Advanced Studies in Pure Mathematics 56, 2009
Singularities — Niigata–Toyama 2007
pp. 133–149

On Nash blow-up of orbifolds

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Abstract.

A short survey on the Nash blow-up of singular varieties, applications and examples in particular for orbifolds, followed by some new results for threefolds.

§1. Introduction

The modification of an algebraic or analytic singular variety, in which every singular point is replaced by the limiting positions of the tangent spaces to nearby smooth points, is called the Nash blow-up (or modification, or transformation).

More precisely, let S be a reduced complex variety of pure dimension d, and Ω^1_S the sheaf of differentials of S. Denote by $\mathbb{G}^d$ the grassmannian $\mathrm{Grass}^d(\Omega^1_S)$ of rank d locally free quotient sheaves of Ω^1_S, and let $\nu : \mathbb{G}^d \to S$ be the canonical morphism from $\mathbb{G}^d$ to S.

Since on the open subset S_{reg} of smooth points of S the restriction of the sheaf Ω^1_S is locally free of rank d, then there is a section $\varphi : S_{reg} \to \mathbb{G}^d$ which is an isomorphism onto its image.

The Nash blow-up $\widetilde{S}$ of S is defined as the scheme closure of $\varphi(S_{reg})$ in $\mathbb{G}^d$. The restriction of ν to $\widetilde{S}$, which will also be denoted by ν, is a proper (and projective) birational morphism, and is an isomorphism on $\varphi(S_{reg})$. The variety $\widetilde{S}$ is canonically equipped with a vector bundle, the quotient sheaf of $\nu^*(\Omega^1_S)$ induced by the universal bundle of $\mathbb{G}^d$, which

Received April 11, 2008.
Revised September 26, 2008.
2000 *Mathematics Subject Classification.* 14J17, 32045, 14L30, 14M25.
Key words and phrases. Nash blow-up, orbifolds, rational singularities, toric varieties.
The author is pleased to acknowledge the organizers, in particular Tatsuo Suwa and Shihoko Ishii, for the invitation to the Fourth Franco–Japanese Symposium and workshop on Singularities.

is called the Nash cotangent bundle.

$$\begin{array}{ccc} \mathbb{G}^d & \hookleftarrow & \widetilde{S} \\ {\scriptstyle\varphi}\uparrow & \searrow & \downarrow{\scriptstyle\nu} \\ S_{reg} & \hookrightarrow & S \end{array}$$

By construction the Nash blow-up of S has the universal property of being minimal among the proper birational models of S equipped with a locally free quotient of maximal rank of Ω^1_S.

Dually, this is also a universal construction that extends the tangent bundle of S_{reg}. If S is an affine variety embedded in $\mathbb{C}^N$, and $\mathrm{Grass}_d(\mathbb{C}^N)$ denotes now the grassmannian of d-dimensional subspaces of $\mathbb{C}^N$, then there is a section σ of $S \times \mathrm{Grass}_d(\mathbb{C}^N)$ over S_{reg} sending each point $p \in S_{reg}$ to the couple $(p, T_p(S))$, where $T_p(S)$ is the linear subspace associated to the tangent space of S at p. The Nash blow-up is the closure of $\sigma(S_{reg})$ in $S \times \mathrm{Grass}_d(\mathbb{C}^N)$, equipped with the projection on the first factor.

An equivalent way to define the Nash blow-up is to consider the sheaf of d-differentials $\Omega^d_S = \wedge^d(\Omega^1_S)$, which is free of rank one over S_{reg}, and the associated projective bundle $\mathbb{P}(\Omega^d_S)$. Then the canonical morphism on S is an isomorphism over S_{reg}. The Nash blow-up is isomorphic to the closure of the inverse image of S_{reg} in $\mathbb{P}(\Omega^d_S)$, since the image of $\mathbb{G}^d$ by the Plücker morphism is closed in $\mathbb{P}(\Omega^d_S)$. There is an equivalent construction in the dual presentation.

The Nash blow-up is a global and intrinsic modification , since it is built from the sheaf of differentials Ω^1_S of the variety S. The scheme structure of the limiting positions of the tangent spaces of S at regular points gives a deep information on the geometry of the singularities of S.

In the following section we review several methods to compute and describe the Nash blow-up or its normalization. Some omitted complete proofs may be found in the references and others will appear elsewhere. A brief survey on some properties and applications is sketched in Section 3, such as a comparison with point blowing-up, the desingularization problem, the local Euler invariant. In Section 4 we give examples and results on the Nash blow-up for surfaces, in particular for orbifold singularities, and we describe some new results and examples in dimension three for toric orbifold singularities. Some non exhaustive references on the subject are given at the end.

§2. Computing the Nash blow-up

The computation of this transformation, though a very natural one, is not particularly easy in general.

The following result proves that locally it may be obtained as the blowing-up of an ideal.

