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Complex and Kähler structures on compact homogeneous manifolds—their existence, classification and moduli problem

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Abstract.

We will survey basic results and recent progress in the existence, classification and moduli problems of complex and Kähler structures on compact homogeneous manifolds. We also state and discuss some related conjectures for further study in this field.

§1. Introduction

In the field of complex geometry one of the primary problems is whether given real manifolds admit certain complex geometrical structure such as complex structures, Kähler structures, or Stein structures. For the case of complex structures, for instance, we have a long standing problem of whether S^6 admits a complex structure. We can consider this problem in a more general setting: we extend the problem to the case of homogeneous manifolds of compact semi-simple Lie groups, including S^6 and $\mathbf{C}P^n$; and propose a closely related conjecture:

Conjecture 1. *A compact homogeneous manifold of compact semi-simple Lie group admits only homogeneous complex structures.*

We see, according to Wang's classification of compact simply connected homogeneous complex manifolds [26], that S^6 admits no homogeneous complex structures; and thus the above conjecture implies that S^6 admits no complex structures. We also know that the complex structure of $\mathbf{C}P^n$ is *rigid*, that is, admits no non-trivial small deformations; and $\mathbf{C}P^2$ admits no complex structures but the original one (which is homogeneous complex). More generally, we have a class of flag manifolds

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including $\mathbf{C}P^n$, which are simply connected, homogeneous Kählerian; and according to Borel [4], they are the only homogeneous manifolds of compact semi-simple Lie groups which admit Kähler structures. On the other hand, according to Samelson [23], compact semi-simple Lie groups of even dimension (being considered as homogeneous manifolds) admit homogeneous complex structures but no Kähler structures (since $b_2 = 0$). It seems that there are not yet known any counter-examples to the above conjecture.

For the case of compact homogeneous manifolds of dimension 4, we have a complete classification of those which admit complex structures (see Section 2 and [14]); in particular we showed that any complex structure on a 4-dimensional compact solvmanifold $\Gamma\backslash G$ (up to finite covering) is *left-invariant*, that is, induced from a left-invariant complex structure on a simply connected solvable Lie group G with lattice Γ. Furthermore, we recently showed (see Section 5 and [15]), based on Nakamura's results [20], that there exists a compact solvmanifold of dimension 6—actually a compact complex solvmanifold of complex dimension 3—which admit a continuous family of non-left-invariant complex structures as small deformations of the original complex structure. This result is important since it implies that there are "abundant" complex structures on compact solvmanifolds of higher dimension, while as mentioned above, it seems that compact homogeneous manifolds of compact semi-simple Lie groups admit only "restricted" complex structures. Concerning left-invariant complex structures on simply connected solvable Lie groups, we have a related conjecture:

Conjecture 2. *Any left-invariant complex structure on a simply connected unimodular solvable (nilpotent) Lie group of dimension $2n$ is Stein (biholomorphic to $\mathbf{C}^n$ respectively).*

It should be noted that an n-dimensional simply connected complex solvable Lie group is biholomorphic to $\mathbf{C}^n$ (cf. [20]), and the conjecture holds for $n = 2$ [21].

We have more decisive results for the classification problem of Kähler structures on compact homogeneous manifolds. For the case of compact solvmanifolds, we have the following result:

Theorem ([14], [15]). *A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus.*

We can express a class of compact Kählerian solvmanifolds in the theorem explicitly as those of the form $\Gamma\backslash G$, where G is a simply connected 2-step solvable Lie group with lattice Γ—they are exactly hyperelliptic surfaces for dimension 4 (see Section 2). For the case of reductive

Lie groups, the classification of compact homogeneous Kähler manifolds suggests the following conjecture to us:

Conjecture 3. *A compact homogeneous manifold of reductive Lie group admits a Kähler structure if and only if it is the product of a complex torus and a flag manifold.*

As mentioned before, a homogeneous manifold of compact semi-simple Lie group admits a Kähler structure if and only if it is a flag manifold. On the other hand, we know that $S^1 \times \Gamma\backslash\widetilde{\mathrm{SL}}_2(\mathbf{R})$, where $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ is the universal covering of $\mathrm{SL}_2(\mathbf{R})$ (which is a simply connected non-compact semi-simple Lie group), and Γ is a lattice of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$, admits a non-Kähler complex structure—defining an elliptic surface [27].

A pseudo-Kähler structure is a pseudo-Hermitian structure with its associated fundamental form ω being closed (see Section 2). It is known that a compact homogeneous pseudo-Kähler manifold is biholomorphic to the product of complex torus and a flag manifold [10]. There exists a compact homogeneous complex pseudo-Kähler solvmanifold which is not homogeneous pseudo-Kählerian, that is, the pseudo-Kähler form ω may not be invariant by the group action [29]. The classification of compact homogeneous complex pseudo-Kähler solvmanifolds is not yet known; at this moment, we have a complete classification for complex dimension 3 (see Section 4 and [15], [16]), and a structure theorem that it is a holomorphic fiber bundle over a complex torus with fiber a complex torus [30], which is, unless trivial, not a principal bundle. Recently Guan [12] has shown a fundamental theorem on cohomology groups of compact solvmanifolds, which could be applied to our problem.

