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On irreducible sextics with non-abelian fundamental group

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Abstract.

We calculate the fundamental groups $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$ for all irreducible plane sextics $B \subset \mathbb{P}^2$ with simple singularities for which π is known to admit a dihedral quotient $\mathbb{D}_{10}$. All groups found are shown to be finite, two of them being of large order: 960 and 21600.

§1. Introduction

1.1. Motivation

Recall that a plane sextic $B \subset \mathbb{P}^2$ is said to be of *torus type* if its equation can be represented in the form $p^3 + q^2 = 0$, where p and q are some homogeneous polynomials of degree 2 and 3, respectively. Essentially, sextics of torus type were introduced by O. Zariski as the ramification loci of cubic surfaces. The fundamental group $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$ of an irreducible sextic B of torus type is known to be infinite; in particular, it is nonabelian. For a long time, no other examples of nonabelian groups were known, which lead M. Oka to a conjecture [8] that the fundamental group of an irreducible sextic that is not of torus type is always abelian. The conjecture was disproved in [3], [4], and for the counterexamples *for which the group π was computed* it turned out to be finite, with the exception of one family with non-simple singularities. Besides, it was also shown that, for each irreducible sextic that is not of torus type, the abelinization of the commutant of π is finite. (This assertion is a restatement of the proved part of Oka's conjecture, related to the Alexander polynomial.) Thus, the following statement seems to be a reasonable replacement for the original conjecture.

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Conjecture 1.1.1. *Let B be an irreducible plane sextic with simple singularities and not of torus type. Then the fundamental group $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$ is finite.*

The fundamental groups of all irreducible sextics with a non-simple singular point are found in [3] (the case of a quadruple point) and [4] (the case of a singular point adjacent to $\mathbf{J}_{10}$). On the other hand, the construction of sextics with simple singularities and nonabelian fundamental group $\pi = \pi_1(\mathbb{P}^2 \smallsetminus C)$ suggested in [3] is rather indirect; it proves that π has a dihedral quotient, but it is not suitable to compute π exactly. In this paper, we attempt to substantiate Conjecture 1.1.1 by computing the groups of some of the curves discovered in [3].

1.2. Principal results

For a group G, denote by $G' = [G, G]$ its commutant, or derived group, and let $G'' = (G')'$ *etc.* We use the notation $\mathbb{Z}_n$ and $\mathbb{D}_n$ for, respectively, the cyclic and dihedral groups of order n.

To shorten the statements, we introduce the term *generalized $\mathbb{D}_{2n}$-sextic* to stand for a plane sextic B whose fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ factors to $\mathbb{D}_{2n}$, $n \geqslant 3$. A *$\mathbb{D}_{2n}$-sextic* is an irreducible generalized $\mathbb{D}_{2n}$-sextic with simple singularities. Recall, see [3], that there are $\mathbb{D}_6$-, $\mathbb{D}_{10}$-, and $\mathbb{D}_{14}$-sextics; all $\mathbb{D}_6$-sextics are of torus type, and the $\mathbb{D}_{10}$-sextics form eight equisingular deformation families, one family for each of the following sets of singularities:

$$4\mathbf{A}_4, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus 2\mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_2,$$
$$\mathbf{A}_9 \oplus 2\mathbf{A}_4, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2, \quad 2\mathbf{A}_9.$$

The objective of the paper is the computation of the fundamental groups of all $\mathbb{D}_{10}$-sextics. The principal result is the following theorem.

Theorem 1.2.1. *Let B be a $\mathbb{D}_{10}$-sextic. Then the group $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$ is finite. Furthermore, one has $\pi = \mathbb{D}_{10} \times \mathbb{Z}_3$ with the following two exceptions:*

(1) *The set of singularities of B is $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$: then* $\operatorname{ord} \pi = 960$ *and one has $\pi''/\pi''' = (\mathbb{Z}_2)^4$ and $\pi''' = \mathbb{Z}_2$.*

(2) *The set of singularities of B is $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$: then* $\operatorname{ord} \pi =$ 21600 *and π'' is the only perfect group of order* 720, *see* [11].

(*In all cases, including the exceptional ones, $\pi/\pi' = \mathbb{Z}_6$ and $\pi'/\pi'' = \mathbb{Z}_5$.*)

According to [3], the $\mathbb{D}_{14}$-sextics form two equisingular deformation families, with the sets of singularities $3\mathbf{A}_6$ and $3\mathbf{A}_6 \oplus \mathbf{A}_1$. For the former family, the group has recently been shown to be $\mathbb{D}_{14} \times \mathbb{Z}_3$, see [7]. The group of the sextics with the set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$ (which are all projectively equivalent) is still unknown.

The groups of the $\mathbb{D}_{10}$-sextics with the sets of singularities $4\mathbf{A}_4$ and $4\mathbf{A}_4 \oplus \mathbf{A}_1$ were found independently by C. Eyral and M. Oka [9].

1.3. Other results and contents

The starting point of the computation is Theorem 2.1.1, which provides an explicit geometric construction for $\mathbb{D}_{10}$-sextics. (In [3], only the existence of $\mathbb{D}_{10}$-sextics with the sets of singularities listed above is proven.) We show that each $\mathbb{D}_{10}$-sextic B is a double covering of a very particular rigid trigonal curve $\bar{B}$ in the Hirzebruch surface Σ_2; the curve $\bar{B}$ has two type $\mathbf{A}_4$ singular points, and various sets of singularities for B are obtained by varying the ramification locus. Theorem 2.1.1 is dealt with in §2.

The fundamental groups are computed in §3: we merely apply the classical approach due to van Kampen to the ruling of the Hirzebruch surface Σ_2. In a few difficult cases, the resulting representations are studied using `GAP` [10].

The concluding §4 is not directly related to Theorem 1.2.1: we use the representation given by Theorem 2.1.1 to produce explicit equations for $\mathbb{D}_{10}$-sextics. The equation depends on three parameters $a, b, c \in \mathbb{C}$, see (4.3.1); we analyze the parameter space and describe the triples (a, b, c) resulting in particular sets of singularities.

1.4. Acknowledgements

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§2. The construction

2.1. The reduction

Recall that any involutive automorphism $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ has a fixed line $L = L_c$ and an isolated fixed point $O = O_c$, and the quotient $\mathbb{P}^2(O)/c$ is the rational geometrically ruled surface Σ_2. (Here, $\mathbb{P}^2(O)$ stands for the plane $\mathbb{P}^2$ with O blown up.) The images in Σ_2 of O and L are, respectively, the exceptional section E and a generic section $\bar{L}$, so that $\mathbb{P}^2(O)$ is the double covering of Σ_2 ramified at $\bar{L} + E$.

Recall also that the semigroup of classes of effective divisors on Σ_2 is generated by E and the class F of a fiber of the ruling. One has $E^2 = -2$, $F^2 = 0$, $F \circ E = 1$, and $K_{\Sigma_2} = -2E - 4F$.

The principal result of this section is the following theorem.

Theorem 2.1.1. *Let $B \subset \mathbb{P}^2$ be a $\mathbb{D}_{10}$-sextic. Then $\mathbb{P}^2$ admits an involution c preserving B and such that O_c does not belong to B and the image of B in $\Sigma_2 = \mathbb{P}^2(O_c)/c$ is a trigonal curve $\bar{B}$, disjoint from E, with two type $\mathbf{A}_4$ singular points.*

Theorem 2.1.1 is proved at the end of this section, in 2.5 below. Theorem 2.1.1 has a partial converse, see Theorem 3.2.11.

Remark 2.1.2. The proof of Theorem 2.1.1 given in 2.5 uses the theory of $K3$-surfaces; we do prove that *each* $\mathbb{D}_{10}$-sextic is symmetric. The fact that each of the eight equisingular deformation families of such sextics admits a symmetric *representative* can easily be established using the results of §3, where each set of singularities listed in the introduction is realized by a symmetric $\mathbb{D}_{10}$-sextic.

2.2. The covering $K3$-surface

Let B be a plane sextic with simple singularities. Consider the double covering $X \to \mathbb{P}^2$ ramified at B and its minimal resolution $\tilde{X}$. Then, $\tilde{X}$ is a $K3$-surface and the deck translation $\tau\colon \tilde{X} \to \tilde{X}$ of the covering is a holomorphic anti-symplectic (*i.e.*, reversing holomorphic 2-forms) involution.

Recall that $H_2(\tilde{X}) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$ is an even unimodular lattice of rank 22 and signature -16. For a singular point P of B, denote by $D_P \subset H_2(\tilde{X})$ the set of classes of exceptional divisors over P; we use the same notation D_P for the incidence graph of these divisors, which is an irreducible Dynkin diagram of the same name **A**–**D**–**E** as the type of P. Note that, for a type **A** singular point, the action of τ_* on D_P is the only nontrivial symmetry of the graph. Let $\Sigma_P \subset H_2(\tilde{X})$ be the sublattice spanned by D_P; it is an irreducible negative definite root system, *i.e.*, negative definite lattice spanned by vectors of square (-2).

