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Characteristic classes along the Japanese singularity road

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Abstract.

The paper presents some recent results obtained by singularity researchers in the area of characteristic classes, in fact a promenade along the Japanese singularity road. Main subjects concern characteristic classes of singular varieties, equivariant Chern classes, bivariant Chern classes and motivic characteristic classes.

§1. Introduction

Aim of this paper is to present some results obtained by singularity researchers in the area of characteristic classes, in fact a promenade along the Japanese singularity road. Singularities are developed in Japan in numerous and various fruitful directions.

Heisuke Hironaka gave a fantastic impulse to singularities not only in Japan but also all over the world. Nowadays, it would be very difficult to list all names of Japanese (and other) singularity researchers. I will not provide any list, that is not my purpose and I am neither qualified nor able to do that. I will concentrate myself on a precise subject, the one of characteristic classes of singular varieties and more precisely on the following four items:

(1) Characteristic classes of singular varieties,
(2) Equivariant Chern classes,
(3) Bivariant Chern classes,
(4) Motivic characteristic classes.

A more complete survey by Jörg Schürmann and Shoji Yokura [SY] provides more details. Related subjects are not in the present paper because well developed elsewhere in surveys. That is the case for residues, for indices of vector fields or for the Chern character on singular varieties (see Suwa [Su4, Su6, Su7]). The interested reader will find expositions

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on indices of vector fields and on characteristic classes in the forthcoming books [Br2, BSS]. I would like to thank the referee for suitable and useful comments.

§2. Characteristic classes of singular varieties

2.1. Schwartz–MacPherson classes

The Chern classes of a (compact) manifold M have been defined by various ways, in particular as being the Chern classs of the tangent bundle TM. In the case of a singular analytic variety X, the tangent bundle does not exist any more. There are different notions of characteristic classes on such a variety, corresponding to different generalisations of the tangent bundle.

The classical notion of Chern classes of analytic complex manifolds has been extended to singular algebraic varieties

— firstly (1965) by M.-H. Schwartz [Sch], using the definition of Chern classes by obstruction theory,

— then (1974) by R. MacPherson [MP], using the notion of Nash transformation and the one of Euler local obstruction.

Here we recall the MacPherson setting:

Let us denote by $\mathbb{F}(X)$ the group of constructible functions on the algebraic complex variety X. The correspondence $\mathbb{F} : X \to \mathbb{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathbb{F}(Y) \to \mathbb{F}(X)$ for a morphism $f : X \to Y$. The functor $\mathbb{F}$ can, in fact, be made a covariant functor $f_* : \mathbb{F}(X) \to \mathbb{F}(Y)$ when considering the pushforward defined on characteristic functions $\mathbf{1}_A$ by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \qquad y \in Y, A \subset X$$

and linearly extended to elements of $\mathbb{F}(X)$. The following result was conjectured by Deligne and Grothendieck in 1969 and proved by MacPherson in 1974:

Theorem 2.1. [MP] *Let $\mathbb{F}$ be the covariant functor of constructible functions and let $H_*(\ ;\mathbb{Z})$ be the usual covariant $\mathbb{Z}$-homology functor. There is a unique natural transformation*

$$c_* : \mathbb{F} \to H_*(\ ;\mathbb{Z})$$

satisfying $c_*(\mathbf{1}_X) = c^*(TX) \cap [X]$ if X is a manifold.

Definition 2.1. *One defines the Chern (Schwartz–MacPherson) class of X by*

$$c_*(X) = c_*(\mathbf{1}_X).$$

One of the ingredients of the MacPherson's Chern class transformation $c_* : \mathbb{F} \to H_*(\ ;\mathbb{Z})$ is the Mather class

$$c_*^M(X) = \nu_*(c^*(\widetilde{T}) \cap [\widetilde{X}]), \tag{1}$$

where $\nu : \widetilde{X} \to X$ is the Nash transformation of X and $\widetilde{T}$ the Nash bundle (see [MP]).

2.1.1. *Generalizations and product formula* On the one hand, Michał Kwieciński [Kw] showed that the Schwartz–MacPherson class $c_*(X)$ of a complex algebraic variety X satisfies the cross-product formula

$$c_*(X \times Y) = c_*(X) \times c_*(Y). \tag{2}$$

On the other hand, Shoji Yokura [Y1] gave various generalizations and modifications of c_* mainly by modifying the definition of pushforward of a constructible function by a proper map. In particular in [Y2], he introduced the twisted MacPherson class $c_{t*}(X)$, element of $H_*(X, \mathbb{Z}[t])$, which reduces to $c_*(X)$ when $t = 1$.