§2. Complex and Kähler structures on compact homogeneous manifolds

A homogeneous manifold M is a differentiable manifold on which a real Lie group G acts transitively. M is a *homogeneous complex manifold*, if M is a complex manifold and the group action is holomorphic.

We will first make some important remarks:

(1) In the case where M is a compact homogeneous complex manifold, we can assume that G is a complex Lie group [26].

(2) A Lie group G, as a homogeneous manifold, admits a homogeneous complex structure J if and only if J is a left-invariant complex structure on G.

(3) A complex structure J on a Lie group G is both left and right-invariant if and only if G is a complex Lie group (w.r.t. J).

The complete classification of 2-dimensional compact homogeneous complex manifolds are known:

Theorem 1 (Tits [25]). *2-dimensional compact homogeneous complex manifolds are biholomorphic to one of the following:*

$$T^2, \mathbf{C}P^2, \mathbf{C}P^1 \times \mathbf{C}P^1, T^1 \times \mathbf{C}P^1, \text{Homogeneous Hopf surface},$$

where T^k denotes a k-dimensional complex torus.

For instance, we have

(1) $\mathbf{C}P^1 = B\backslash G = (B \cap H)\backslash H$, where $G = \mathrm{SL}_2(\mathbf{C})$, $H = \mathrm{SU}_2(\mathbf{C})$ and B is a Borel subgroup of G:

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}) \right\}.$$

(2) A homogeneous Hopf surface S is by definition W/Γ_γ, where $W = (\mathbf{C}^2 - \{O\})$ and Γ_γ is a group of automorphisms on W generated by the multiplication by $\gamma\,(\neq 1)$. S is diffeomorphic to $\mathrm{U}_2(\mathbf{C}) = \mathrm{SU}_2(\mathbf{C}) \rtimes S^1 \cong S^3 \times S^1$, and $S = B_\gamma\backslash G$, where B_γ is the subgroup of B with $\alpha = \gamma^k, \delta = \gamma^{-k}\,(k \in \mathbf{Z})$. Note that S has a structure of a holomorphic T^1-bundle over $\mathbf{C}P^1$.

M is a *homogeneous complex Kähler manifold*, if M is a homogeneous complex manifold which admits a Kähler structure. M is a *homogeneous Kähler manifold*, if M is a homogeneous complex manifold which admits a Kähler structure invariant by the group action. The following theorem (Theorem 2) is well known, which was first proved by Matsushima for homogeneous Kähler cases, and later by Borel–Remmert for homogeneous complex Kähler cases.

Theorem 2 (Matsushima [17], Borel–Remmert [9]). *A compact homogeneous complex Kähler manifold is biholomorphic to the product of a complex torus and a homogeneous rational manifold (which is a compact simply connected algebraic manifold).*

Let $M = \Gamma\backslash G$ be a compact homogeneous complex manifold, where G is a simply connected complex Lie group with discrete subgroup Γ. Then, Theorem 2 implies that M admits a Kähler structure if and only if M is a complex torus. In particular, the only compact homogeneous complex Kähler solvmanifold is a complex torus.

Let M be a symplectic manifold with symplectic form ω. If M admits a complex structure J such that $\omega(JX, JY) = \omega(X, Y)$ for any vector fields X, Y on M, we call (ω, J) a *pseudo-Kähler* structure on M. For a pseudo-Kähler structure (ω, J), we have a pseudo-Hermitian structure (g, J) defined by $g(X,Y) = \omega(X, JY)$. In other words, a pseudo-Kähler (Kähler) structure is a pseudo-Hermitian (Hermitian) structure (g, J) with its closed fundamental form ω, where ω is defined by $\omega(X,Y) = g(JX,Y)$ for any vector fields X, Y.

The following theorem (Theorem 3) asserts that Theorem 2 holds also for homogeneous pseudo-Kähler cases. However, as we will see in Section 4, it does not hold for compact homogeneous complex pseudo-Kähler manifolds.

Theorem 3 (Dorfmeister and Guan [10]). *A compact homogeneous pseudo-Kähler manifold is biholomorphic to the product of a complex torus and a homogeneous rational manifold.*

A compact solvmanifold M can be written as, up to finite covering,

$$M = \Gamma \backslash G,$$

where G is a simply connected real solvable Lie group and Γ is a lattice of G. A complex structure J on M is *left-invariant complex structure*, if it is deduced from a left-invariant complex structure on G.

A left-invariant complex structure J on G can be considered as a linear automorphism of $\mathfrak{g}$, that is $J \in \mathrm{GL}(\mathfrak{g}, \mathbf{R})$, such that $J^2 = -I$; and the integrability condition is satisfied:

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

vanishes for $X, Y \in \mathfrak{g}$.