Theorem 1. *Let S be an integral affine variety defined by a prime ideal $I \subset \mathbb{C}[X_1, \cdots, X_N]$. Let $f_1, f_2, \cdots, f_c \in I$ be such that $df_1 \wedge df_2 \wedge \cdots \wedge df_c \neq 0$ generically, with $c = N - d$, the codimension of S. Then the Nash blow-up $\widetilde{S}$ of S is obtained by the blowing-up with center the ideal generated by the images in $\mathbb{C}[X_1, \cdots, X_N]/I$ of the minors of order c of the jacobian matrix of $f_1, \cdots, f_c$.*

Note that it is not required that the f_i, $i = 1, \cdots, c$ generate I (see [G1]). For a slightly more general statement where S is only supposed to be reduced and equidimensional see [N].

The base field may be replaced by any characteristic zero field K, considering the Kähler differentials for S.

Note that the subscheme of S whose ideal is the center of the blowing-up giving the Nash blow-up of S may be strictly bigger than the singular locus of S.

An immediate consequence of this theorem is the following.

Corollary 1. *If S is a complete intersection, then the Nash blow-up is obtained by blowing up the jacobian ideal. In particular then this is true for hypersurfaces.*

If S is not a complete intersection, then we may take an embedding of S in a complete intersection W. Then the Nash blow-up of S is the strict transform of S in the Nash blow-up of W.

The computation of the blowing-up of the jacobian ideal, or the ideal given in the preceding theorem may not be easy.

In the case of a surface S we have another method to obtain the Nash blow-up by blowing-up closed points. Recall that a morphism $\pi : X \longrightarrow S$ is called a desingularization or resolution of singularities of S if π is proper and birational and X is smooth. Usually π is also required to be an isomorphism over the regular open subset S_{reg} of S.

Theorem 2. *Let $\pi : X \longrightarrow S$ be a desingularization of a surface S. Then there is a smooth surface $\widehat{X}$ obtained by a finite sequence of closed point blowing-ups of X giving a desingularization $\widehat{\pi} : \widehat{X} \longrightarrow \widetilde{S}$ of the Nash blow-up $\widetilde{S}$ of S.*

This follows from the resolution of indeterminacies for rational maps of surfaces. For the computation we may apply a relative construction to $\pi^*(\Omega^1_S))$ over X to obtain a locally free rank two quotient sheaf and a birational morphism $\bar{\nu} : \bar{X} \longrightarrow X$. By the universal property of the Nash blow-up the morphism $\pi \circ \bar{\nu}$ factorizes through $\widetilde{S}$. If $\bar{X}$ is singular, let $\widehat{X}$ be a desingularization of $\bar{X}$ and $\widehat{\nu} : \widehat{X} \longrightarrow X$ the induced birational morphism. The two surfaces are smooth, so $\widehat{\nu}$ is a finite composition of closed point blowing-ups and $\widehat{X}$ is a desingularization of $\widetilde{S}$. (see [G5]).

This method gives also precise information on the singularities of the Nash blow-up $\widetilde{S}$ or its normalization, and it may be iterated to obtain data on the following Nash blow-ups.

Usually the practical use of this method is in the cases where one may describe the morphism π from the data of the smooth surface X and the exceptional divisor D of π. In some cases the data of the weighted dual graph associated to D is enough, but in the general case this data is not sufficient. By considering π as a morphism contracting the divisor D, then the uniqueness of such a contraction morphism is assured only if the contracted surface S is normal. This is the reason why the normalized Nash blow-up is the natural object to consider and for which we may obtain effective results with this point of view.

It is well adapted in particular for rational singularities, but it may be applied to more general surface singularities. We will give some examples for orbifold singularities.

Another method that may be useful to compute the Nash blow-up for higher dimensional singularities, is the adaptation of the first one for toric singularities, as follows.

Let $S = S_\sigma$ be a toric variety of dimension d associated to a cone σ in $N_{\mathbb{R}}$, with the usual notations (see [TE]).

Theorem 3. *The Nash blow-up of a toric variety S_σ is obtained by the blowing-up with center the ideal in $\mathbb{C}[\sigma^\vee \cap M]$ generated by the elements $\chi^{\sum_1^d e_i}$ such that the $e_i \in \sigma^\vee \cap M$ are $\mathbb{R}$-linearly independent.*

For the proof see [G1]. The description as a toric variety of this blow-up in terms of a fan gives the *normalized* Nash blow-up, since the semigroups considered here are saturated. The fan is computed as the polar fan associated to the convex hull of the integer points inducing the above ideal.

We give in the last section several results on Nash blow-up of toric singularities obtained with the last two methods.

§3. Some properties and applications

Given an isolated singular point $s \in S$, the blowing-up of s and the Nash blow-up may be reasonably compared. The first one means geometrically to take the limiting positions of secants to S through s, and the (reduced) exceptional fiber is the projective tangent cone to s. The second one means to take the limiting positions of the tangent spaces to nearby smooth points, and the exceptional fiber is the Nash fiber.

As shown in the following examples the Nash blow-up is not smooth in general, as well as the point blowing-up. A natural question is if it is possible to resolve singularities by iterating Nash blow-ups.