Denote $\Sigma = \bigoplus \Sigma_P$, the summation running over all singular points P of B, and let $S = \Sigma \oplus \mathbb{Z}h \subset H_2(\tilde{X})$, where $h \in H_2(\tilde{X})$ is the class realized by the pull-back of a generic line in $\mathbb{P}^2$. One has $h^2 = 2$, and the sums above are orthogonal. Let $\tilde{\Sigma} \subset \tilde{S} \subset H_2(\tilde{X})$ be the primitive hulls of Σ and S, respectively. The finite index extension $\tilde{S} \supset S$ is determined by its *kernel* $\mathcal{K}$, which is an isotropic subgroup of the discriminant form $\operatorname{discr} S$. (For the definition of the discriminant form and its relation to lattice extensions, see V. V. Nikulin [13].) As shown in [2], if B is irreducible, then $\mathcal{K} \subset \operatorname{discr} \Sigma$. According to [3], it is the kernel $\mathcal{K}$ that essentially enumerates the dihedral quotients of $\pi_1(\mathbb{P}^2 \smallsetminus B)$.

2.3. Symmetric sextics

Let B be a plane sextic with simple singularities, and denote by $\tilde{X}$ the covering $K3$-surface. Let $\tilde{c}\colon \tilde{X} \to \tilde{X}$ be a holomorphic symplectic (*i.e.*, preserving holomorphic 2-forms) involution. As is known, $\tilde{c}$ has eight fixed points, and the quotient $Y = \tilde{X}'/\tilde{c}$ is again a $K3$-surface, where $\tilde{X}'$ is $\tilde{X}$ with the fixed points of $\tilde{c}$ blown up.

Since the projection $\tilde{X} \to \mathbb{P}^2$ is the map defined by the linear system $h \in \operatorname{Pic} \tilde{X}$, the two involutions $\tilde{c}$, τ commute if and only if the induced automorphism $\tilde{c}_*$ of $H_2(\tilde{X})$ preserves h. In this case, $\tilde{c}$ descends to an involution $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ which preserves B. Let $O = O_c$ and $L = L_c$. In what follows, *we always assume that B does not contain L as a component.* We fix the notation $\bar{L}$ and $\bar{B}$ for the images of L and B, respectively, in Σ_2.

Alternatively, if $\tilde{c}_*(h) = h$, then τ descends to an anti-symplectic involution $\tau_Y : Y \to Y$, and the quotient Y/τ_Y blows down to Σ_2.

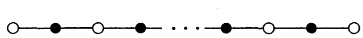
Lemma 2.3.1. *Let O, B, $\bar{B}$,* etc. *be as above. If $O \notin B$, then $\bar{B} \in |3E + 6F|$ is a trigonal curve disjoint from E. If $O \in B$, then $\bar{B} \in |2E + 6F|$ is a hyperelliptic curve, $\bar{B} \circ E = 2$.*

Proof. The branch locus of the ramified covering $Y \to \Sigma_2$ consists of $\bar{B}$, $\bar{L}$, and, if $O \in B$, the exceptional section E. On the other hand, since Y is a $K3$-surface, the branch locus is an anti-bicanonical curve, *i.e.*, it belongs to $|4E + 8F|$. Since $\bar{L} \in |E + 2F|$, the statement follows. Q.E.D.

Lemma 2.3.2. *Let P be a c-invariant type $\mathbf{A}_p$ singular point of B, and let $\bar{P} \in \Sigma_2$ be its image. Assume that $p > 1$ and that $\tilde{c}_*$ acts trivially on D_P. Then $P \in L$ and $\bar{P}$ is a type $\mathbf{A}_{2p+1}$ singular point of $\bar{B} + \bar{L}$,* i.e., *a point of $(p+1)$-fold intersection of $\bar{L}$ and $\bar{B}$ at a smooth branch of $\bar{B}$.*

Conversely, given a point $\bar{P}$ of $(p+1)$-fold intersection, $p \geqslant 1$, of $\bar{L}$ and $\bar{B}$ smooth for $\bar{B}$, the pull-back of $\bar{P}$ is a type $\mathbf{A}_p$ singular point of the double covering of $\bar{B}$ ramified at $\bar{L} + E$.

Proof. Since each curve in D_P is preserved by $\tilde{c}$ (as a set), the intersection points of the curves must be fixed by $\tilde{c}$. Furthermore, each of the two outermost curves must contain one more fixed point of $\tilde{c}$. Blowing up the fixed points, one obtains a sequence of rational curves with the following incidence graph:



(Here, ∘ and • stand, respectively, for (-1)- and (-4)-curves; the vertices in the diagram alternate and their total number is $2p+1$.) The (-1)-curves are in the fixed point set of $\tilde{c}$, and the (-4)-curves are not. Hence, the projection of the exceptional divisor to Y is a sequence of $(2p+1)$ rational curves whose incidence graph D is $\mathbf{A}_{2p+1}$. The involution τ_Y induced from τ acts on D as the only nontrivial symmetry; in particular, it is nontrivial on the middle curve. Thus, the projection to Y/τ_Y is a chain of p (-2)-curves ending in a (-1)-curve; it blows down to a single type $\mathbf{A}_{2p+1}$ singular point of the branch locus. As a consequence, one concludes that $P \neq O$, as otherwise the exceptional divisor would blow down to a (-2)-curve (the exceptional section E) rather than to a single point (*cf.* the proof of Lemma 2.3.3).

The converse statement can be proved by analyzing a local equation of $\bar{B} + \bar{L}$, so that $\bar{L} = \{y = 0\}$, and substituting $y \mapsto y^2$. Q.E.D.

The next lemma deals partially with the case when a singular point P of B coincides with O. In this case, we define the image $\bar{P}$ of P as the image of the intersection of the the proper transform of B in $\mathbb{P}^2(O)$ and the exceptional divisor over O. This definition gives rise to a single point $\bar{P} \in E \subset \Sigma_2$ unless P is of type $\mathbf{A}_1$. In the latter exceptional case, with an abuse of the language, we will say that $\bar{P} \subset E$ is a pair of points. (Note that this approach agrees with the standard convention to identify $\mathbf{D}_3 = \mathbf{A}_3$ and $\mathbf{D}_2 = 2\mathbf{A}_1$.)

Lemma 2.3.3. *Let P be a c-invariant type $\mathbf{A}_p$ singular point of B, and let $\bar{P} \subset \Sigma_2$ be its image. Assume that either $p = 1$ or $p > 1$ and $\tilde{c}_*$ acts nontrivially on D_P. Then $p = 2k - 1$ is odd and either*

(1) *$P \in L$ and $\bar{P}$ is a type $\mathbf{D}_{k+2}$ (type $\mathbf{A}_3$ if $p = 1$) singular point of $\bar{B} + \bar{L}$, or*

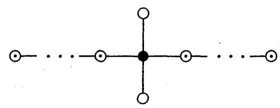
(2) *$P = O$ and $\bar{P}$ is a type $\mathbf{D}_{k+1}$ singular point (a type $\mathbf{A}_3$ singular point if $p = 2$ or a pair of type $\mathbf{A}_1$ singular points if $p = 1$) of $\bar{B} + E$.*

The type $\mathbf{D}_s$, $s \geqslant 3$, singular point above is formed by the section $\bar{L}$ or E intersecting $\bar{B}$ with multiplicity 2 at a type $\mathbf{A}_{s-3}$ singular point of $\bar{B}$.

Conversely, any type $\mathbf{D}_{k+2}$, $k \geqslant 1$, singular point of $\bar{B} + \bar{L}$ (respectively, type $\mathbf{D}_{k+1}$, $k \geqslant 1$, singular point of $\bar{B} + E$) as above gives rise to a c-invariant type $\mathbf{A}_{2k+1}$ singular point of the double covering B of $\bar{B}$ ramified at $\bar{L} + E$.

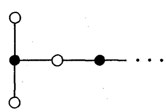
Proof. If p were even, the two middle curves in the exceptional divisor over P would intersect transversally at a fixed point of $\tilde{c}$. Since $\tilde{c}$ is symplectic, it cannot transpose two such curves. (The differential $d\tilde{c}$ at each fixed point is the multiplication by (-1).) Hence, p is odd.

The middle curve in the exceptional divisor is fixed by $\tilde{c}$; hence, it contains two fixed points. Blowing them up, one obtains a collection of rational curves with the following incidence graph:

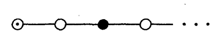


(Here, $\circ$, $\odot$, and $\bullet$ stand, respectively, for (-1)-, (-2)-, and (-4)-curves; the total number of curves is $p+2 = 2k+1$, the action of $\tilde{c}_*$ on the graph is the horizontal symmetry, and $\tilde{c}$ fixes the two (-1)-curves pointwise.) The projection of this divisor to Y is a collection of $(k+2)$ rational curves whose incidence graph is $\mathbf{D}_{k+2}$ (the 'short' edges corresponding to the two (-1)-curves in $\tilde{X}$).

Since τ_Y is anti-symplectic, whenever two invariant curves intersect transversally at a fixed point, exactly one of them is fixed by τ_Y pointwise. Hence, the action of τ_Y on the exceptional divisor is determined by its action on the curve with three neighbors in the incidence graph. Depending on whether this action is trivial or not, the projection of the exceptional divisor to Y/τ_Y has one of the following two incidence graphs:



or



Now, depending on the parity of k, these graphs blow down either to a single type $\mathbf{D}_{k+2}$ singular point of the branch locus (if the last vertex corresponds to a (-1)-curve) or to a (-2)-curve with a type $\mathbf{D}_{k+1}$ singular point on it (if the last vertex corresponds to a (-4)-curve); in the latter case, the remaining (-2)-curve is the exceptional section $E \subset \Sigma_2$.

The converse statement is proved by analyzing a local equation. Q.E.D.