We have a complete list of complex structures on compact solvmanifolds of dimension 4, all of which are left-invariant:

Theorem 4 ([14]). *A complex surface is diffeomorphic to a solvmanifold of dimension 4 if and only if it is one of the following surfaces: Complex torus, Hyperelliptic surface, Inoue Surface of type S^0, Primary Kodaira surface, Secondary Kodaira surface, Inoue Surface of type $S^{\pm}$. Furthermore, every complex structure on each of these complex surfaces (considered as solvmanifolds) is left-invariant.*

We can express each of these complex structures as a linear automorphism J of $\mathfrak{g}$. In the following list, for each surface the Lie algebra $\mathfrak{g}$ of G has a basis $\{X_1, X_2, X_3, X_4\}$ with only nonzero brackets specified.

Except for (6), the complex structure J is defined by

$$JX_1 = X_2, JX_2 = -X_1, JX_3 = X_4, JX_4 = -X_3.$$

(1) Complex Tori
$[X_i, X_j] = 0$ $(1 \le i < j \le 4)$;

(2) Hyperelliptic Surfaces
$[X_4, X_1] = -X_2, [X_4, X_2] = X_1$;

(3) Inoue Surfaces of Type S^0
$[X_4, X_1] = aX_1 - bX_2, [X_4, X_2] = bX_1 + aX_2, [X_4, X_3] = -2aX_3$, where $a, b\ (\neq 0) \in \mathbf{R}$;

(4) Primary Kodaira Surfaces
$[X_1, X_2] = -X_3$;

(5) Secondary Kodaira Surfaces
$[X_1, X_2] = -X_3, [X_4, X_1] = -X_2, [X_4, X_2] = X_1$;

(6) Inoue Surfaces of Type S^+ and S^-
$[X_2, X_3] = -X_1, [X_4, X_2] = X_2, [X_4, X_3] = -X_3$, and,

$$JX_1 = X_2, JX_2 = -X_1, JX_3 = X_4 - qX_2, JX_4 = -X_3 - qX_1.$$

Example 1 (Hyperelliptic Surfaces). Let $G =: (\mathbf{C} \times \mathbf{R}) \rtimes \mathbf{R}$, where the action $\phi : \mathbf{R} \to \mathrm{Aut}(\mathbf{C} \times \mathbf{R})$ is defined by

$$\phi(t)((z, s)) = (e^{\sqrt{-1}\eta t} z, s),$$

where $\eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi$ or $\frac{1}{3}\pi$.

Since the action on the second factor $\mathbf{R}$ is trivial, the multiplication of G is defined on $\mathbf{C}^2$ as follows:

$$(w_1, w_2) \cdot (z_1, z_2) = (w_1 + e^{\sqrt{-1}\eta t} z_1, w_2 + z_2),$$

where $t = \mathrm{Re}\, w_2$.

We can see that there exist seven isomorphism classes of lattices Γ of G, which correspond to seven classes of hyperelliptic surfaces.

Example 2 (Primary Kodaira Surfaces). Let $G = N \times \mathbf{R}$ be the nilpotent Lie group, where

$$N = \left\{ \begin{pmatrix} 1 & x & s \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, s \in \mathbf{R} \right\},$$

which has the lattice Γ_n with $s = \dfrac{z}{n}, x, y, z \in \mathbf{Z}$.

Taking the coordinate change Φ from $N \times \mathbf{R}$ to $\mathbf{R}^4$:

$$\Phi : ((x, y, s), t) \longrightarrow (x, y, 2s - xy, 2t + \frac{1}{2}(x^2 + y^2)),$$

and regarding $\mathbf{R}^4$ as $\mathbf{C}^2$, the group operation on G can be expressed as

$$(w_1, w_2) \cdot (z_1, z_2) = (w_1 + z_1, w_2 - \sqrt{-1}\bar{w}_1 z_1 + z_2).$$

Let M be a solvmanifold of the form $\Gamma\backslash G$. M is *of completely solvable type*, if the adjoint representation of $\mathfrak{g}$ has only real eigenvalues. M is *of rigid type*, if the adjoint representation of $\mathfrak{g}$ has only pure imaginary (including 0) eigenvalues. We note that

(1) It is clear that M is both of completely solvable and of rigid type if and only if $\mathfrak{g}$ is nilpotent, that is, M is a nilmanifold.

(2) A hyperelliptic surface can be characterized as a solvmanifold of dimension 4 of rigid type which admits a Kähler structure [14].