For curve singularities, by iteration one obtains eventually with both kinds of modifications the same result, the desingularization of the curve, i.e. its normalization (see [N] for the Nash blow-up).

For surface singularities, there is a result by Zariski showing that an iteration of point blowing-up and normalization gives a desingularization (see [Z]). The method based in Theorem 2 for obtaining the normalized Nash blow-up may be applied in particular for rational singularities (see [G5]). On the other hand Hironaka proved that after a finite number of Nash blow-ups the singularities obtained are rational (see [H]) of a special kind, the so called sandwiched singularities. This singularities are resolved by normalized Nash blow-ups, by applying the above method (see [Sp]). Then for singular surfaces there is a result which can be compared to Zariski's method, by Nash blow-up instead of point blowing-up, and by normalization.

The Nash blow-up may be applied to non isolated singularities and higher dimensional singularities, to obtain resolutions as we shall see in some examples. It is not known if surfaces singularities may be resolved by iterating the Nash blow-up without normalization. For higher dimensions the desingularization problem by iteration of Nash blow-up or normalized Nash blow-up is open.

Another idea is to combine both Nash and point blowing-up, to have both natural fiber bundles obtained by extending the tangent bundle and the normal bundle on the fiber over a singular point, related to the secants or "radial" structure. The reduced structure is given by the limits of secants and of tangent spaces (see [W]).

Sometimes the (normalized) Nash blow-up $\widetilde{S}$ of S dominates already the blowing-up S' of s. For surfaces this happens when the tangent cone at s does not have a plane as a component (see [Sn]) and this is equivalent to saying that the pull back on $\widetilde{S}$ of hyperplane sections through s have no base points.

In general $\widetilde{S}$ does not dominate S'. Let $\bar{\sigma} : \bar{S}' \longrightarrow \widetilde{S}$ be the blowing-up of the fiber $\nu^{-1}(s)$ of s in $\widetilde{S}$. Then $\nu \circ \bar{\sigma}$ factorizes through σ and we have a commutative diagram.

$$\begin{array}{ccc} \bar{S}' & \xrightarrow{\bar{\sigma}} & \widetilde{S} \\ {\scriptstyle \nu'}\downarrow & & \downarrow{\scriptstyle \nu} \\ S' & \xrightarrow{\sigma} & S \end{array}$$

This mixed construction is also applied for example to obtain a formula for the local Euler obstruction (see [G3]). By definition the Chern class of a smooth variety is the Chern class of its tangent bundle. For singular varieties, there are generalizations like the (Schwartz–MacPherson) Chern classes which are defined by means of the Nash cotangent bundle and the local Euler obstruction. A report on Chern classes of singular varieties is far beyond the scope of this short survey. This is an important application of the Nash blow-up that has many ramifications and has been applied in different contexts.

§4. Examples and results

1) If S is smooth, then the Nash blow-up $\widetilde{S}$ is isomorphic to S. In fact the converse statement is also true, over any characteristic zero field (see [Li], [N]).

2) Let S be the union of two 2-planes in $\mathbb{C}^4$ intersecting in only one point, which is singular. Then the Nash blow-up simply separates the two planes, and the fiber over the singular point is given by two points. This is an example of a not complete intersection variety.

This may be computed by the the blowing-up centered on the ideal generated by the maximal minors of the jacobian matrix of a generically differentially independent system, as in the method of Theorem 1.

Compare with the blowing-up of the singular point where the fiber contains two lines.

3) If S is the affine cone over a smooth projective plane curve $\mathcal{C}$, then the singularity at the vertex is isolated, and the exceptional fiber in the Nash blow-up is the *dual* curve of $\mathcal{C}$.

Indeed, the tangent plane is constant along any generatrix of the cone, and its projection, at infinity, is the tangent line to $\mathcal{C}$ at the point determined by the chosen generatrix. So the set of limiting positions of tangent planes near the vertex is in bijection with the tangent lines to $\mathcal{C}$, i.e. its dual.

4) The preceding affine cone coincides with the tangent cone at the vertex.

This is an example of what happens also for the general case of surfaces, in which the dual variety to the *projective tangent cone* at a singular point x is always a part of the exceptional Nash fiber over x. For 2-dimensional hypersurfaces in general the Nash fiber contains also pencils of planes through special lines of the tangent cone. For surfaces which are not hypersurfaces we may have in the Nash fiber other components of families of planes which are not projective lines. (see [G2], [Le]).

5) Let S be the "*Whitney–Cartan umbrella*" defined by the equation $x^2 = y^2 z$ in $\mathbb{C}^3$.

This is an example of non normal singular surface. The Nash blow-up may be computed by blowing up the jacobian ideal $J = (x, yz, y^2)$. The result is a smooth surface $\widetilde{S}$, but the Nash fibre over the origin is a smooth rational curve corresponding to a pencil of planes with a common axis, and with an immersed point given by the dual to the tangent cone, not reduced since it is a double plane. The fiber over each other singular point of S has only two points corresponding to the two planes of the tangent cone. The minimal resolution of S is obtained by contracting the rational curve in $\widetilde{S}$.

