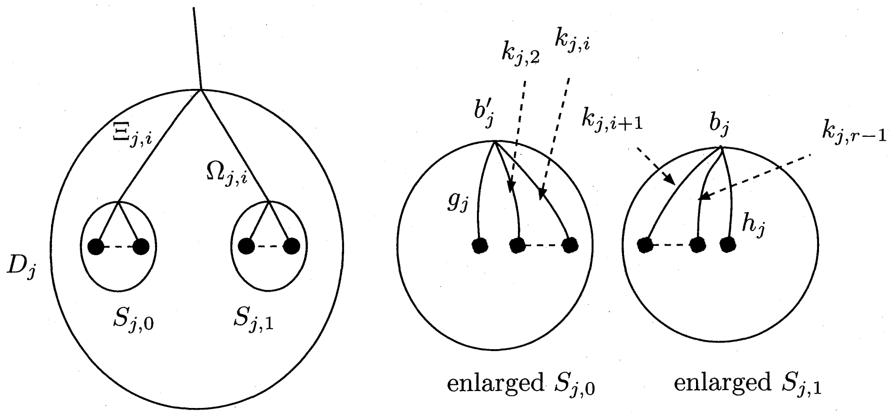

Fig. 10. Generators $k_{j,i}$ in L_{η_0} and ε_0-disks $S_{j,0}, S_{j,1}$

$1, \dots, r$. Note that a_i is a lasso for the component $C(\tau_i)$. Then the relations $(R_1) \sim (R_{r-1})$ reduces to:

$$(Ab_2) \quad a_1 a_2 \cdots a_r = a_2 \cdots a_r a_1 = \cdots = a_r a_1 a_2 \cdots a_{r-1}.$$

Put $\Omega = a_1 a_2 \cdots a_r$. Now we replace the last generator a_r by Ω. Then (Ab_2) can be read as

$$(Ab_3) \qquad \Omega a_i = a_i \Omega, \quad i = 1, 2, \dots, r-1.$$

Thus Ω is in the center of the fundamental group $\pi_1(\mathbb{C}^2 - C(\vec{\tau}))$ and we can present $\pi_1(\mathbb{C}^2 - C(\vec{\tau}))$ as follows.

$$(5) \qquad \pi_1(\mathbb{C}^2 - C(\vec{\tau})) = \langle a_1, \dots, a_{r-1}, \Omega \mid (Ab_3), R_1, \dots, R_b \rangle$$

where $R_1, \dots, R_b$ are possible other relations to be added.

Proposition 14. *The words of the relations $R_1, \dots, R_b$ do not contain Ω.*

Proof. Observe that the images of $a_1, \dots, a_{r-1}, \Omega$ are free generators of $H_1(\mathbb{C}^2 - C(\vec{\tau})) \cong \mathbb{Z}^r$. Thus if R_j contains Ω, the total summation of the exponents of Ω is zero. As Ω commutes with other generators, we can eliminate Ω in R_j. Q.E.D.

Thus we observe that $\pi_1(\mathbb{C}^2 - C(\vec{\tau})) \cong \mathbb{Z} \times F'$ where

$$F' := \langle a_1, \ldots, a_{r-1} \,|\, R_1, \ldots, R_b \rangle, \quad \mathbb{Z} = \langle \Omega \rangle.$$

4.2. The proof of Main Theorem 7

Recall that

$$C(\tau): \quad (1-\tau)\, f(x,y) + \tau\, g(x,y) = 0, \ \tau \in \mathbb{C}.$$

We consider the d^2-fold branched covering map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\varphi(x,y) = (f(x,y), g(x,y))$. Let us consider the pencil of lines $L(\tau), \tau \in \mathbb{C}$. Define a central line arrangement $L(\vec{\tau})$ by

$$L(\vec{\tau}) = L(\tau_1) \cup L(\tau_2) \cup \cdots \cup L(\tau_r), \ \vec{\tau} = (\tau_1, \ldots, \tau_r).$$

Now it is immediate from the definition that $\varphi^{-1}(L(\vec{\tau})) = C(\vec{\tau})$. Thus we get a homomorphism which is canonically surjective (see Appendix):

$$\varphi_\sharp : \quad \pi_1(\mathbb{C}^2 - C(\vec{\tau})) \to \pi_1(\mathbb{C}^2 - L(\vec{\tau})) \cong G^{(r)}.$$

Now we can prove that the other relations $R_1, \ldots, R_b$ are empty. Consider an abstract group G:

$$G := \langle \hat{a}_1, \ldots, \hat{a}_{r-1}, \hat{\Omega} \,|\, \hat{a}_i \hat{\Omega} = \hat{\Omega} \hat{a}_i, \ i = 1, \ldots, r-1 \rangle.$$

The relation (4) implies that we have a canonical surjective homomorphism $\psi : G \to \pi_1(\mathbb{C}^2 - C(\vec{\tau}))$ which is defined by $\hat{a}_i \mapsto a_i$ and $\hat{\Omega} \mapsto \Omega$. Let $F \subset G$ be the free group of rank $r-1$ with generators $\hat{a}_1, \ldots, \hat{a}_{r-1}$. We consider the surjective homomorphism $\Psi : G \to F(r-1)$ which is the composite $p \circ \varphi_\sharp \circ \psi$ where $p : G^{(r)} \cong \mathbb{Z} \times F(r-1) \to F(r-1)$ is the canonical projection homomorphism. Let

$$\Theta = \Psi \circ \iota_1 : F \to G \to \pi_1(\mathbb{C}^2 - L(\vec{\tau})) \to F(r-1)$$

where ι_1, ι_2 are the canonical inclusion homomorphisms.