For completeness, we also consider the case of type $\mathbf{D}$ and $\mathbf{E}$ singular points.

Lemma 2.3.4. *Let P be a c-invariant singular point of B of type $\mathbf{D}$ or $\mathbf{E}$, and let $\bar{P} \in \Sigma_2$ be its image. Then $\tilde{c}_*$ acts nontrivially on D_P; in particular, P is of type $\mathbf{D}_q$, $q \geqslant 4$, or $\mathbf{E}_6$. Furthermore, $P \in L$ and $\bar{P}$ is, respectively, a type $\mathbf{D}_{2q-2}$ or $\mathbf{E}_7$ singular point of $\bar{B} + \bar{L}$; in the former case, $\bar{P}$ is a simple node of $\bar{B}$ with one of the branches tangent to $\bar{L}$.*

Conversely, if $\bar{P}$ is a cusp of $\bar{B}$ tangent to $\bar{L}$ (a type $\mathbf{E}_7$ singularity of $\bar{B} + \bar{L}$), then the pull-back of $\bar{P}$ is a type $\mathbf{E}_6$ singular point of the double covering B of $\bar{B}$ ramified at $\bar{L} + E$. If $\bar{P}$ is a simple node of $\bar{B}$

with one of the branches intersecting $\bar{L}$ with multiplicity $r \geqslant 2$, then the pull-back of $\bar{P}$ is a type $\mathbf{D}_{r+2}$ singular point of the double covering B.

Proof. If $\tilde{c}_*$ acted trivially on D_P, then the curve with three neighbors in the diagram would have three fixed points and thus it would be fixed by $\tilde{c}$. Hence, the action is nontrivial. This observation rules out type $\mathbf{E}_7$ and $\mathbf{E}_8$ singular points. The further analysis is completely similar to the proof of Lemma 2.3.3, with the additional simplification that the diagram in Y is asymmetric and, hence, the curve with three neighbors is fixed by c_Y. We omit the details. Q.E.D.

2.4. Construction of the involution

All $\mathbb{D}_{10}$-sextics are described in [3]; any such curve has 'essential' set of singularities $4\mathbf{A}_4$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4$, or $2\mathbf{A}_9$ plus, possibly, a few other singular points of type $\mathbf{A}_1$ or $\mathbf{A}_2$.

Let B be a $\mathbb{D}_{10}$-sextic. Pick a point $P = P_i$ of type $\mathbf{A}_4$ (respectively, a point $P = Q_k$ of type $\mathbf{A}_9$), choose an orientation of its (linear) graph D_P, and denote by $e_{i1}, \ldots, e_{i4}$ (respectively, $f_{k1}, \ldots, f_{k9}$) the vertices of D_P, numbered consecutively according to the chosen orientation. Let e_{ij}^* (respectively, f_{kj}^*) be the dual basis for $\Sigma_P^* \subset \Sigma_P \otimes \mathbb{Q}$. Note that $e_{ij}^* = -e_{i,5-j}^* \bmod \Sigma_P$ and $f_{ij}^* = -f_{i,10-j}^* \bmod \Sigma_P$. According to [3], under an appropriate numbering of the singular points and appropriate orientation of their graphs D_P, the kernel $\mathcal{K}$ of the extension $\tilde{\Sigma} \supset \Sigma$ is the cyclic group $\mathbb{Z}_5$ generated by the residue $\gamma = \bar{\gamma} \bmod \Sigma$, where $\bar{\gamma}$ is given by

$$e_{11}^* + e_{21}^* + e_{32}^* + e_{42}^*, \quad f_{14}^* + e_{11}^* + e_{21}^*, \quad \text{or} \quad f_{14}^* + f_{22}^*$$

(for the set of essential singularities $4\mathbf{A}_4$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4$, or $2\mathbf{A}_9$, respectively).

Define an involution $c_S \colon S \to S$ as follows:

- $h \mapsto h$,
- $x \mapsto x$ for $x \in \Sigma_P$ for P a singular point other than $\mathbf{A}_4$ or $\mathbf{A}_9$,
- $e_{1j} \leftrightarrow e_{2,5-j}$, $e_{3j} \leftrightarrow e_{4,5-j}$, $j = 1, \ldots, 4$,
- $f_{kj} \leftrightarrow f_{k,10-j}$, $j = 1, \ldots, 9$.

It is immediate that c_S acts identically on the p-primary part of $\operatorname{discr} S$ for any prime $p \neq 5$, and the action of c_S on the 5-primary part $\operatorname{discr} S \otimes \mathbb{F}_5$ (which can be regarded as an $\mathbb{F}_5$-vector space) has two dimensional (-1)-eigenspace which contains $\mathcal{K}$ as a maximal isotropic subgroup, so that $\mathcal{K}^\perp/\mathcal{K}$ can be identified with the $(+1)$-eigenspace of c_S. Hence, c_S extends to an involution $\tilde{c}_S \colon \tilde{S} \to \tilde{S}$, the latter acts identically on $\operatorname{discr} \tilde{S}$, and the direct sum $\tilde{c}_S \oplus \mathrm{id}_{S^\perp}$ extends to an involution $\tilde{c}_* \colon H_2(X) \to H_2(X)$.

By construction, $\tilde{c}_*$ preserves h and, since it acts identically on the transcendental lattice $(\operatorname{Pic} X)^\perp \subset \tilde{S}^\perp$, it also preserves classes of holomorphic forms. Furthermore, $\tilde{c}_*$ preserves the positive cone of X. (Recall that the positive cone is an open fundamental polyhedron $V^+ \subset (\operatorname{Pic} X) \otimes \mathbb{R}$ of the group generated by reflections defined by vectors $x \in \operatorname{Pic} X$ with $x^2 = -2$; it is uniquely characterized by the requirement that $V^+ \cdot e > 0$ for any exceptional divisor $e \in \bigcup_P D_P$ and that the closure of V^+ should contain h.) Due to the description of the fine period space of marked $K3$-surfaces given in A. Beauville [1], $\tilde{c}_*$ is induced by a unique involutive automorphism $\tilde{c}\colon X \to X$, which is symplectic and commutes with the deck translation τ. The descent of $\tilde{c}$ to $\mathbb{P}^2$ is the involution c the existence of which is asserted by Theorem 2.1.1.

2.5. Proof of Theorem 2.1.1

The involution $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ is constructed in 2.4. Due to Lemmas 2.3.1–2.3.3 (and the description of the singularities of B and the action of $\tilde{c}_*$ on the set of exceptional divisors, see 2.4), the image $\bar{B} \subset \Sigma_2$ is either a trigonal curve with the set of singularities $2\mathbf{A}_4$ (and then $O \notin B$) or a hyperelliptic curve with the set of singularities $2\mathbf{A}_4$ or $\mathbf{A}_4 \oplus \mathbf{A}_3$. (Recall that, by the definition of a $\mathbb{D}_{10}$-sextic, B is assumed irreducible; hence, it cannot contain the fixed line L as a component.) The latter possibility is ruled out by the fact that the genus of a nonsingular curve in $|2E + 6F|$ is 3. Q.E.D.

§3. Calculation of the groups

3.1. The curve $\bar{B}$

The trigonal curve $\bar{B} \subset \Sigma_2$ with two type $\mathbf{A}_4$ singular points is a maximal trigonal curve in the sense of [5]. Up to automorphism of Σ_2, such a curve is unique; its skeleton $\operatorname{Sk} \subset \mathbb{P}^1$ (see [5]) is shown in Figure 1. One can observe that the skeleton is symmetric with respect to the dotted grey line (the real structure $z \mapsto \bar{z}$ on $\mathbb{P}^1$) and, properly drown, it is also symmetric with respect to the holomorphic involution $z \mapsto -1/z$. Hence, the curve $\bar{B}$ can be chosen real and symmetric with respect to a real holomorphic involution of Σ_2 (see §4 below for explicit equations). Furthermore, all singular fibers of $\bar{B}$ (two cusps and two vertical tangents) are also real.

Alternatively, $\bar{B}$ can be obtained as a birational transform of a plane quartic C with the set of singularities $\mathbf{A}_4 \oplus \mathbf{A}_2$, see Figure 2. (Up to automorphism of $\mathbb{P}^2$, such a quartic is also unique.) In the figure, the line (P_0P_1) is tangent to C at P_0, and the transformation consists in blowing P_0 up twice and blowing down the transform of (P_0P_1) and

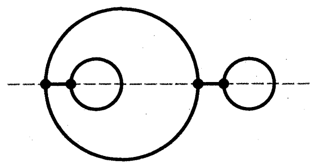


Fig. 1. The skeleton of $\bar{B}$

one of the exceptional divisors over P_0. Lines through P_0 other than (P_0P_1) transform to fibers of Σ_2; this observation gives one a fairly good understanding of the geometry of $\bar{B}$, see, *e.g.*, Figure 4.