6) Let S in $\mathbb{C}^3$ be a singularity of type $\mathbb{A}_2$ defined by the equation $xy = z^3$.

Then $\widetilde{S}$, obtained by blowing-up the jacobian ideal $J = (x, y, z^2)$, has two singular points, each one of the type of an affine cone over the rational curve of degree 3 in $\mathbb{P}^3$. The second (following) Nash blow-up is smooth.

This is an example of a quotient singularity, by a cyclic group of order 3 acting on $\mathbb{C}^2$, i.e. an *orbifold.*

7) For the more general cases of two-dimensional orbifolds we have precise results of the *normalized* Nash blow-up of quotients by finite subgroups of $SL(2, \mathbb{C})$, which give the rational double points, also called Klein or DuVal singularities (see [A], [B], [G5]). We will now describe the result for the tetrahedral case $\mathbb{E}_6$, where the Nash blow-up finds an orbit corresponding to both the associated cube and octahedron.

Let G be the binary tetrahedral group acting on $\mathbb{C}^2$, with only the origin as a fixed point, and $q : \mathbb{C}^2 \longrightarrow S = \mathbb{C}^2/G$ the quotient morphism. The surface S has a singularity of type $\mathbb{E}_6$.

Let $\sigma : \widehat{\mathbb{C}}^2 \longrightarrow \mathbb{C}^2$ be the blowing-up of the origin of $\mathbb{C}^2$, and D the exceptional projective line in $\widehat{\mathbb{C}}^2$. Since the origin is a fixed point of G, there is a pull back of the action of G on $\widehat{\mathbb{C}}^2$ and the center Z of G acts trivially on D , so we get the natural action of the order 12 tetrahedral rotation group G/Z on the sphere $S^2 \approx D$.

The quotient $\widehat{q} : \widehat{\mathbb{C}}^2 \longrightarrow \widehat{S} = \widehat{\mathbb{C}}^2/G$ gives a surface $\widehat{S}$ with three singular points, $\widehat{P}_1$ and $\widehat{P}_2$ of type $\mathbb{A}_2$, and $\widehat{Q}_2$ of type $\mathbb{A}_1$, contained in the line $\widehat{D}$ image of D, and corresponding to the three exceptional orbits given respectively by the 4 vertices of a tetrahedron T, the 4 vertices of the dual tetrahedron T^* and the 6 vertices of the octahedron O, intersection of T and T^*.

$$\begin{array}{ccccccc} & & & & X & \supset & L \\ & & & & \downarrow & & \downarrow \\ D & \subset & \widehat{\mathbb{C}}^2 & \xrightarrow{\widehat{q}} & \widehat{S} & \supset & \widehat{D} \\ & & {\scriptstyle\sigma}\downarrow & & \downarrow & & \\ & & \mathbb{C}^2 & \xrightarrow{q} & S & & \end{array}$$

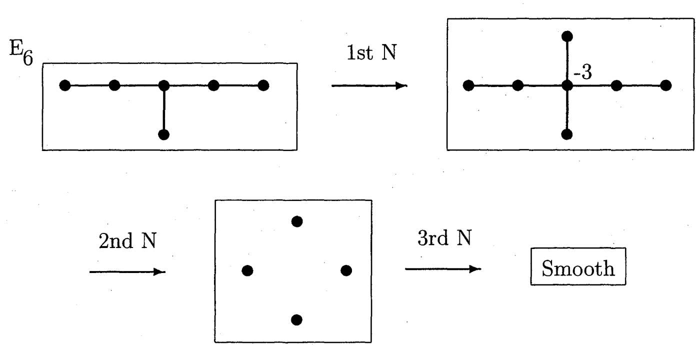
By resolving the three singular points $\widehat{P}_1, \widehat{P}_2$ and $\widehat{Q}_2$ of $\widehat{S}$ we obtain the minimal desingularization X of S, with an exceptional fibre with the $\mathbb{E}_6$ configuration.

The central line L of this $\mathbb{E}_6$ configuration projects isomorphically on the line $\widehat{D}$. The intersections of the three branches with the central line L are the three points P_1, P_2 and Q_2 which project to the singular points of $\widehat{S}$. It turns out that the rational map $\pi : X \dashrightarrow \widetilde{S}$ from X into the Nash blow-up $\widetilde{S}$ is not defined only in point Q_1 of L (i.e. a base point of the polar curve), such that the cross ratio $(Q_1, Q_2, P_1, P_2) = -1$.

Note that this point is invariant by the automorphism of the configuration $\mathbb{E}_6$ exchanging P_1 andP_2 and fixing Q_2.

Let $\widehat{Q}_1$ be the projection of Q_1 on the projective line $\widehat{D}$. Then this forth point $\widehat{Q}_1$ is the image of a regular G-orbit of 12 points in D, corresponding to the central points of the edges of the octahedron O, or the intersection points of O with the cube given by the vertices of T and T^*. The blowing-up of $Q_1 \in L$ in X gives the forth line cutting the central line of the exceptional fiber in the resolution of the (normalized) Nash blow-up of the $\mathbb{E}_6$ singularity obtained by the method of Theorem 2.