$$\begin{tikzcd}
G \cong \mathbb{Z} \times F \arrow[r, "\psi"] & \pi_1(\mathbb{C}^2 - C(\vec{\tau})) \arrow[r, "\varphi_\sharp"] & \pi_1(\mathbb{C}^2 - L(\vec{\tau})) \arrow[d, "\cong"] & \\
F = \langle \hat{a}_1, \ldots, \hat{a}_{r-1} \rangle \arrow[u, "\iota_1"] \arrow[r, "\psi'"] & F' \arrow[u, "\iota_2"] & \mathbb{Z} \times F(r-1) \arrow[r, "p"] & F(r-1)
\end{tikzcd}$$

Note that the image of the first factor $\mathbb{Z} = \langle \hat{\Omega} \rangle$ of G by $\varphi_\sharp \circ \psi$ is in the center of $\mathbb{Z} \times F(r-1)$, which is the first factor $\mathbb{Z} = \langle \omega \rangle$. This implies that $\langle \hat{\Omega} \rangle$ is in the kernel of Ψ and the surjectivity of the mapping Θ' follows immediately where $\Theta' : G/\mathrm{Ker}\,\Psi \to F(r-1)$. As $F = G/\langle \hat{\Omega} \rangle$ and $\mathrm{Ker}\,\Psi \supset \langle \hat{\Omega} \rangle$, the canonical homomorphism $F \to G/\mathrm{Ker}\,\Psi$ is surjective, and therefore the surjectivity of Θ' implies the surjectivity of Θ. By the definition of the relations $R_1, \dots, R_b$, the kernel of Θ contains the normal subgroup generated by $N = \langle \hat{R}_1, \dots, \hat{R}_b \rangle$ where $\hat{R}_j$ is the word of $\hat{a}_1, \dots, \hat{a}_{r-1}$ which is defined by replacing a_i by $\hat{a}_i$ in R_j. However by the Hopfian property of free groups (Theorem 2.13, [7]), $\Theta : F \to F(r-1)$ is an isomorphism. Thus this implies $R_1, \dots, R_b$ are empty and $\varphi_\sharp$ is an isomorphism. Q.E.D.

4.3. Appendix: Surjectivity of $\varphi_\sharp$

Consider $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$, $(x, y) \mapsto (u, v) = (f(x, y), g(x, y))$. The locus of the critical points of φ, denoted by $Cr(\varphi)$, is a curve of degree $(d-1)^2$ defined by

$$Cr(\varphi) = \left\{ \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 0 \right\}$$

and the critical value set (=bifurcating set) Σ_φ is a curve which is defined by the image of $Cr(\varphi)$ by φ. Now $\Sigma_\varphi \cap L(\vec{\tau})$ consists of a finite number of points. Choose a generic line $M : au + bv + c = 0$ in the base space so that $M \cap L(\vec{\tau}) \cap \Sigma_\varphi = \emptyset$ and M is transverse to $L(\vec{\tau})$. Choose a base point $\bar{b}_0 \in M - M \cap L(\vec{\tau})$ so that $\bar{b}_0 \notin \Sigma_\varphi$. We may assume that $\varphi(b_0) = \bar{b}_0$. Now consider the inverse image

$$F := \varphi^{-1}(M) = \{af(x, y) + bg(x, y) + c = 0\}.$$

Note that F is an irreducible smooth curve of degree d, provided a, b, c are generic enough. Then the restriction $\varphi : F \to M$ is a branched covering of degree d^2, branched at $M \cap \Sigma_\varphi$. Now choose a set of lassos $q_1, \dots, q_r$ for $L(\vec{\tau})$ in M so that they are generators of $\pi_1(M - M \cap L(\vec{\tau}))$ and these lassos do not pass through $\Sigma_\varphi \cap M$ and the image of $[q_j]$ in $\pi_1(M - M \cap \Sigma_\varphi)$ is trivial for any $j = 1, \dots, r$. Now consider the lift $\tilde{q}_j$ in F which starts from b_0. By the assumption, it is easy to see that $\tilde{q}_j$ is closed, and thus it defines an element of $\pi_1(F - F \cap C(\vec{\tau}))$. This implies that the homomorphism

$$\pi_1(F - F \cap C(\vec{\tau})) \to \pi_1(M - M \cap L(\vec{\tau}))$$

is surjective. The surjectivity of $\varphi_\sharp$ follows from the diagram of the canonical homomorphisms,

$$\begin{array}{ccc}
\pi_1(F - F \cap C(\vec{\tau})) & \to & \pi_1(M - M \cap L(\vec{\tau})) \\
\downarrow & & \downarrow \\
\pi_1(\mathbb{C}^2 - C(\vec{\tau})) & \xrightarrow{\varphi_\sharp} & \pi_1(\mathbb{C}^2 - L(\vec{\tau}))
\end{array}$$

as the right vertical homomorphism is surjective by van Kampen–Zariski principle.

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