Example 3. Let $G =: \mathbf{C}^l \rtimes \mathbf{R}^{2k}$, where the action $\phi : \mathbf{R}^{2k} \rightarrow \mathrm{Aut}(\mathbf{C}^l)$ is defined by

$$\phi(\bar{t}_i)((z_1, z_2, \ldots, z_l)) = (e^{\sqrt{-1}\,\eta_1^i\, t_i} z_1, e^{\sqrt{-1}\,\eta_2^i\, t_i} z_2, \ldots, e^{\sqrt{-1}\,\eta_l^i\, t_i} z_l),$$

where $\bar{t}_i = t_i e_i$ (e_i: the i-th unit vector in $\mathbf{R}^{2k}$), and $e^{\sqrt{-1}\,\eta_j^i}$ is the s_i-th root of unity, $i = 1, \ldots, 2k, j = 1, \ldots, l$.

If an abelian lattice $\mathbf{Z}^{2l}$ of $\mathbf{C}^l$ is preserved by the action ϕ on $\mathbf{Z}^{2k}$, then $M = \Gamma\backslash G$ defines a solvmanifold of rigid type, where $\Gamma = \mathbf{Z}^{2l} \rtimes \mathbf{Z}^{2k}$ is a lattice of G.

The Lie algebra $\mathfrak{g}$ of G is the following:

$$\mathfrak{g} = \{X_1, X_2, \ldots, X_{2l}, X_{2l+1}, \ldots, X_{2l+2k}\}_{\mathbf{R}},$$

where the bracket multiplications are defined by

$$[X_{2l+2i}, X_{2j-1}] = -X_{2j}, [X_{2l+2i}, X_{2j}] = X_{2j-1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$, and all other brackets vanish.

The canonical left-invariant complex structure is defined by

$$JX_{2j-1} = X_{2j}, JX_{2j} = -X_{2j-1},$$

$$JX_{2l+2i-1} = X_{2l+2i}, JX_{2l+2i} = -X_{2l+2i-1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$.

Example 4. Let $G =: \mathbf{C}^l \rtimes \mathbf{R}^{2k}$, where the action $\phi : \mathbf{R}^{2k} \to \mathrm{Aut}(\mathbf{C}^l)$ is defined by

$$\phi(\bar{t}_i)((z_1, z_2, \ldots, z_l)) = (e^{2\pi\sqrt{-1}\,t_i} z_1, e^{2\pi\sqrt{-1}\,t_i} z_2, \ldots, e^{2\pi\sqrt{-1}\,t_i} z_l),$$

where $\bar{t}_i = t_i e_i$ (e_i: the i-the unit vector in $\mathbf{R}^{2k}$), $i = 1, \ldots, 2k$. Then, $\mathbf{Z}^{2n} \backslash G$ is a solvmanifold diffeomorphic to a torus T^{2n} $(n = k + l)$.

A compact solvmanifold M in Example 3 is a finite quotient of a complex torus and has a structure of a complex torus bundle over a complex torus, admitting a canonical Kähler structure. We could have shown the converse that if a compact solvmanifold admits a Kähler structure, then it must be of this type:

Theorem 5 ([14], [15]). *A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus.*

We note that

(1) Since Kählerian solvmanifolds (as defined in Example 3) are of rigid type, it follows that a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus. This is the so-called Benson–Gordon conjecture ([6]).

(2) We know [5], [13] that a compact nilmanifold admits a Kähler structure if and only if it is a complex torus; and this result holds also for bimeromorphic Kähler structures [13]. We see that Theorem 5 also holds for bimeromorphic Kähler structures, since the proof is based on this result and a result of Arapura and Nori [1] that a polycyclic Kähler group must be almost nilpotent.

(3) As noted in the paper [15], the Benson–Gordon conjecture (stated in (1)) can be proved directly from the above results on Kählerian nilmanifolds and polycyclic Kähler groups, together with a result of Auslander [2] that for a compact solvmanifold $\Gamma \backslash G$, the Lie algebra $\mathfrak{g}$ of G is of rigid type if and only if Γ is almost nilpotent (where G is a simply connected solvable Lie group with discrete subgroup Γ): If $M = \Gamma \backslash G$ admits a Kähler structure and $\mathfrak{g}$ is of completely solvable type, then Γ is almost nilpotent. Hence $\mathfrak{g}$ is both of rigid type and of completely solvable type; and thus $\mathfrak{g}$ is nilpotent. Therefore, M is a compact Kählerian nilmanifold, that is, a complex torus. There is a recent paper (by Baues and Cortés [3]) discussing the Benson–Gordon

conjecture and other relevant topics from more topological point of view.

Concerning Kähler structures on a compact homogeneous manifold of compact semi-simple Lie group, we have a fundamental theorem:

Theorem 6 (Borel [4], Goto [11]). *Let G be a compact real semi-simple Lie group and D is a closed subgroup which is the centralizer of a toral subgroup of G. Then, $M = D\backslash G$ (of even-dimension) admits a homogeneous Kähler structure, which is a simply connected and projective algebraic manifold. Conversely, if a compact homogeneous manifold of compact semi-simple Lie group admits a Kähler structure, it must be of the above form, admitting a homogeneous Kähler structure.*

We note that

(1) $M = D\backslash G$ has a homogeneous complex structure $P\backslash G_{\mathbf{C}}$, where $G_{\mathbf{C}}$ is the complexification of G, and P is a parabolic subgroup of $G_{\mathbf{C}}$ which contains a Borel subgroup B of $G_{\mathbf{C}}$.