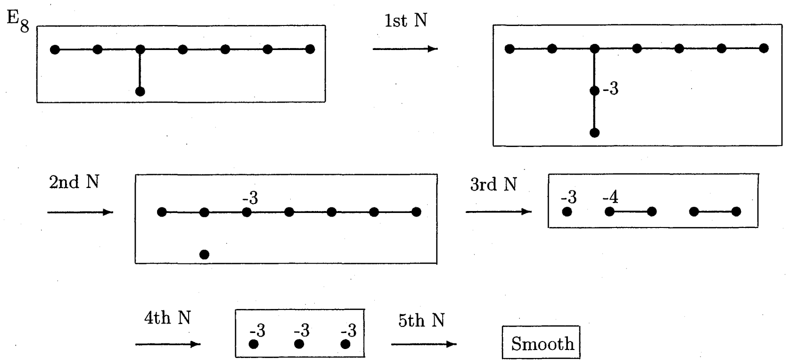
The desingularization of the $\mathbb{E}_6$ singularity is obtained after three normalized Nash blow-ups. In the following diagram are given the kind of singularities obtained in each step, by the weighted dual graph of their minimal desingularizations. The self intersection of a divisor is given as a weight of the corresponding vertex only if it is not -2.



The desingularization of the $\mathbb{E}_6$ singularity obtained by this iteration of normalized Nash blow-ups is not the minimal desingularization.

The minimal desingularization equipped with a fiber bundle extending the tangent bundle of S_{reg} is the minimal desingularization of the Nash blow-up.

8) For the icosahedral singularity $\mathbb{E}_8$ the desingularization is obtained with five normalized Nash blow-ups, as follows.



Note that the singularity of the first normalized Nash blow-up of $\mathbb{E}_8$ (and of $\mathbb{E}_6$) is not an orbifold singularity (since the dual graph of the minimal resolution is not a subgraph of the $\mathbb{A}-\mathbb{D}-\mathbb{E}$ types).

9) Nash blow-up of two-dimensional toric orbifolds.

Any two-dimensional normal toric singularity is an orbifold since it may be obtained as a quotient by a finite cyclic subgroup of $GL(2,\mathbb{C})$.

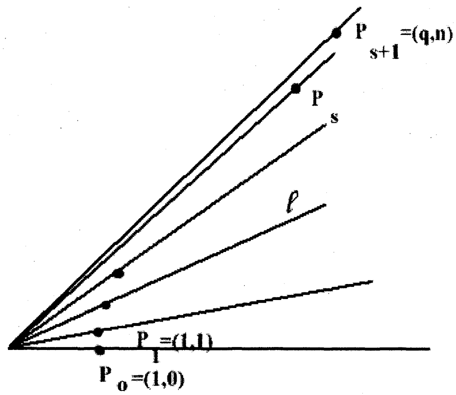
With the usual notations, let S_σ or $S_{(q,n)}$ be the affine toric variety associated to the cone $\sigma = \langle(1,0),(q,n)\rangle$ in $N_\mathbb{R}$, with q and n integers such that $1 \le q < n$, $\gcd(q,n) = 1$. The Hilbert basis or minimal generating system $G = \{P_0, \cdots, P_{s+1}\}$ of the semigroup $\sigma \cap N$ is given by the Jung–Hirzebruch continued fraction

$$n/(n-q) = a_1 \underline{-1/}\overline{a_2 - 1} \cdots - 1/a_s, \quad \text{with} \quad a_i \ge 2 \ \ \forall i, \quad \text{by}$$

$$P_0 = (1,0), \ P_1 = (1,1) \quad \text{and} \quad P_{i+1} = a_i P_i - P_{i-1} \quad \text{for} \quad 1 \le i \le s.$$

In the case $q = 0$, and $n = 1$, then σ is a regular cone, S_σ is smooth and G contains the two extremal vectors.

The minimal desingularization X of S_σ is given by the fan cut out from σ by the lines ℓ through the points of G, and the dual graph Γ is a chain with s vertices.



We denote by L_i, $1 \leq i \leq s$, the ordered vertices of the chain and the smooth rational curves they represent. The self intersection of L_i in X is $-a_i$. The graph Γ is represented also by the Newton polygon from P_1 to P_s of the convex hull of the semigroup generated by G. Each segment of the Newton polygon has either a central vertex or a central edge. The minimal desingularization of the Nash blow-up of S_σ and the normalized Nash blow-up are described as follows.