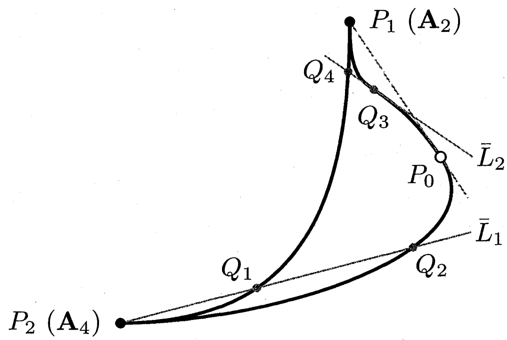


Fig. 2. The quartic C with the set of singularities $\mathbf{A}_4 \oplus \mathbf{A}_2$

3.2. Van Kampen's method

To calculate the fundamental group, we fix a real curve $\bar{B}$ as in 3.1 and choose an appropriate real section $\bar{L}$ intersecting $\bar{B}$ at real points. (The choice of real curves facilitates the computation of the braid monodromy.) We start with applying a version of Zariski–van Kampen's method [12] to the group $\pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E \cup \bar{L}))$. The modifications necessary to use this approach for curves in Σ_2 are explained further in this section. A more detailed and formal account of the observations outlined here (the choice of a section, the relation at infinity, and the fact that one of the braid relations can be ignored) can be found in my recent paper [6].

Let $F_1, \ldots, F_k$ be the singular fibers of $\bar{B} + \bar{L}$ (*i.e.*, the fibers intersecting $\bar{B} + \bar{L}$ at less than four points). Under the assumptions, they are all real. In the figures below, the curves $\bar{B}$ and $\bar{L}$ are shown, respectively, in black and grey, and the singular fibers are the vertical grey dotted lines.

Fix a real nonsingular fiber F_∞ intersecting $\bar{B}$ at one real point, and consider the affine part $\Sigma_2 \smallsetminus (E \cup F_\infty)$, *cf.* Figure 4. Pick a real nonsingular fiber F intersecting $\bar{B}$ at three real points and a generic real section S disjoint from E (*i.e.*, $S \in |E + 2F|$). We identify S with the base of the ruling. Let $x = S \cap F$, $x_\infty = S \cap F_\infty$, and $x_i = S \cap F_i$, $i = 1, \ldots, k$. Assume that S is *proper* in the following sense: there is a segment $I \subset S_\mathbb{R}$ containing x and all x_i, $i = 1, \ldots, k$, and disjoint from $\bar{B}$ and $\bar{L}$. In the figures, we assume that I lies above $\bar{B}$ and $\bar{L}$.

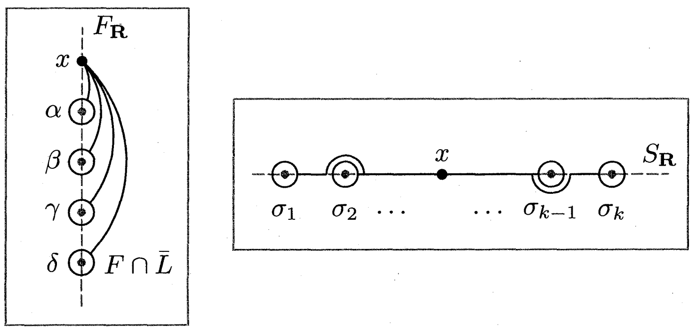


Fig. 3. The basis α, β, γ, δ and the loops σ_i

Remark 3.2.1. The existence of a proper section is straightforward: one can take for S a small real perturbation of the divisor $E + 2F_\infty$. Alternatively, in appropriate affine coordinates (x, y) such that F_∞ is the fiber over $x = \infty$, one can take for S any real section $x = \text{const} \gg 0$. In the terminology of [6], the sections in question are proper sections over a small closed regular neighborhood Δ of the projection of I to the base of the ruling. According to [6], such sections do exist and over Δ they are all homotopic in the class of proper sections.

Let $G = \pi_1(F \smallsetminus (\bar{B} \cup E \cup \bar{L}), x)$, and let α, β, γ, δ be the basis for G shown in Figure 3, left. (All loops are oriented in the counterclockwise direction.) The generator δ plays a special rôle in the sequel (*cf.* Proposition 3.2.7): *we always assume that δ is a loop around $F \cap \bar{L}$.* Let, further, $\sigma_1, \ldots, \sigma_k$ be the basis for the group $\pi_1(S \smallsetminus \{x_1, \ldots, x_k\, x_\infty\}, x)$ shown in Figure 3, right: each σ_i is a small circle about x_i connected to x by a real segment $l_i \subset S_\mathbb{R}$ bent to circumvent the other singular fibers in the counterclockwise direction.

Definition 3.2.2. The *braid monodromy* along a loop σ_i, $i = 1, \ldots, k$, is the automorphism $m_i \colon G \to G$ resulting from dragging F along σ_i while keeping the base point in S. (Since S is proper, m_i is indeed a braid.)

Denote by $\rho \in G$ the class of a large circle in $F \smallsetminus (\bar{B} \cup E \cup \bar{L})$ encompassing all points of intersection of F and $\bar{B} \cup \bar{L}$. In the basis shown in Figure 3, left, one has $\rho = \alpha\beta\gamma\delta$; see also Remark after Proposition 3.2.3.

Proposition 3.2.3. *The group* $\Pi = \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E \cup \bar{L}))$ *is given by*

$$\Pi = \bigl\langle \alpha, \beta, \gamma, \delta \bigm| m_i = \mathrm{id},\ i = 1, \ldots, k, \quad \rho^2 = 1 \bigr\rangle, \tag{3.2.4}$$

where each braid relation $m_i = \mathrm{id}$ *should be understood as a quadruple of relations* $m_i(\alpha) = \alpha$, $m_i(\beta) = \beta$, $m_i(\gamma) = \gamma$, $m_i(\delta) = \delta$.

Remark 3.2.5. Sometimes, in order to satisfy the assumption that δ corresponds to the point $F \cap \bar{L}$, the generators α, β, γ, δ should be reordered by inserting δ between α and β or β and γ, *cf.* 3.3 and 3.4. This order is important for the precise form of the expression for ρ, *i.e.*, for the relation at infinity in Proposition 3.2.3. However, in each case we show that, modulo the braid relations, δ commutes with all subsequent generators and in (3.2.4) one can still assume that $\rho = \alpha\beta\gamma\delta$.

Proof of Proposition 3.2.3. The presentation (3.2.4) is the essence of van Kampen's method, see [12], applied to the ruling of Σ_2. The only statement that needs proof is the *relation at infinity* $\rho^2 = 1$, resulting from the patching of the fiber at infinity F_∞. This relation is $[\partial D] = 1$, where $D \subset S$ is a small disk around $F_\infty \cap S$. If the base fiber F is sufficiently close to F_∞, then $[\partial D] = \rho^2$, as in this case one can take for S a small perturbation of $E + 2F$. In general, one can drag F to a fiber F' close to ∞, transforming a large circle in F to one in F' and keeping it away from $\bar{B} \cup E \cup \bar{L}$. On the other hand, due to the properness of S, the translation homomorphism between any two nonsingular fibers is a braid; hence, it leaves the product $\alpha\beta\gamma\delta$ (in the appropriate order, the the remark above) invariant and the resulting relation has the same form for any fiber F. Q.E.D.

In sections 3.3–3.6 below, we attempt to calculate Π using Proposition 3.2.3. To find the braid monodromy m_i, we represent it as the *local braid monodromy* along a small circle surrounding x_i, conjugated by the translation homomorphism along the real path l_i connecting x_i to x. The former is well known: it can be found by considering model equations. For the latter, we choose the models so that, at each moment, all but at most two points of the curve are real; in this case, the resulting braids are written down directly from the pictures.

The following lemma facilitates the calculation by reducing the number of fibers to be considered.

Lemma 3.2.6. *In the presentation* (3.2.4), (*any*) *one of the braid relations* $m_i = \mathrm{id}$ *can be ignored.*

Proof. The product $\sigma_1 \ldots \sigma_k$ is the class of a large circle encompassing all singular fibers. (Recall that F_∞ is assumed nonsingular.) Hence, $m_k \circ \ldots \circ m_1$ is the so called *monodromy at infinity*, which is known to be the conjugation by $(\alpha\beta\gamma\delta)^2$. In view of the last relation in (3.2.4), each m_i is a composition of the others. Q.E.D.

For the rest of this section, we fix the notation α, β, γ, δ for generators of the group $\Pi = \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E \cup \bar{L}))$, chosen as explained above. The generator δ plays a special rôle in the passage to $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$.

Proposition 3.2.7. *The fundamental group* $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$ *is the kernel of the homomorphism* $\Pi/\delta^2 \to \mathbb{Z}_2$, $\alpha, \beta, \gamma \mapsto 0$, $\delta \mapsto 1$.

Remark 3.2.8. Here and below, Π/δ^2 stands for the quotient of Π by the normal subgroup normally generated by δ^2. In other words, one adds an extra relation $\delta^2 = 1$ to the presentation (3.2.4).

Proof of Proposition 3.2.7. The statement is an immediate consequence of the construction: one considers the appropriate double covering of $\Sigma_2 \smallsetminus (\bar{B} \cup E \cup \bar{L})$ (the one 'ramified at $\bar{L} + E$') and patches L upstairs. Q.E.D.

Lemma 3.2.9. *If* δ *is a central element of* Π/δ^2, *then* $\pi_1(\mathbb{P}^2 \smallsetminus B) = \mathbb{D}_{10} \times \mathbb{Z}_3$.