Michał Kwieciński and Shoji Yokura [KY] showed that the cross-product formula $c_{t*}(X \times Y) = c_{t*}(X) \times c_{t*}(Y)$ holds for the twisted MacPherson class.

2.1.2. *Verdier–Riemann–Roch theorem* The Chern class satisfies the Verdier–Riemann–Roch theorem, that we describe explicitly below (see [Y4]):

Let $f : X \to Y$ be a smooth map, one denotes by $f^!$ the smooth pullback in homology (see [SGA6]). If Y is a manifold, $f^!$ corresponds to the Gysin map obtained from the contravariant map f^* modulo Poincaré duality maps (here $\bullet \cap [Y]$ is an isomorphism), that is

$$\begin{array}{ccc} H^*(Y) & \xrightarrow{f^*} & H^*(X) \\ \downarrow \bullet\cap[Y] & & \downarrow \bullet\cap[X] \\ H_{n-*}(Y) & \xrightarrow{f^!} & H_{m-*}(X). \end{array} \tag{3}$$

Theorem 2.2. *Let $f : X \to Y$ be a smooth map, T_f the vector bundle of tangent spaces to the fibres of f. Then one has*

$$c_*(f^*\beta) = c^*(T_f) \cap f^! c_*(\beta) \qquad \beta \in \mathbb{F}(Y)$$

i.e. commutativity of the diagram

$$\begin{array}{ccc} \mathbb{F}(Y) & \xrightarrow{f^*} & \mathbb{F}(X) \\ \downarrow c_* & & \downarrow c_* \\ H_*(Y) & \xrightarrow{c^*(T_f)\cap f^!} & H_*(X). \end{array} \tag{4}$$

2.2. Fulton–Johnson classes

Let X be an Isolated Complete Intersection Singularity (ICIS), for example a hypersurface, embedded in a complex analytic manifold M. The tangent bundle to the regular part of X cannot be extended to all of X. Nevertheless, the normal bundle to the regular part of X can be extended as a bundle N defined to all of X. Let us consider the virtual bundle

$$\tau_X = TM|_X - N$$

and define the Fulton–Johnson class [Fu] as the homological Chern class of the virtual bundle:

$$c_*^{FJ}(X) = c_*(\tau_X) := c^*(\tau_X) \cap [X]. \tag{5}$$

Note that, in fact, the Fulton–Johnson classes are defined in a more general context (see [Fu]).

The comparison between Schwartz–MacPherson and Fulton–Johnson classes is a natural question, that has been studied by many authors. That is precisely the subject of the next section.

2.3. Milnor classes

Let X be an ICIS with an isolated singularity at x and let F_x be the Milnor fibre of X at x. It is well-known [Mi] that the Euler–Poincaré characteristic of F_x is equal to

$$\chi(F_x) = 1 + (-1)^n \mu(X, x)$$

where $n = \dim_{\mathbb{C}} X$ and $\mu(X, x)$ is the Milnor number of X at x.

Tatsuo Suwa proved in 1996 the following result using a formula in [SS]

Theorem 2.3. [Su1] *If Let X be a n-dimensional compact complex variety and suppose that the singularities of X are isolated points $\{x_i\}_{1\leq i\leq q}$ where X is a local complete intersection. Then the difference $c_*(X) - c_*^{FJ}(X)$ lies in dimension 0 and is equal to*

$$c_*(X) - c_*^{FJ}(X) = (-1)^{n+1} \sum_{i=1}^{q} \mu(X, x_i)$$

That is a motivation for the following definition:

Definition 2.2. *Let X be a complete intersection, the difference $c_*(X) - c_*^{FJ}(X)$ is called Milnor class of X and denoted by $\mathcal{M}(X)$.*