Theorem 4. *Let $S = S_\sigma$ be a two-dimensional toric singularity and X its minimal desingularization. With the above notations, we have the following results.*

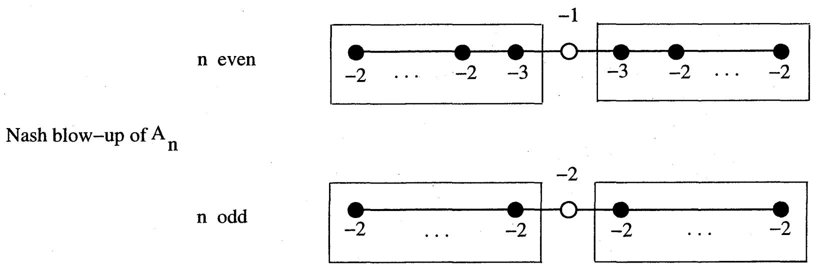
a) The minimal desingularization $\bar{X}$ of the Nash blow-up $\widetilde{S}_\sigma$ is obtained by blowing-up the points in X represented by the central edges of Γ.

b) The normalized Nash blow-up of S is obtained by contracting the strict transform $\bar{L}_i$ of L_i in $\bar{X}$ into a point if an only if $a_i = 2$, $s > 1$ and L_i is not central, or $a_i = 3$, $1 < i < s$.

The degree of the Nash cotangent bundle on each exceptional component may be also obtained ([G1], [G5]).

The fan that gives the normalized Nash blow-up is cut out in σ by the lines passing through points in G, or points of the form $P_i + P_{i+1}$, other than those associated to contracted divisors. Then this fan is given by the lines through P_i if $a_i > 3$, or if $a_i = 3$ for $i = 1$ or $i = s$, or if $a_i = 2$ and P_i is central in its face in the Newton polygon, or through $P_i + P_{i+1}$ if P_i and P_{i+1} are the extremities of a central edge in the Newton polygon.

For example if $q = 1$, i.e. in the case $\mathbb{A}_n$, for $n > 1$ we obtain the following dual graphs for the minimal desingularization of the Nash blow-up. The black vertices are associated to the exceptional divisors, and the white vertices are associated to divisors not contracted by the desingularization morphism.



It follows that the singularities obtained are also toric, and that there are strictly fewer exceptional components in the minimal desingularization of each singularity . Then we obtain:

Corollary 2. *A desingularization of S_σ is obtained by a finite number of iterations of normalized Nash blow-ups.*

In fact the number of normalized Nash blow-ups required to resolve the singularity may be given precisely in terms of the Jung–Hirzebruch continued fraction of $n/(n-q)$. It is much less (asymptotically logarithmic) than the number of blowing-ups centered at the maximal ideals required to obtain a desingularization.

10) On Nash blow-up of three-dimensional toric orbifolds.

We describe here some results on the three-dimensional toric case of singularities obtained as a quotient of $\mathbb{C}^3$ by a finite abelian subgroup of $GL(3, \mathcal{C})$. There is not a complete description as in the two-dimensional case.

In general the toric singularities which are quotients by (abelian) groups are associated with simplicial cones, and are called V-manifolds.

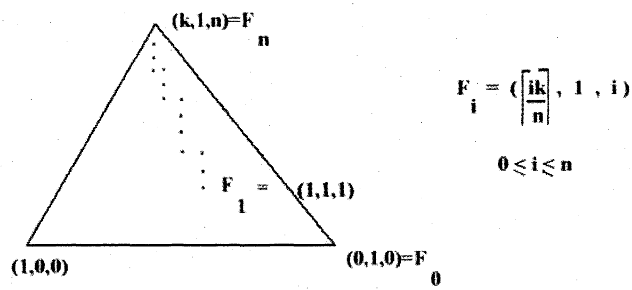
Let us consider first the terminal toric orbifold singularities S_σ, in the sense of Mori theory. In dimension two, terminal implies regular, so these examples exist only in dimension at least three. In this case the tetrahedron with vertices the origin and the extremal primitive vectors of the simplicial cone σ is simple. This means that the only integer points in the tetrahedron are the vertices.

C1) Then, using the White–Frumkin terminal lemma (see [O]), we find that there is a basis of the free abelian group $N \simeq \mathbb{Z}^3$ such that $\sigma = \langle (1,0,0), (0,1,0), (k,1,n) \rangle$, with k and n integers, $\gcd(k,n) = 1$, $1 \leq k < n$.

C2) This presentation of σ is not unique, but up to permutation of coordinates or the extremal primitive vectors defining the simplicial cone, there are essentially only two other representations , by replacing the third vector by $(n-k, 1, n)$ or $(p, n-p, n)$ with $pk \equiv 1(n)$.

C3) For such a cone σ in $N_{\mathbb{R}}$, then the dual in $M_{\mathbb{R}}$, with the usual notations, is $\sigma^{\vee} = \langle (n,0,-k), (0,n,-1), (0,0,1) \rangle$.

C4) The Hilbert basis of the semigroup $\sigma \cap N$ is given by the extremal primitive vectors, and the points $F_i = (\lceil ik/n \rceil, 1, i)$, $0 < i < n$.