Proof. Since δ is central, the relation at infinity $(\alpha\beta\gamma\delta)^2 = 1$ (or similar, see Remark after Proposition 3.2.3) turns into $(\alpha\beta\gamma)^2 = 1$ in Π/δ^2. Furthermore, each braid relation has the form $w^{-1}xw = y$ or $w^{-1}\delta w = \delta$, where x and y are among α, β, γ and w is a word in α, β, γ, δ. (Recall that δ is the generator corresponding to the separate component $\bar{L}$ of $\bar{B} + \bar{L}$.) Since δ is a central element, it can be cancelled out, turning each such relation either to a tautology or to a braid relation for the group $\bar{\pi} := \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E)) = \mathbb{D}_{10} \times \mathbb{Z}_3$ (the latter group is calculated in [4]). Furthermore, after this simplification, each relation for Π/δ^2 contains either only α, β, γ or only δ and, using the fact that δ is central again, one concludes that Π/δ^2 is the direct product of its subgroup $\mathbb{Z}_2$ generated by δ and the subgroup generated by α, β,

γ, the latter being isomorphic to $\bar{\pi}$ *via* the isomorphism sending each generator α, β, γ of Π/δ^2 to the generator of $\bar{\pi}$ of the same name. (As shown above, in these generators the two groups have the same set of relations.) Hence, $\Pi/\delta^2 = (\mathbb{D}_{10} \times \mathbb{Z}_3) \times \mathbb{Z}_2$, and Proposition 3.2.7 applies. Q.E.D.

Corollary 3.2.10. *Let B be a $\mathbb{D}_{10}$-sextic with the set of singularities $4\mathbf{A}_4$. Then one has $\pi_1(\mathbb{P}^2 \smallsetminus B) = \mathbb{D}_{10} \times \mathbb{Z}_3$.*

Proof. The curve B is the double covering of $\bar{B}$ ramified at a section $\bar{L}$ transversal to $\bar{B}$. In this case, δ is a central element of the group $\pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E \cup \bar{L}))$ (*cf.* Section 3.3 and Figure 4 below for a much less generic situation), and the statement follows from Lemma 3.2.9. Q.E.D.

Theorem 3.2.11. *Let $\bar{B} \subset \Sigma_2$ be a trigonal curve with two type $\mathbf{A}_4$ singular points, and let $p\colon \mathbb{P}^2 \to \Sigma_2/E$ be the double covering ramified at the vertex E/E and a section $\bar{L}$ disjoint from E. Then $p^{-1}(\bar{B})$ is a generalized $\mathbb{D}_{10}$-sextic.*

Proof. By perturbing $\bar{L}$ to a section transversal to $\bar{B}$, one perturbs B to a generic $\mathbb{D}_{10}$-sextic as in Corollary 3.2.10. Q.E.D.

3.3. The set of singularities $2\mathbf{A}_9$

The sextic B is the double covering of $\bar{B}$ ramified at a section $\bar{L}$ passing through both cusps of $\bar{B}$, see Figure 4; one can take for $\bar{L}$ the transform of the secant $\bar{L}_1$ shown in Figure 2.

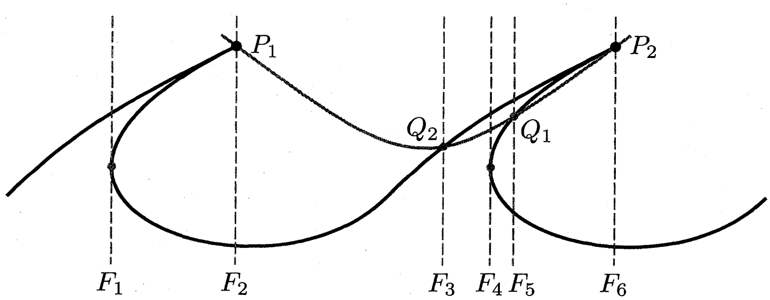


Fig. 4. The set of singularities $2\mathbf{A}_9$

We choose the generators α, δ, β, γ in a nonsingular fiber between F_4 and F_5. Then, there are relations $[\delta, \beta] = 1$, $\beta = \gamma$, and $[\delta, \alpha] = 1$ (from F_5, F_4, and F_3, respectively); due to Lemma 3.2.9, the group is $\mathbb{D}_{10} \times \mathbb{Z}_3$.

3.4. The set of singularities $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$

The curve B is the double covering of $\bar{B}$ ramified at a section $\bar{L}$ double tangent to $\bar{B}$, see Figure 5. (The existence of a double tangent section whose position with respect to $\bar{B}$ is as shown in the figure is rather obvious geometrically: one moves a sufficiently sharp parabola to achieve two tangency points. An explicit construction of a pair $(\bar{B}, \bar{L})$ using equations is found in Section 4.7 and Figure 11 below.)

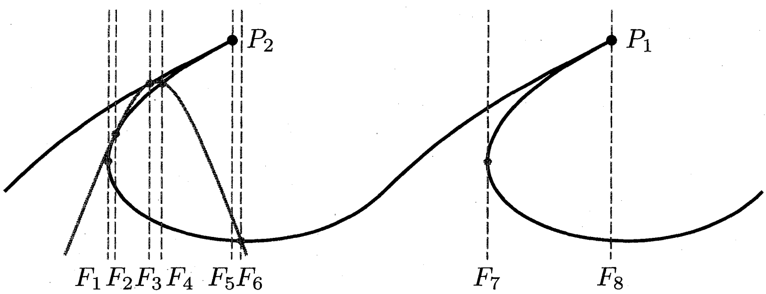


Fig. 5. The set of singularities $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$

We choose the generators α, β, δ, γ in a nonsingular fiber between F_4 and F_5. Ignoring the cusp F_8, see Lemma 3.2.6, the relations for Π are

(3.4.1)	$(\alpha\beta)^2\alpha = \beta(\alpha\beta)^2$	(from the cusp F_5),
(3.4.2)	$[\beta, \delta] = [\gamma, \delta] = 1$	(from F_4 and F_6),
(3.4.3)	$(\alpha\delta)^2 = (\delta\alpha)^2$	(from F_3),
(3.4.4)	$(\alpha^{-1}\delta\alpha\beta)^2 = (\beta\alpha^{-1}\delta\alpha)^2$	(from F_2),
(3.4.5)	$\gamma = (\alpha^{-1}\delta\alpha)^{-1}\beta(\alpha^{-1}\delta\alpha)$	(from the tangent F_1),
(3.4.6)	$\gamma^{-1}\alpha\beta\alpha^{-1}\gamma = (\alpha\beta)\alpha(\alpha\beta)^{-1}$	(from the tangent F_7),
(3.4.7)	$(\alpha\beta\delta\gamma)^2 = 1$	(patching F_∞).

(The relations are simplified using (3.4.2).) The corresponding group π given by Proposition 3.2.7 was analyzed using the `GAP` software package [10], see Figure 6. (To simplify the input, we analyze the group Π/δ^2; the passage to the index 2 subgroup π is immediate.) According to `GAP`, π is an iterated semi-direct product $\mathbb{Z}_3 \times ((((\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2)$, and the abelian factors of its derived series are as stated in Theorem 1.2.1. (Here, Q_8 is the order 8 subgroup $\{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$.)

```
gap> f:=FreeGroup("a","b","c","d");;
gap> g:=f / [
> (f.1*f.2)^2*f.1/(f.2*(f.1*f.2)^2),
> f.2*f.4/(f.4*f.2), f.3*f.4/(f.4*f.3),
> (f.1*f.4)^2/(f.4*f.1)^2,
> (f.1^-1*f.4*f.1*f.2)^2/(f.2*f.1^-1*f.4*f.1)^2,
> (f.1^-1*f.4*f.1)^-1*f.2*(f.1^-1*f.4*f.1)/f.3,
> f.3^-1*f.1*f.2*f.1^-1*f.3/((f.1*f.2)*f.1*(f.1*f.2)^-1),
> (f.1*f.2*f.3*f.4)^2,
> f.4^2];
<fp group on the generators [ a, b, c, d ]>
gap> Size(g);
1920
gap> DerivedSeriesOfGroup(g);
[ <fp group of size 1920 on the generators [ a, b, c, d ]>,
  Group([ b*a^-1, b^-1*a, c*a^-1, c^-1*a, a*b*a^-2, a*c*a^-2,
d*a*d^-1*a^-1 ])
    , Group([ c*b^-1, c^-1*b, a*c*b^-1*a^-1, a*c^-1*b*a^-1, a^-1*b^2*a^-1,
      a^-1*c*b*a^-1, a^-1*c^-1*b*a, b^2*a^-2, b*c*a^-2, b^-1*a^2*b^-1,
      b^-2*a^2, b^-1*c^-1*a^2, d*a*d^-1*a^-1, a^2*c*b^-1*a^-2, a*b^2*a^-3,
      a*b^-1*a^2*b^-1*a^-1, a^-2*b^-2*a^-2, a^-1*b^-1*a^2*b*a^-1,
      a^-1*b^-1*a^-2*b^-1*a^-1 ]), Group(<60 generators>),
  Group(<111 generators>) ]
gap> List(DerivedSeriesOfGroup(g),Size);
[ 1920, 160, 32, 2, 1 ]
gap> List(DerivedSeriesOfGroup(g),AbelianInvariants);
[ [ 2, 2, 3 ], [ 5 ], [ 2, 2, 2, 2 ], [ 2 ], [  ] ]
gap> StructureDescription(g);
"C6 x ((((C2 x Q8) :  C2) :  C5) :  C2)"
```

Fig. 6. GAP output for the set of singularities $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$

3.5. The set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$

The curve B is the double covering of $\bar{B}$ ramified at a section $\bar{L}$ inflection tangent to $\bar{B}$, see Figure 7. One can take for $\bar{L}$ the transform of the tangent $\bar{L}_2$ shown in Figure 2; an explicit construction using equations is found in Section 4.8 and Figure 11.