(2) It is known (Samelson [23], Wang [26]) that any even-dimensional compact semi-simple Lie group admits a homogeneous complex structure but no Kähler structures.

(3) It is known (Burstall et al. [8]) that if a compact inner Riemannian symmetric manifold admits a Hermitian structure (which is compatible with the given metric), then it is Hermitian symmetric. In particular, S^6 (considered as a compact inner Riemannian symmetric manifold) admits no complex structures compatible with the given metric.

§3. The classification of 3-dimensional compact complex solvmanifolds

Let M be a 3-dimensional compact complex solvmanifold. Then, M can be written as $\Gamma\backslash G$, where Γ is a lattice of a simply connected unimodular complex solvable Lie group G (cf. [7]).

The Lie algebra $\mathfrak{g}$ of G is unimodular (i.e. the trace of $\mathrm{ad}(X) = 0$ for every X of $\mathfrak{g}$), which is one of the following types:

(1) Abelian Type: $[X, Y] = [Y, Z] = [X, Z] = 0$.

(2) Nilpotent Type: $[X, Y] = Z,\ [X, Z] = [Y, Z] = 0$.

(3) Non-Nilpotent Type: $[X, Y] = -Y,\ [X, Z] = Z,\ [Y, Z] = 0$.

(1) ***Abelian Type***: $G = \mathbf{C}^3$

A lattice Γ of G is generated by a basis of $\mathbf{C}^3$ as a vector space over $\mathbf{R}$.

(2) ***Nilpotent Type***: $G = \mathbf{C}^2 \rtimes \mathbf{C}$ with the action ϕ defined by

$$\phi(x)(y,z) = (y, z + xy),$$

or in the matrix form,

$$G = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbf{C} \right\}.$$

A lattice Γ of G can be written as

$$\Gamma = \Delta \rtimes \Lambda,$$

where Δ is a lattice of $\mathbf{C}^2$ and Λ is a lattice of $\mathbf{C}$.

Since an automorphism $f \in \mathrm{Aut}(\mathbf{C})$ defined by $f(x) = \alpha x, \alpha \neq 0$ can be extended to an automorphism $F \in \mathrm{Aut}(G)$ defined by $F(x,y,z) = (\alpha x, \alpha^{-1} y, z)$, we can assume that Λ is generated by 1 and λ ($\lambda \notin \mathbf{R}$) over $\mathbf{Z}$.

Since Δ is preserved by $\phi(1)$ and $\phi(\lambda)$, we see that Δ is generated by $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (0, \alpha_1), (0, \alpha_2)$ over $\mathbf{Z}$, where β_1 and β_2 are arbitrary complex numbers, and α_1 and α_2 are linearly independent over $\mathbf{R}$ such that (α_1, α_2) is an eigenvector of some $A \in \mathrm{GL}(2, \mathbf{Z})$ with the eigenvalue λ.

Conversely, for any $A \in \mathrm{GL}(2, \mathbf{Z})$ with non-real eigenvalue λ, we can define a lattice Γ of G.

(3) ***Non-Nilpotent Type***: $G = \mathbf{C}^2 \rtimes \mathbf{C}$ with the action ϕ defined by

$$\phi(x)(y,z) = (e^x y, e^{-x} z),$$

or in the matrix form,

$$G = \left\{ \left(\begin{array}{cccc} e^x & 0 & 0 & y \\ 0 & e^{-x} & 0 & z \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbf{C} \right\}.$$

A lattice Γ of G can be written as $\Gamma = \Delta \rtimes \Lambda$, where Δ is a lattice of $\mathbf{C}^2$, and Λ is a lattice of $\mathbf{C}$ generated by λ and μ over $\mathbf{Z}$.

Since Δ is preserved by $\phi(\lambda)$ and $\phi(\mu)$, we see that Δ is generated by $(\alpha_i, \beta_i), i = 1, 2, 3, 4$ over $\mathbf{Z}$ such that

$$\gamma^{-1}\alpha_i = \textstyle\sum_{j=1}^4 a_{ij}\alpha_j,\ \gamma\beta_i = \sum_{j=1}^4 a_{ij}\beta_j,$$

$$\delta^{-1}\alpha_i = \textstyle\sum_{j=1}^4 b_{ij}\alpha_j,\ \delta\beta_i = \sum_{j=1}^4 a_{ij}\beta_j,$$

where $\gamma = e^\lambda, \delta = e^\mu$, and $A = (a_{ij}), B = (b_{ij}) \in \mathrm{SL}_4(\mathbf{Z})$ are semi-simple and mutually commutative. In other words, we have simultaneous eigenvectors $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbf{C}^4$ of A and B with eigenvalues γ^{-1}, γ and δ^{-1}, δ respectively.