Let M be a smooth analytic complex manifold of dimension $n+1$ and L a holomorphic line bundle on M. Let $f \in H^0(M, L)$ be a holomorphic section of L, then $X = f^{-1}(0)$ is a hypersurface. Let us define the following constructible functions on X:

$$\chi : X \to \mathbb{Z} \qquad \chi(x) = \chi(F_x)$$

$$\mu : X \to \mathbb{Z} \qquad \mu = (-1)^{n-1}(\chi - \mathbf{1}_X)$$

In a preprint [Y3], an extended and revised version of which was published later as [Y5], Shoji Yokura studied Milnor classes and conjectured the following result, which has been proved by Adam Parusiński and Piotr Pragacz [PP]:

Theorem 2.4. *In the previous situation, let L_X be the restriction of the bundle L to X, then one has*

$$c_*(X) - c_*^{FJ}(X) = c(L_X)^{-1} \cap c_*(\mu).$$

2.3.1. *Localization* The localization technique, using Chern–Weil theory, formulated explicitely by Tatsuo Suwa in his book [Su2], allows one to compute Milnor classes precisely in some cases.

Let X be a locally complete intersection with virtual tangent bundle τ_X. By using suitable stratified vector fields, the so-called radial vector fields defined by M.-H. Schwartz (see [BS, Sch]), one can define [BLSS1] the localized Schwartz class and the localized virtual class, for each component S of the singular locus of X. They lie in $H_*(S)$ and their difference is denoted by $\mu(X, S)$. One has:

Proposition 2.1. [BLSS1] *The Milnor class $\mathcal{M}(X)$ is equal to*

$$\mathcal{M}(X) = \sum_S \mu(X, S). \tag{6}$$

If S is a non-singular l-dimensional component of the singular set of X, the l-th Milnor class $\mu_l(X, S)$ is equal to $\mu_H(p)[S]$, where H denotes a normal slice to S in the ambient space and $\mu_H(p)$ is the Milnor number of $X \cap H$ at $p \in S$.

If X has only isolated singularities, formula 6 is the previous expression of $\mathcal{M}(X)$ as the sum of Milnor numbers of X at singular points (Theorem 2.3).

2.3.2. *Product formula* In [OY], Toru Ohmoto and Shoji Yokura prove a product formula for the Milnor class:

Proposition 2.2. *Let $\{(M_i, E_i, s_i, X_i)\}$, $1 \le i \le r$, be a finite collection of compact complex analytic manifolds M_i of dimension $n_i + k_i$,* *holomorphic vector bundles E_i of rank k_i over M_i, regular holomorphic sections f_i of the bundles E_i and n_i-dimensional local complete intersections X_i, the zeros of the f_i. Then the Milnor class $\mathcal{M}(X_1 \times \cdots \times X_r)$ is given by the sum*

$$\sum(-1)^{n_1\varepsilon_1 + \cdots n_r\varepsilon_r}(P_1(X_1) \times \cdots P_r(X_r))$$

where P_i can be either the Milnor class $\mathcal{M}$ or the Schwartz–MacPherson class c_*, the sum is taken over all the r-uples $(P_1, \ldots, P_r)$ different from $(c_*, \ldots, c_*)$ and $\varepsilon_i = 0$ if $P_i = \mathcal{M}$ and $\varepsilon_i = 1$ if $P_i = c_*$.

2.3.3. *Other comparisons* In [OSY], Toru Ohmoto, Tatsuo Suwa and Shoji Yokura show that, for an n-dimensional compact (strong) local complete intersection with isolated singularities, the Mather class (1) and the Fulton class (5) differ by the sum of the nth polar multiplicities at the singularities. This is a consequence of the Suwa formula (Theorem 2.3) and the fact that, in this case, the Euler local obstruction (another ingredient of the MacPherson construction) in a singular point is equal to the Milnor number of a generic hyperplane section.

§3. Equivariant Chern classes

Let G be a (complex reductive linear) algebraic group acting on a possibly singular complex algebraic variety X over $\mathbb{C}$, or more generally over a field of characteristic 0. The G-equivariant (Chow) homology group H_*^G (resp. A_*^G) is defined by Totaro [To], Edidin–Graham [EG] using an algebraic version of the Borel construction. It admits the equivariant fundamental class $[X]_G \in H_{2n}^G(X)$ ($n = \dim X$) so that $\cdot \cap [X]_G : H_G^i(X) \to H_{2n-i}^G(X)$ is an isomorphism if X is non-singular.