We choose the generators α, β, γ, δ in a nonsingular fiber between F_4 and F_5. Ignoring the vertical tangent F_1, see Lemma 3.2.6, the relations for Π are

(3.5.1)	$\beta = \gamma$	(from the tangent F_4),
(3.5.2)	$[\alpha, \gamma\delta\gamma^{-1}] = 1$	(from F_3),
(3.5.3)	$(\gamma\delta)^3 = (\delta\gamma)^3$	(from F_5),
(3.5.4)	$[\alpha\beta, \delta_1] = 1$	(from the cusp F_6),

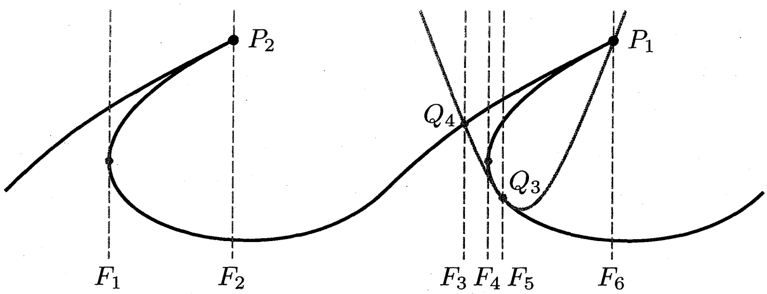


Fig. 7. The set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$

(3.5.5) $\qquad \delta_1(\alpha\beta)^2\alpha = \beta(\alpha\beta)^2\delta_1 \qquad$ (from the cusp F_6),

(3.5.6) $\qquad (\beta_2\gamma)^2\beta_2 = \gamma(\beta_2\gamma)^2 \qquad$ (from the cusp F_2),

(3.5.7) $\qquad (\alpha\beta\gamma\delta)^2 = 1 \qquad$ (patching F_∞),

where $\delta_1 = (\gamma\delta\gamma)\delta(\gamma\delta\gamma)^{-1}$ and $\beta_2 = (\alpha\gamma\delta^{-1}\gamma^{-1})\beta(\alpha\gamma\delta^{-1}\gamma^{-1})^{-1}$. The resulting group π, see Proposition 3.2.7, was analyzed using GAP [10], see Figure 8. Its derived series is as stated in Theorem 1.2.1.

3.6. The sets of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$ and $4\mathbf{A}_4 \oplus \mathbf{A}_2$

For the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, we perturb the inflection tangency point Q_3 in Figure 7 to a simple tangency point and a point of transversal intersection. Then, (3.5.3) is replaced with $[\gamma, \delta] = 1$ and one obtains $[\alpha, \delta] = 1$ (from (3.5.2)), $\delta_1 = \delta$, and $[\beta, \delta] = 1$ (from (3.5.4)); due to Lemma 3.2.9, the resulting group π is $\mathbb{D}_{10} \times \mathbb{Z}_3$.

For the set of singularities $4\mathbf{A}_4 \oplus \mathbf{A}_2$, the intersection point P_1 in Figure 7 is perturbed to two points of transversal intersection. Then, (3.5.4) and (3.5.5) turn into $[\alpha, \delta_1] = [\beta, \delta_1] = 1$ and $(\alpha\beta)^2\alpha = \beta(\alpha\beta)^2$, respectively. Using GAP [10] shows that the resulting group π is $\mathbb{D}_{10} \times \mathbb{Z}_3$. (Since the group in question factors to the group $\mathbb{D}_{10} \times \mathbb{Z}_3$ corresponding to the set of singularities $4\mathbf{A}_4$, see Corollary 3.2.10, it suffices to show that the order of Π/δ^2 equals 60; for details, see Figure 9.)

3.7. The sets of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4$ and $4\mathbf{A}_4 \oplus \mathbf{A}_1$

These sextics can be obtained by small perturbations from sextics with the sets of singularities, *e.g.*, $2\mathbf{A}_9$ (see 3.3) and $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$ (see 3.6), respectively; the resulting groups have already been shown to be isomorphic to $\mathbb{D}_{10} \times \mathbb{Z}_3$.

```
gap> f:=FreeGroup("a","b","c","d");;
gap> g:=f / [
> f.2/f.3,
> f.1*f.3*f.4*f.3^-1/(f.3*f.4*f.3^-1*f.1),
> (f.3*f.4)^3/(f.4*f.3)^3,
> f.1*f.2*(f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1/
> ((f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1*f.1*f.2),
> (f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1*(f.1*f.2)^2*f.1/
> ((f.2*f.1)^2*f.2*(f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1),
> ((f.1*f.2*f.4^-1)*f.2*(f.1*f.2*f.4^-1)^-1*f.3)^2*
> (f.1*f.2*f.4^-1)*f.2*(f.1*f.2*f.4^-1)^-1/
> (f.3*((f.1*f.2*f.4^-1)*f.2*(f.1*f.2*f.4^-1)^-1*f.3)^2),
> (f.1*f.2*f.3*f.4)^2,
> f.4^2];
<fp group on the generators [ a, b, c, d ]>
gap> Size(g);
43200
gap> DerivedSeriesOfGroup(g);
[ <fp group of size 43200 on the generators [ a, b, c, d ]>,
  Group([ b*a^-1, b^-1*a, a*b*a^-2, a^-1*b^-1*a^2, d*a*d^-1*a^-1,
      d*a^-1*d^-1*a, d*b*d^-1*a^-1 ]), Group(<18 generators>) ]
gap> List(DerivedSeriesOfGroup(g),Size);
[ 43200, 3600, 720 ]
gap> List(DerivedSeriesOfGroup(g),AbelianInvariants);
[ [ 2, 2, 3 ], [ 5 ], [  ] ]
```

Fig. 8. GAP output for the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$

§4. The equations

The calculations in this section (substitution, factorization, discriminants, and system solving) are straightforward but rather tedious. Most calculations were performed using Maple.

4.1. A few remarks

To simplify the equations below, we choose an affine chart of Σ_2 so that one of the type $\mathbf{A}_4$ singular points of $\bar{B}$ is over $x = \infty$. This should not be confused with the fiber F_∞ in §3, where the equations of this section are not used. If the equations were to be used for plotting, one would need to apply an appropriate Möbius transformation in the x-coordinate first. Alternatively, using the more formal approach of [6], one can argue that the braid monodromy can be computed over a closed disk containing all but one singular fibers of $\bar{B} + \bar{L}$; hence, one of the singular fibers can be placed over $x = \infty$ and ignored.

```
gap> f:=FreeGroup("a","b","c","d");;
gap> g:=f / [
> f.2/f.3,
> f.1*f.3*f.4*f.3^-1/(f.3*f.4*f.3^-1*f.1),
> (f.3*f.4)^3/(f.4*f.3)^3,
> f.1*f.2*(f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1/
> ((f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1*f.1*f.2),
> (f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1*(f.1*f.2)^2*f.1/
> ((f.2*f.1)^2*f.2*(f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1),
> ((f.1*f.2*f.4^-1)*f.2*(f.1*f.2*f.4^-1)^-1*f.3)^2*
> (f.1*f.2*f.4^-1)*f.2*(f.1*f.2*f.4^-1)^-1/
> (f.3*((f.1*f.2*f.4^-1)*f.2*(f.1*f.2*f.4^-1)^-1*f.3)^2),
> (f.1*f.2*f.3*f.4)^2,
> f.1*(f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1/
> ((f.3*f.4*f.3)*f.4*(f.3*f.4*f.3)^-1*f.1),
> f.4^2];
<fp group on the generators [ a, b, c, d ]>
gap> Size(g);
60
gap> DerivedSeriesOfGroup(g);
[ <fp group of size 60 on the generators [ a, b, c, d ]>,
Group([ b*a^-1, b^-1*a, a*b*a^-2, a^-1*b^-1*a^2, d*a*d^-1*a^-1,
d*a^-1*d^-1*a, d*b*d^-1*a^-1 ]), Group(<17 generators>) ]
gap> List(DerivedSeriesOfGroup(g),Size);
[ 60, 5, 1 ]
gap> List(DerivedSeriesOfGroup(g),AbelianInvariants);
[ [ 2, 2, 3 ], [ 5 ], [ ] ]
gap> StructureDescription(g);
"C6 x D10"
```

Fig. 9. GAP output for the set of singularities $4\mathbf{A}_4 \oplus \mathbf{A}_2$

As a by-product, we show that the equisingular moduli spaces of all $\mathbb{D}_{10}$-sextics are unirational. (Indeed, in each case below, it is straightforward that the conditions imposed on the triples (a, b, c) define a rational subvariety in $\mathbb{C}^3$.) In particular, each equisingular moduli space is connected and nonempty. These facts were proved in [3] arithmetically; the existence of $\mathbb{D}_{10}$-sextics realizing each of the eight sets of singularities listed in Section 1.2 can also be derived from the results of J.-G. Yang [14], using the arithmetic characterization of $\mathbb{D}_{2n}$-sextics found in [3].