C5) The Jung–Hirzebruch continued fractions of n/k and $n/(n-k)$ are used to describe the Hilbert basis $G(\sigma^{\vee})$ of the dual cone semigroup $\sigma^{\vee} \cap M$. Let $\{P_0, \cdots, P_{s+1}\}$ and $\{Q_0, \cdots, Q_{t+1}\}$ be defined by

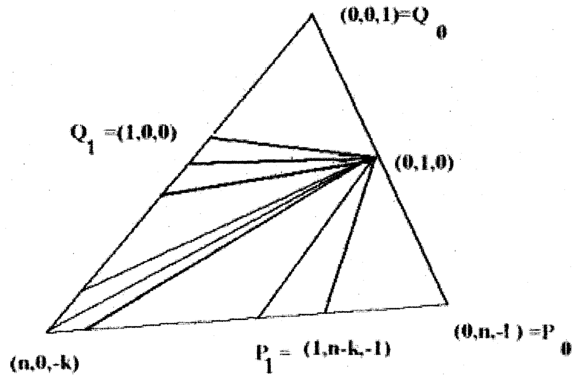
$$n/(n-k) = a_1 \underline{-1/}\overline{a_2 - 1} \cdots - 1/a_s, \quad \text{with} \quad a_i \geq 2 \;\; \forall i,$$

$$P_0 = (0,n,-1),\ P_1 = (1, n-k, -1),\ P_{i+1} = a_i P_i - P_{i-1} \quad \text{for} \quad 1 \leq i \leq s.$$

$$n/k = b_1 \underline{-1/}\overline{b_2 - 1} \cdots - 1/b_t, \quad \text{with} \quad b_i \geq 2 \;\; \forall i,$$

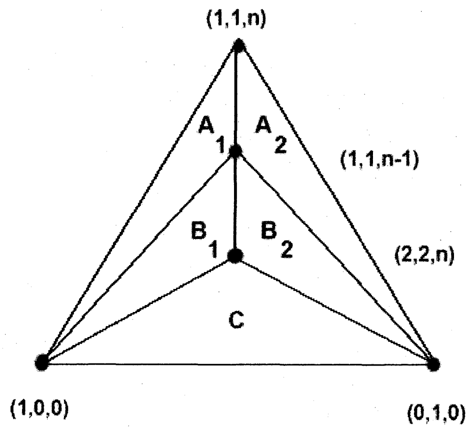
$$Q_0 = (0,0,1),\ Q_1 = (1,0,0),\ Q_{i+1} = b_i Q_i - Q_{i-1} \quad \text{for} \quad 1 \leq i \leq t.$$

The three extremal primitive vectors belong to these sequences as P_0, Q_0 and $P_{s+1} = Q_{t+1} = (n,0,-k)$. The Hilbert basis $G(\sigma^{\vee})$ of $\sigma^{\vee} \cap M$ is the set of the P_i's, the Q_i's and the point $(0,1,0)$. Note that the P_i's belong to the wall of $\sigma^\vee$ generated by P_0 and P_{s+1}, the Q_i's to the wall generated by Q_0 and Q_{t+1}, and $(0,1,0)$ to the wall generated by P_0 and Q_0.



C6) In particular we obtain that if σ is a terminal cone, then the Hilbert basis $G(\sigma^\vee)$ of the dual is contained in the walls of $\sigma^\vee$. The converse is not true in general.

C7) We compute the Nash blow-up in the case $k = 1$ by the method of Theorem 3. The fan obtained is the following.



The cones denoted by A_1 and A_2 are regular.

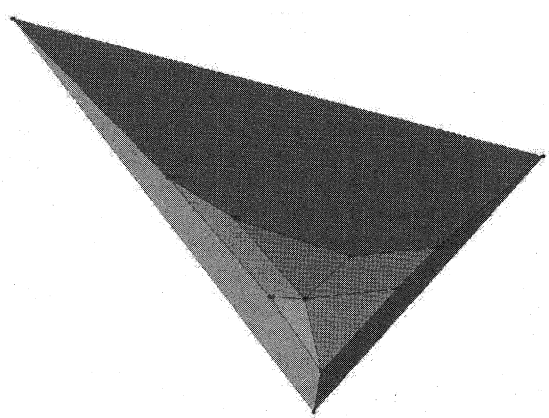
The cones denoted by B_1 and B_2 are isomorphic, if n is odd, to $\langle(1,0,0),(0,1,0),(0,2,n-2)\rangle$, and to $\langle(1,0,0),(0,1,0),(0,1,n/2-1)\rangle$

if n is even. They may be resolved by iteration of normalized Nash blow-ups, as in the two-dimensional case. Though not terminal cones, the structure of their Hilbert bases is similar to the two dimensional case, belonging to the non regular cone facet, plus the opposite extremal primitive vector.

The last cone of the fan, denoted by C, is isomorphic, if n is even, to $\langle(1,0,0),(0,1,0),(1,1,n/2)\rangle$, and to $\langle(1,0,0),(0,1,0),(2,2,n)\rangle$ if n is odd. This last cone is not terminal.