Conversely, for any mutually commutative, semi-simple matrices $A, B \in \mathrm{SL}(4, \mathbf{Z})$ with eigenvalues γ^{-1}, γ and δ^{-1}, δ respectively, take simultaneous eigenvectors $\alpha, \beta \in \mathbf{C}^4$ of A and B. Then, $(\alpha_i, \beta_i), i = 1, 2, 3, 4$ are linearly independent over $\mathbf{R}$, defining a lattice of Δ preserved by $\phi(\lambda)$ and $\phi(\mu)$ ($\lambda = \log\gamma, \mu = \log\delta$).

Since λ and μ are linearly independent over $\mathbf{R}$, we have either $|\gamma| \neq 1$ or $|\delta| \neq 1$. And if, for instance, $|\gamma| \neq 1$ and $\gamma \notin \mathbf{R}$, then A has four distinct eigenvalues $\gamma^{-1}, \gamma, \overline{\gamma}^{-1}, \overline{\gamma}$.

For the case where both A and B have real eigenvalues γ^{-1}, γ and δ^{-1}, δ respectively, take simultaneous non-real eigenvectors $\alpha, \beta \in \mathbf{C}^4$ for them; then we see that $(\alpha_i, \beta_i), i = 1, 2, 3, 4$ are linearly independent over $\mathbf{R}$, defining a lattice Δ of $\mathbf{C}^2$ preserved by $\phi(\lambda)$ and $\phi(\mu)$.

Example 5. The Iwasawa manifold is obtained by putting $\lambda = \sqrt{-1}, \alpha_1 = \alpha_2 = 0, \beta_1 = 1, \beta_2 = \sqrt{-1}$.

Example 6. Take $A \in \mathrm{SL}(4, \mathbf{Z})$ with four non-real eigenvalues $\gamma, \gamma^{-1}, \overline{\gamma}, \overline{\gamma}^{-1}$; for instance,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -3 & 1 \end{pmatrix},$$

with the characteristic polynomial given by

$$\det(tI - A) = t^4 - t^3 + 3t^2 - t + 1.$$

For the lattice Λ of $\mathbf{C}$ generated by λ ($\lambda = \log\gamma$) and $\mu = k\pi\sqrt{-1}$ ($k \in \mathbf{Z}$), and the lattice Δ of $\mathbf{C}^2$ generated by $(\alpha_i, \beta_i), i = 1, 2, 3, 4$, we can define a lattice $\Gamma = \Delta \rtimes \Lambda$ of G, where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbf{C}^4$ are eigenvectors of A with eigenvalue γ, γ^{-1}.

Example 7 (Nakamura [20]). Take $A \in \mathrm{SL}_2(\mathbf{Z})$ with two real eigenvalues γ^{-1}, γ, $\gamma \neq \pm 1$, and their real eigenvectors $(a_1, a_2), (b_1, b_2) \in \mathbf{R}^2$. Then, for any $\epsilon \notin \mathbf{R}$ (e.g. $\epsilon = \sqrt{-1}$), $(a_1, a_2, a_1\epsilon, a_2\epsilon)$ and $(b_1, b_2, b_1\epsilon, b_2\epsilon)$ are non-real eigenvectors for $A \oplus A \in \mathrm{SL}_4(\mathbf{Z})$ with eigenvalues γ^{-1}, γ.

For the lattice Λ of $\mathbf{C}$ generated by λ ($\lambda = \log \gamma$) and $\mu = k\pi\sqrt{-1}$ ($k \in \mathbf{Z}$), and the lattice Δ of $\mathbf{C}^2$ generated by $(a_1, b_1), (a_2, b_2), (a_1\epsilon, b_1\epsilon)$, $(a_2\epsilon, b_2\epsilon)$, we can define a lattice $\Gamma = \Delta \rtimes \Lambda$ of G.

Theorem 7 (Winkelmann [28]). *Let G be a simply connected complex solvable linear algebraic group with lattice Γ. Then, we have*

$$\dim H^1(\Gamma \backslash G, \mathcal{O}) = \dim H^1(\mathfrak{g}, \mathbf{C}) + \dim W,$$

where $\mathcal{O}$ denotes the structure sheaf of M, $\mathfrak{n}$ the nilradical of $\mathfrak{g}$, and W the maximal linear subspace of $[\mathfrak{g}, \mathfrak{g}]/[\mathfrak{n}, \mathfrak{n}]$ for which $\mathrm{Ad}(\xi)$ on W is a real semi-simple linear endomorphism for any $\xi \in \Gamma$.

We have $\dim H^1(\mathfrak{g}, \mathbf{C}) = \dim \mathfrak{g} - \dim [\mathfrak{g}, \mathfrak{g}]$, and $\mathrm{Ad}(\xi)|W$ is diagonalizable over $\mathbf{R}$.