4.2. The curve $\bar{B}$

In appropriate affine coordinates (x, y) in Σ_2 the trigonal curve $\bar{B}$ with two type $\mathbf{A}_4$ singular points is given by the Weierstraß equation

$$f(x, y) = 4y^3 - 3yp(x) + q(x) = 0, \text{ where} \tag{4.2.1}$$

$$p(x) = x^4 - 12x^3 + 14x^2 + 12x + 1,$$
$$q(x) = (x^2 + 1)(x^4 - 18x^3 + 74x^2 + 18x + 1).$$

The discriminant of (4.2.1) with respect to y is

$$\Delta = (2)^{10}(3)^6 x^5 (x^2 - 11x - 1);$$

it has two 5-fold roots $x = 0$ and $x = \infty$ (the singular points of $\bar{B}$) and two simple roots $x_\pm = 11/2 \pm 5\sqrt{5}/2$ (the two vertical tangents).

Remark 4.2.2. The point $x = \infty$ counts as a 5-fold root of Δ as the 'predicted' degree of Δ is 12. Originally, equation (4.2.1) was obtained by an appropriate birational coordinate change from the equation

$$y^2 - 2x^2 y + x^4 - 4x^3 y$$

of the quartic with the set of singularities $\mathbf{A}_4 \oplus \mathbf{A}_2$, see Figure 2.

The curve $\bar{B}$ is rational; it can be parametrized as follows

$$(4.2.3)\quad x(t) = \frac{t^2(t-1)}{t+1}, \qquad y(t) = \frac{(t^2+1)(t^4 - 2t^3 - 6t^2 + 2t + 1)}{2(t+1)^2}.$$

The special points on the curve correspond to the following values of the parameter:

$$(4.2.4)\quad \begin{array}{ll} t_0 = 0, \quad t_\infty = \infty & \text{(the cusps)}, \\ t'_0 = 1, \quad t'_\infty = -1 & \text{(the points under the cusps)}, \\ t_\pm = -\frac{1}{2} \mp \frac{1}{2}\sqrt{5} & \text{(the tangency points over } x = x_\pm), \\ t'_\pm = 2 \pm \sqrt{5} & \text{(the other points over } x = x_\pm). \end{array}$$

Both the curve and the parametrization are real, as are all singular fibers of $\bar{B}$. Furthermore, $\bar{B}$ is invariant under the automorphism $x \mapsto -1/x$, $y \mapsto y/x^2$. In the t-line, this transformation corresponds to the automorphism $t \mapsto -1/t$.

4.3. Generic sextics

Due to Theorem 2.1.1, any $\mathbb{D}_{10}$-sextic is given by an affine equation of the form

$$(4.3.1)\qquad f(x, y^2 + ax^2 + bx + c) = 0,$$

where $f(x, y)$ is the polynomial given by 4.2.1 and $y = ax^2 + bx + c$ is the equation of the section $\bar{L}$ constituting the branch locus. Conversely, from Theorem 3.2.11 it follows that any curve B given by (4.3.1) is a $\mathbb{D}_{10}$-sextic provided that it is irreducible and all its singularities are simple. Note that B is reducible (splits into two cubics interchanged by the involution on $\mathbb{P}^2$) if and only if, at each point of intersection of $\bar{B}$ and $\bar{L}$, the local intersection index is even. Hence, in view of the classification of sections given below, B is reducible if and only if it has the (non-simple) set of singularities $\mathbf{Y}_{1,1}^1 \oplus \mathbf{A}_9$, see 4.4; such a curve splits into two cubics with a common cusp.

The set of singularities of a sextic B given by (4.3.1) with a generic triple (a, b, c) (so that $\bar{L}$ is transversal to $\bar{B}$) is $4\mathbf{A}_4$. In Sections 4.4–4.8 below, we discuss the possible degenerations of the section $\bar{L}$ and express them in terms of the triple (a, b, c). (Sometimes, the condition is stated using an extra parameter t, as an attempt to eliminate t results in a multi-line `Maple` output.) For each degeneration, we use Lemmas 2.3.2 and 2.3.3 to indicate the set of singularities of the corresponding sextic B. The results should be understood as follows: a sextic B given by (4.3.1) has a certain set of singularities Σ if and only if the triple (a, b, c) satisfies the condition corresponding to Σ but does not satisfy any condition corresponding to an immediate degeneration of Σ (see the adjacency diagram shown in Figure 10).

$$\begin{tikzcd}
 & & (\mathbf{W}_{12} \oplus 2\mathbf{A}_4) \arrow[d] & \\
 & & (\mathbf{Y}_{1,1}^1 \oplus 2\mathbf{A}_4) \arrow[d] & (\mathbf{Y}_{1,1}^1 \oplus \mathbf{A}_9) \arrow[l] \arrow[d] \\
 & 4\mathbf{A}_4 & \mathbf{A}_9 \oplus 2\mathbf{A}_4 \arrow[l] & 2\mathbf{A}_9 \arrow[l] \\
4\mathbf{A}_4 \oplus 2\mathbf{A}_1 \arrow[r] & 4\mathbf{A}_4 \oplus \mathbf{A}_1 \arrow[u] & \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1 \arrow[l] \arrow[u] & \\
 & 4\mathbf{A}_4 \oplus \mathbf{A}_2 \arrow[u] & \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2 \arrow[l] \arrow[u] &
\end{tikzcd}$$

Fig. 10. Immediate adjacencies of sets of singularities

For completeness, we also mention (parenthetically in Figure 10) the sextics B given by (4.3.1) whose singularities are not simple; this is the case if and only if the triple (a,b,c) is as in (4.4.2) below. In Arnol'd's notation, B may only have a non-simple singular point of one of the following two types: $\mathbf{Y}^1_{1,1}$ (transversal intersection of two cusps) or $\mathbf{W}_{12}$ (semiquasihomogeneous singularity of type $(4,5)$).

4.4. Sections through singular points

A section $y = ax^2 + bx + c$ passes through one of the cusps of $\bar{B}$ (the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4$) if and only if
(4.4.1)
$$c = \frac{1}{2} \quad \text{(the cusp at } t = 0) \qquad \text{or} \qquad a = \frac{1}{2} \quad \text{(the cusp at } t = \infty).$$

Hence, the section passes through both cusps (the set of singularities $2\mathbf{A}_9$) if and only if $a = c = 1/2$.

Further degenerations considered here produce sextics with a non-simple singular point. The section is tangent to $\bar{B}$ at a cusp (the set of singularities $\mathbf{Y}^1_{1,1} \oplus 2\mathbf{A}_4$) if and only if
(4.4.2)
$$c = \frac{1}{2},\ b = 3 \quad \text{(at } t = 0) \qquad \text{or} \qquad a = \frac{1}{2},\ b = -3 \quad \text{(at } t = \infty).$$

It passes through the other cusp (the set of singularities $\mathbf{Y}^1_{1,1} \oplus \mathbf{A}_9$; this sextic is reducible) if and only if $a = c = 1/2$ and $b = \pm 3$. Finally, the section passes through a cusp with local intersection index 5 (the set of singularities $\mathbf{W}_{12} \oplus 2\mathbf{A}_4$) if and only if

$$(4.4.3) \qquad (a,b,c) = \left(-\frac{11}{2}, 3, \frac{1}{2}\right) \quad \text{or} \quad (a,b,c) = \left(\frac{1}{2}, -3, -\frac{11}{2}\right).$$

Such a section cannot pass through the other cusp.

A section passing through both cusps of $\bar{B}$ or tangent to $\bar{B}$ at a cusp does not admit any degenerations other than described above. Indeed, if $a = c = 1/2$ (respectively, $c = 1/2$ and $b = 3$), then, restricting the original polynomial $f(x,y)$ to the section and reducing the trivial factor x^2 (respectively, x^4), one obtains a polynomial in x whose discriminant is $16(b-3)^3(b+3)^3$ (respectively, $12(2a-1)^5$).

Remark 4.4.4. According to [3], $\mathbb{D}_{10}$-sextics are characterized by the existence of two conics in a very special position with respect to the type $\mathbf{A}_4$ and $\mathbf{A}_9$ singular points of the curve. These conics are the pull-backs of the two sections $y = ax^2 + bx + c$ with $a = c = 1/2$ and $b = \pm 3$, each section being tangent to $\bar{B}$ at one of its cusps and passing through the other cusp.

4.5. Digression: other generalized $\mathbb{D}_{10}$-sextics

From 4.4, it follows that the double covering construction also produces representatives of the two families of irreducible generalized $\mathbb{D}_{10}$-sextics with a quadruple singular point, see [3]. (In each family, symmetric curves form a codimension one subset.) It is worth mentioning that the remaining classes of irreducible generalized $\mathbb{D}_{10}$-sextics, those with the sets of singularities $\mathbf{J}_{2,0} \oplus 2\mathbf{A}_4$, $\mathbf{J}_{2,1} \oplus 2\mathbf{A}_4$, and $\mathbf{J}_{2,5} \oplus \mathbf{A}_4$, see [4], are also related to the trigonal curve $\bar{B} \subset \Sigma_2$ with two type $\mathbf{A}_4$ singular points: they are obtained from $\bar{B}$ by a birational transformation rather than double covering.