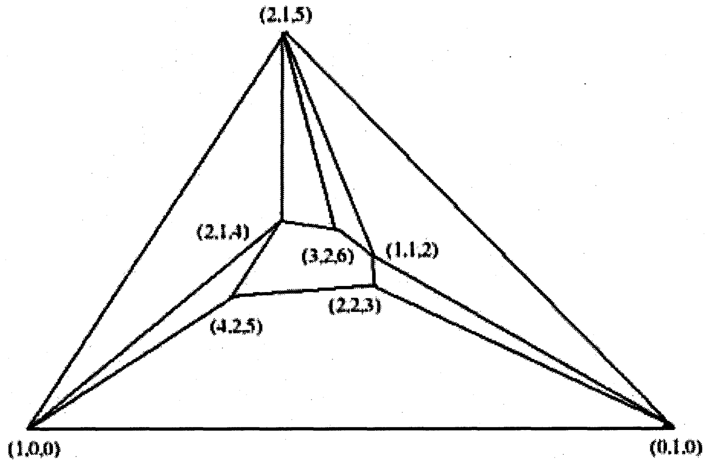
C8) A second family of examples, similar to the two dimensional affine cone over a rational curve in $\mathbb{P}^n$ is the three dimensional cone $\sigma = \langle(1,0,0),(0,1,0),(n-1,n-1,n)\rangle$. The Hilbert basis contains only one more point besides the vertices, the Nash fan is the elementary subdivision of σ induced by this point, and it is a regular fan. This is a (highly) non complete intersection example. The normalized Nash blow-up is smooth.

C9) As a last example, consider a terminal cone with $k=2$. The Newton polytope in $\sigma^\vee$ corresponding to the ideal described in Theorem 3 seen from the origin looks like the following.



The Nash fan induced in the dual space contains a cone which is not simplicial, so the corresponding affine toric variety is not an orbifold.

As it was the case with the normalized Nash blow-up of the two dimensional orbifold singularities $\mathbb{E}_6$ and $\mathbb{E}_8$.



References

[A] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math., **88** (1966), 129–136.

[B] E. Brieskorn, Rationale singularitäten komplexer Flächen Invent. Math., **4** (1968), 336–358.

[G1] G. Gonzalez-Sprinberg, Eventails en dimension 2 et Transformation de Nash, Secrétariat Mathématique de l'Ecole Norm. Sup., Paris, 1977.

[G2] ———, Calcul de l'invariant local d'Euler pour les singularités quotient de surfaces, C. R. Acad. Sci. Paris Sér. A-B, **288** (1979), 989–992.

[G3] ———, L'obstruction locale d'Euler et le théorème de MacPherson, Astérisque, **82-83** (1981), 7–23.

[G4] ———, Cycle maximal et invariant d'euler local des singularités isolées de surfaces, Topology, **21** (1982), 401–408.

[G5] ———, Résolution de Nash des points doubles rationnels, Ann. Inst. Fourier, **32** (1982), 111–178.

[G6] ———, Désingularisation des surfaces par des modifications de Nash normalisées, Séminaire Bourbaki, Astérisque, **145-146** (1987), 187–207.

[H] H. Hironaka, On Nash blowing-up, In: Arithmetic and Geometry II, Progr. Math., **36**, Birkhäuser, 1983, pp. 103–111.

[Li] J. Lipman, On the Jacobian ideal of the module of differentials, Proc. Amer. Math. Soc., **21** (1969), 422–426.

[Le] Lê D. Tráng, Limites d'espaces tangents et obstruction d'Euler des surfaces, Astérisque, **82-83** (1981), 45–69.

[LT] Lê D. Tráng and B. Teissier, Limites d'espaces tangents en géométrie analytique, Comment. Math. Helv., **63** (1988), 540–578.

[N] A. Nobile, Some properties of the Nash blowing-up, Pacific J. Math., **60** (1975), 297–305.

[O] T. Oda, Convex Bodies and Algebraic Geometry, Ergeb. Math. Grenzgeb. (3), **15**, Springer-Verlag, 1988.

[P] R. Piene, Ideals associated to a desingularization, In: Algebraic Geometry, Lecture Notes in Math., **732**, Springer-Verlag, 1979, pp. 503–517.

[R] O. Riemenschneider, Deformationen von Quotientensingularitäten (nach zyklischen Gruppen), Math. Ann., **209** (1974), 211–248.

[Sn] J. Snoussi, The Nash modification and hyperplane sections on surfaces, Bull. Braz. Math. Soc. (N.S.), **36** (2005), 309–317.

[Sp] M. Spivakovsky, Sandwiched singularities and desingularization of surfaces by normalized Nash transformations, Ann. of Math. (2), **131** (1990), 411–491.

[T] B. Teissier, The hunting of invariants in the geometry of discriminants, Real and Complex Singularities, Proc. Oslo, 1977.

[TE] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings I, Lecture Notes in Math., **339**, Springer-Verlag, 1973.

[W] H. Whitney, Tangents to an analytic variety, Ann. of Math. (2), **81** (1965), 469–549.

[Z] O. Zariski, The reduction of singularities of an algebraic surface, Ann. of Math. (2), **40** (1939), 639–689.

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