Applying the Winkelmann's formula above and our classification of 3-dimensional compact complex solvmanifolds, we can determine $h^1(M) = \dim H^1(M, \mathcal{O})$ completely:

(1) Abelian Type: $\dim W = 0$, $h^1 = 3$;

(2) Nilpotent Type: $\dim W = 0$, $h^1 = 2$;

(3a) Non-Nilpotent Type with either γ or $\delta \notin \mathbf{R}$: $\dim W = 0$, $h^1 = 1$.

(3b) Non-Nilpotent Type with $\gamma, \delta \in \mathbf{R}$: $\dim W = 2$, $h^1 = 3$;

Example 8. We see that Example 6 is of type (3a), and Example 7 is of type (3b).

§4. Pseudo-Kähler structures on a 3-dimensional compact complex solvmanifold

We can see from Theorem 3 that a compact solvmanifold admits a homogeneous pseudo-Kähler if and only if it is a complex torus. Yamada gave the first example of homogeneous complex pseudo-Kähler non-toral solvmanifold; and showed the following fundamental result:

Theorem 8 (Yamada [29], [30]). *Let M be an n-dimensional compact complex solvmanifold which admits a pseudo-Kähler structure. Then, we have $h^1(M) \geq n$; and M has a structure of a complex torus bundle over a complex torus.*

We remark that Winkelmann's formula implies that if we have $h^1 \geq n$ then $[\mathfrak{n}, \mathfrak{n}] = 0$; and thus the Mostow fibration gives a structure of a complex torus bundle over a complex torus.

Theorem 9 ([16]). *A 3-dimensional compact complex solvmanifold M admits a pseudo-Kähler structure if and only if it is of abelian type, or of non-nilpotent type with $\gamma, \delta \in \mathbf{R}$.*

Proof (Sketch). If M is of type (2) or (3a), then M admits no pseudo-Kähler structures. Therefore, it suffices to show that M of type (3b) admits a pseudo-Kähler structure.

We have $\gamma, \delta \in \mathbf{R}$ if and only if Λ is generated by $\lambda = a+k\pi\sqrt{-1}, \mu = b + l\pi\sqrt{-1}$, where $a, b \in \mathbf{R}$ and $k, l \in \mathbf{Z}$.

We can construct a pseudo-Kähler structure ω on $\Gamma\backslash G$ in the following:

$$\omega = \sqrt{-1}dx \wedge d\overline{x} + dy \wedge d\overline{z} + d\overline{y} \wedge dz,$$

or using Maure–Cartan forms, $\omega_1, \omega_2, \omega_3$,on G,

$$\omega = \sqrt{-1}\omega_1 \wedge \overline{\omega_1} + e^{-2\,\mathrm{Im}(x)\sqrt{-1}}\,\omega_2 \wedge \overline{\omega_3} + e^{2\,\mathrm{Im}(x)\sqrt{-1}}\,\overline{\omega_2} \wedge \omega_3,$$

where $\omega_1 = dx, \omega_2 = e^x\, dy, \omega_3 = e^{-x}\, dz$. Q.E.D.

Concerning pseudo-Kähler structures on compact complex nilmanifolds, we have

Theorem 10 (Kodaira [20]). *Let M be an n-dimensional compact complex nilmanifold, and denote by r the number of linearly independent closed holomorphic 1-forms on M. Then, we have $h^1(M) = r$, and $r = n$ holds if and only if M is a complex torus.*

In particular, applying Theorem 8, we see that a non-toral compact complex nilmanifold admits no pseudo-Kähler structures.

We have the following result on holomorphic principal fiber bundles over a complex torus with fiber a complex torus:

Theorem 11 (Murakami [19]). *A holomorphic principal fiber bundle over a complex torus with fiber a complex torus is a compact 2-step nilmanifold with a left-invariant complex structure: and it has a holomorphic connection if and only if it is a compact complex nilmanifold.*

We see in particular that a holomorphic principal bundle over a complex torus with fiber a complex torus admits no pseudo-Kähler structures.

§5. Small deformations and non-left invariant complex structures on a compact complex solvmanifold

Let G be a connected simply connected Lie group of dimension $2m$, and $\mathfrak{g}$ the Lie algebra of G.

Lemma 1. *An almost complex structure J on $\mathfrak{g}$ is integrable if and only if the subspace $\mathfrak{h}_J$ of $\mathfrak{g}_{\mathbf{C}}$ generated by $X+\sqrt{-1}JX (X \in \mathfrak{g})$ is a complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ such that $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_J \oplus \overline{\mathfrak{h}_J}$.*

Lemma 2. *Let V be a real vector space of dimension $2m$. Then, for a complex subspace W of $V \otimes \mathbf{C}$ such that $V \otimes \mathbf{C} = W \oplus \overline{W}$, there exists a unique $J_W \in \mathrm{GL}(V, \mathbf{R})$, ${J_W}^2 = -I$ such that $W = \{X + \sqrt{-1}J_W X | X \in V\}_{\mathbf{C}}$.*

There exists one to one correspondence between complex structures J on $\mathfrak{g}$ and complex Lie subalgebras $\mathfrak{h}$ such that $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h} \oplus \overline{\mathfrak{h}}$, given by $J \to \mathfrak{h}_J$ and $\mathfrak{h} \to J_{\mathfrak{h}}$.