4.6. Simple and double tangents

Let

$$s(t) = ax^2(t) + bx(t) + c, \tag{4.6.1}$$

where $x(t)$ is given by (4.2.4). Solving $s(t) = y(t)$ and $s'(t) = y'(t)$, one concludes that a section $y = ax^2 + bx + c$ is tangent to $\bar{B}$ at a point $(x(t), y(t)) \in \bar{B}$ (the set of singularities $4\mathbf{A}_4 \oplus \mathbf{A}_1$) if and only if

$$\begin{aligned} a &= \frac{-b(t^3+2t^2-1)+t^5-5t^3-5t^2-3}{2t^2(t-1)(t^2+t-1)}, \\ c &= -\frac{bt^2(t^3-2t+1)+3t^5+5t^3-5t^2+1}{2(t+1)(t^2+t-1)} \end{aligned} \tag{4.6.2}$$

for some $b \in \mathbb{C}$ and $t \in \mathbb{C} \smallsetminus \{0, \pm 1, t_\pm\}$ or

(4.6.3)

$$t = 1 \text{ and } (b, c) = (-6, -1) \qquad \text{or} \qquad t = -1 \text{ and } (a, b) = (-1, 6).$$

This section passes through the cusp of $\bar{B}$ at $t = 0$ (the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$) if and only if $c = 1/2$, see 4.4; in this case

(4.6.4)

$$a = \frac{t^4 - t^3 - 2t^2 + 3t + 11}{2(t-1)^2(t^2+t-1)}, \quad b = -\frac{3(t^3+2t-1)}{(t-1)(t^2+t-1)}, \quad c = \frac{1}{2}.$$

The section passes through the cusp of $\bar{B}$ at $t = \infty$ (another implementation of the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$) if and only if $a = 1/2$, see 4.4; in this case

(4.6.5)

$$a = \frac{1}{2}, \quad b = -\frac{3(t^3+2t^2+1)}{(t+1)(t^2+t-1)}, \quad c = -\frac{11t^4 - 3t^3 - 2t^2 + t + 1}{2(t+1)^2(t^2+t-1)}.$$

Relations (4.6.4) and (4.6.5) still hold for the exceptional values $t = -1$ and $t = 1$, respectively, *cf.* (4.6.3).

There is a somewhat unexpectedly simple relation between the two tangency points of a section double tangent to $\bar{B}$. We state it below as a separate lemma. Denote by $\epsilon_\pm$ the roots of the polynomial t^2+3t+1. One has $\epsilon_\pm = (-3\pm\sqrt{5})/2 = t_\pm/t'_\pm$. Note that $\epsilon_+\epsilon_- = 1$.

Lemma 4.6.6. *Let $\bar{B} \subset \Sigma_2$ be the trigonal curve parametrized by (4.2.3), and let $t_1, t_2 \in \mathbb{C}\smallsetminus\{0, t_\pm\}$ be two distinct values of the parameter. Then, there is a section tangent to $\bar{B}$ at both points $(x(t_i), y(t_i))$, $i = 1, 2$, if and only if $t_2/t_1 = \epsilon_\pm$.*

Proof. Assume that $t_1, t_2 \neq \pm 1$. (The case when one of t_1, t_2 takes an exceptional value ± 1 is treated similarly using (4.6.3).) Let a_1, a_2 and c_1, c_2 be the coefficients a and c in (4.6.2) evaluated at $t = t_1, t_2$, respectively. Then $a_1 - a_2 = c_1 - c_2 = 0$; hence, $(a_1-a_2)t_1^2t_2^2(t_1-1)(t_2-1) + (c_1-c_2)(t_1+1)(t_2+1) = 0$. The latter expression takes the form

$$\frac{3(t_1-t_2)^3(t_1^2+3t_1t_2+t_2^2)}{(t_1^2+t_1-1)(t_2^2+t_2-1)} = 0,$$

and, taking into account the restrictions on t_1, t_2, one obtains $t_2/t_1 = \epsilon_\pm$. For the converse statement, one observes that, if $t_2/t_1 = \epsilon_\pm$, then the linear system $a_1 = a_2$, $c_1 = c_2$ in one variable b has a solution (given by (4.6.7) below). Q.E.D.

Thus, a section $y = ax^2 + bx + c$ is double tangent to $\bar{B}$ (the set of singularities $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$) if and only if

$$(4.6.7) \qquad b = b_\pm = -\frac{3(t^2+(3\epsilon_\mp+1)t-\epsilon_\mp)(t^2-\epsilon_\pm t-\epsilon_\mp)(t+t_\pm)}{(t-t_\pm)(t-t_\mp)^2(t-t'_\pm)^2}.$$

Here, the two tangency points are at t and $\epsilon_\pm t$; the expressions for a and c are obtained by a direct substitution to (4.6.2). This relation still holds if $t = \pm 1$.

A section tangent to $\bar{B}$ at a smooth point does not admit any other degenerations. Indeed, sections passing through both singular points of $\bar{B}$ or tangent to $\bar{B}$ at a singular point are considered in 4.4, and sections inflection tangent to $\bar{B}$ are treated in 4.8 below. A section cannot be tangent to $\bar{B}$ at three smooth points at $t = t_1$, t_2, and t_3, as then one would have $t_2/t_1 = \epsilon_\pm$, $t_3/t_2 = \epsilon_\pm$, and $t_3/t_1 = \epsilon_\pm$; this system is incompatible unless $t_1 = t_2 = t_3 = 0$. Finally, a double tangent cannot pass through a singular point, say at $t = 0$, as substituting $t = t_1$ and $t = t_2$ to (4.6.4) and eliminating a and b, one obtains a system in (t_1, t_2) which has no solutions with $t_1 \neq t_2$.

4.7. Digression: the curve in Figure 5

The pair $(\bar{B}, \bar{L})$ used to calculate the fundamental group of a sextic B with the set of singularities $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$, see 3.4 and Figure 5, can be obtained from (4.6.7) and (4.6.2) with the following values of the parameters: $t_1 = t = 1/2$ and $t_2 = \epsilon_+ t_1 \approx -0.191$. Then one has $a \approx -161.05$, $b \approx -13.93$, and $c \approx 0.0448$. The two points of transversal intersection of $\bar{B}$ and the section $y = ax^2 + bx + c$ are at $t \approx 0.281$ (over $x \approx -0.0442$) and $t \approx 1.101$ (over $x \approx 0.0585$). This section is shown in Figure 11.

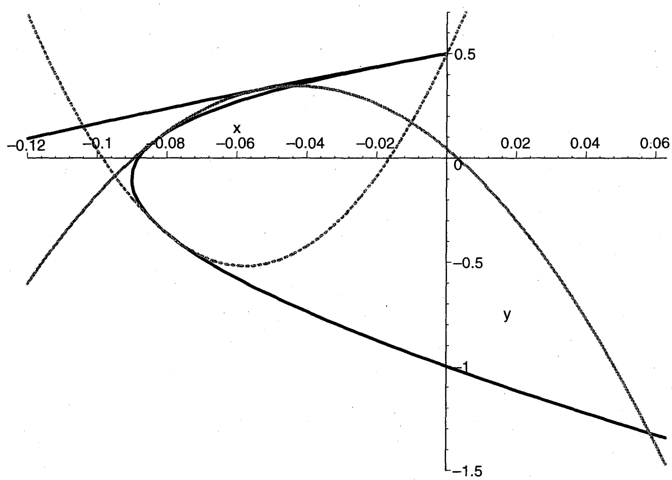


Fig. 11. `Maple` plot of the curve $\bar{B}$ (black), a double tangent (solid grey), and an inflection tangent through a singular point (dotted grey)

4.8. Inflection tangents

A section $y = ax^2 + bx + c$ is inflection tangent to $\bar{B}$ at a point $(x(t), y(t)) \in \bar{B}$ (the set of singularities $4\mathbf{A}_4 \oplus \mathbf{A}_2$) if and only if

$$
\begin{aligned}
a &= \frac{t^6 + 3t^5 - 5t^3 + 12t + 11}{2(t^2 + t - 1)^3}, \\
b &= -\frac{3(t^2 + 1)(t^4 + 3t^3 - t^2 - 3t + 1)}{(t^2 + t - 1)^3}, \\
c &= -\frac{11t^6 - 12t^5 + 5t^3 - 3t + 1}{2(t^2 + t - 1)^3},
\end{aligned} \tag{4.8.1}
$$

$t \in \mathbb{C} \smallsetminus \{0, t_\pm\}$. (To see this, one should solve for (a, b, c) the system $s(t) = y(t)$, $s'(t) = y'(t)$, $s''(t) = y''(t)$, where $s(t)$ is the section given by (4.6.1).) This inflection tangent passes through one of the cusps of $\bar{B}$ (the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$) if and only if $t = 3/4$ (the cusp at $t = 0$) or $t = -4/3$ (the cusp at $t = \infty$). The corresponding values of (a, b, c) are

$$(4.8.2)\quad (a, b, c) = \left(\frac{3077}{10}, \frac{177}{5}, \frac{1}{2}\right) \quad \text{and} \quad (a, b, c) = \left(\frac{1}{2}, -\frac{177}{5}, \frac{3077}{10}\right),$$

respectively. The section corresponding to $t = 3/4$ is shown in Figure 11.

An inflection tangent at a smooth point of $\bar{B}$ cannot have any other degenerations. Indeed, after clearing the denominators and reducing the trivial factor $(u - t)^3$, the equation $y(u) = ax(u)^2 + bx(u) + c$ with a, b, and c given by (4.8.1) and $x(\,\cdot\,)$, $y(\,\cdot\,)$ as in (4.2.3) has solution $u = t$ only for $t = 0$ or $t_\pm$ (hence, no quadruple intersection points), and the discriminant of the above equation with respect to u is, up to a constant coefficient, $t^2(3t+4)(4t-3)(t^2+t-1)^3$ (hence, no other tangency points).

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