For a complex structure J, the complex Lie subgroup H_J of $G_{\mathbf{C}}$ corresponding to $\mathfrak{h}_J$ is closed, simply connected, and $H_J \backslash G_{\mathbf{C}}$ is biholomorphic to $\mathbf{C}^m$. The canonical inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ induces an inclusion $G \hookrightarrow G_{\mathbf{C}}$, and $\Gamma = G \cap H_J$ is a discrete subgroup of G. We have the following canonical map $g = i \circ \pi$:

$$G \xrightarrow{\pi} \Gamma \backslash G \overset{i}{\hookrightarrow} H_J \backslash G_{\mathbf{C}},$$

where π is a covering map, and i is an inclusion. The left-invariant complex structure J on G is the one induced by g from an open set $U = \mathrm{Im}\, g \subset \mathbf{C}^m$. For the details of the above argument we refer to the paper [24].

Let G be a 3-dimensional complex solvable Lie group of non-nilpotent type, and $\mathfrak{g}$ its Lie algebra. Recall that $\mathfrak{g}$ has a basis X, Y, Z over $\mathbf{C}$ with bracket multiplication defined by

$$[X,Y] = -Y,\ [X,Z] = Z,\ [Y,Z] = 0.$$

Let $\mathfrak{g}_{\mathbf{R}}$ denote the real Lie algebra underlying $\mathfrak{g}$, and $\mathfrak{g}_{\mathbf{C}}$ the complexification of $\mathfrak{g}_{\mathbf{R}}$, that is, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}_{\mathbf{R}} \oplus \sqrt{-1}\mathfrak{g}_{\mathbf{R}}$.

Let J_0 be the original complex structure with its associated complex subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_{\mathbf{C}}$ such that $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_0 \oplus \overline{\mathfrak{h}_0}$, and H_0 the complex subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{h}_0$.

Lemma 3 ([16]). *For any complex structure J on G with its associated complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbf{C}}$ such that $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h} \oplus \overline{\mathfrak{h}}$, there exists a complex automorphism of Lie algebras $\Phi : \mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ such that $\Phi \circ \tau_0 = \tau \circ \Phi$ and $\Phi(\mathfrak{h}_0) = \mathfrak{h}$, where τ_0 and τ are the conjugations with respect to J_0 and J respectively.*

As a consequence we have

Theorem 12 ([16]). *Let G be a 3-dimensional simply connected complex solvable Lie group of non-nilpotent type. Then, any left-invariant complex structure on G is biholomorphic to $\mathbf{C}^3$.*

Proof. The complex automorphism of Lie algebras Φ induces a complex automorphism of Lie group $\Psi : G_{\mathbf{C}} \to G_{\mathbf{C}}$ such that $q \circ \Psi = \tilde{\Psi} \circ q_0$, which send H_0 to H biholomorphically;

$$\begin{array}{ccccc} (G, J_0) & \stackrel{i}{\hookrightarrow} & G_{\mathbf{C}} & \stackrel{q_0}{\to} & H_0\backslash G_{\mathbf{C}} \\ & & \Psi\downarrow & & \tilde{\Psi}\downarrow \\ (G, J) & \stackrel{i}{\hookrightarrow} & G_{\mathbf{C}} & \stackrel{q}{\to} & H\backslash G_{\mathbf{C}} \end{array}$$

Here, we have $\Gamma = G \cap H_0 = \{0\}$, and $g_0 = q_0 \circ i$ is a biholomorphic map to $H_0\backslash G_{\mathbf{C}} = \mathbf{C}^3$. Q.E.D.

We can also see that all left-invariant complex structures on a 3-dimensional simply connected complex solvable Lie group are biholomorphic to $\mathbf{C}^3$.

Nakamura constructed small deformations of 3-dimensional compact complex solvmanifolds; and showed in particular that there exists a continuous family of complex structures on those of type (3b) whose universal coverings are not Stein (as noted in the paper, this construction is actually due to Kodaira).

Theorem 13 ([16]). *There exists a continuous family of non-left-invariant complex structures on a 3-dimensional compact complex solvmanifold of non-nilpotent type with $h^1 = 3$.*

We note that small deformations of a 3-dimensional compact complex nilmanifold (Iwasawa manifold) are all left-invariant (due to Salamon [22]). We conjecture that this also holds for higher dimension. Recently, there appears a preprint (by McLaughlin et al. [18]) which proves the conjecture for a more general class of left-invariant complex structures on compact nilmanifolds.

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