Let $\mathbb{F}^G(X)$ the group of G-invariant constructible functions, *i.e.* one has $\alpha(g.x) = \alpha(x)$ for any $g \in G$ and $x \in X$. Toru Ohmoto [Oh1] showes the following:

Theorem 3.1. *There is a unique natural transformation c_*^G from the G-equivariant constructible function functor $\mathbb{F}^G$ to the G-equivariant homology functor H_*^G*

$$c_*^G : \mathbb{F}^G(X) \to H_*^G(X)$$

such that, when X is non-singular, then one has

$$c_*^G(\mathbf{1}_X) = c^G(TX) \cap [X]_G$$

where $c^G(TX)$ is the G-equivariant total Chern class of the tangent bundle of X.

The Ohomoto construction uses the following equivariant Verdier–Riemann–Roch formula (for smooth morphisms) :

Theorem 3.2.

$$c_*^G(f^*\beta) = c^*(T_f^G) \cap f^! c_*^G(\beta) \qquad \beta \in \mathbb{F}^G(Y)$$

In [Oh2], Toru Ohmoto introduces, as an application, variants of "orbifold Chern classes" for quotient varieties via finite groups. Those are related to so-called "stringy Chern classes" which are defined using motivic integrations (see[Al, dFLNU]). A typical example is the symmetric product $S^n X$, for which Ohmoto shows various generating function formulae of orbifold Chern classes (including formulae for the Euler characteristics of Hilbert schemes of points on a surface).

§4. Bivariant Chern classes

4.1. Bivariant theories

In general, a bivariant theory is the data of a group associated to every morphism $X \to Y$, of three operations and with a series of axioms giving the compatibility properties between the three operations.

4.1.1. *Bivariant homology theory* Let us recall an example of bivariant theory: the homology bivariant theory. One defines

Definition 4.1. *The bivariant homology theory associates to each map $f : X \to Y$ a graded abelian group $H_*(X \xrightarrow{f} Y)$ defined by*

$$H_k(X \xrightarrow{f} Y) = H^{n-k}(Y \times M, Y \times M \setminus X_\phi)$$

where $n = \dim M$, f factors through an open embedding (f, ϕ)

$$X \overset{(f,\phi)}{\hookrightarrow} (Y, M) \xrightarrow{pr_1} Y \tag{7}$$

and $X_\phi = (f, \phi)(X)$.

Particular cases

Let us remark that one has the two following particular cases:

$$H_*(X \xrightarrow{id} X) = H^*(X), \qquad\qquad H_*(X \to \{pt\}) = H_*(X).$$

Operations

Three natural operations are defined on the bivariant homology groups (see [FM]):

• Product

$$\bullet : H_*(X \to Y) \times H_*(Y \to Z) \longrightarrow H_*(X \to Z).$$

• Direct image
If one has a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

then one has the direct image

$$f_* : H_*(X \to Z) \longrightarrow H_*(Y \to Z).$$

• Pullback
If one has a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

then one has the pullback

$$g^* : H_*(X \to Y) \longrightarrow H_*(X' \to Y').$$

A series of properties provide different type of compatibility of the three operations, which are the compatibility axioms in the general theory.

Direct image and Pullback operations gives classical covariance of homology and contravariance of cohomology.

Let $f : X \to Y$ smooth of relative dimension d, then there is $[f] \in H^{-d}(X \xrightarrow{f} Y)$ canonical orientation such that

$$[f] \bullet \eta = f^! \eta \qquad \eta \in H_*(Y), f^! \eta \in H_*(X)$$

4.1.2. *Bivariant constructible functions* Another classical example of bivariant theory is given by the groups of constructible functions:

Definition 4.2. [FM, Br1, Sab] *The group of bivariant constructible functions* $\mathbb{F}(X \to Y)$ *is the set of constructible functions* $\alpha : X \to \mathbb{Z}$ *on* X *satisfying the following local Euler condition:*

$$\alpha(x) = \chi(\mathbb{B}_\varepsilon \cap f^{-1}(z); \alpha)$$

where $\mathbb{B}_\varepsilon$ *is a small ball centered in* x*, the point* z *is close to* $f(x)$ *and*

$$\chi(K; \alpha) = \sum n \, \chi(K \cap \alpha^{-1}(n)).$$

4.2. Grothendieck transformations

Given two bivariant theories, say $\mathbb{K}$ and $\mathbb{H}$, a Grothendieck transformation $\gamma : \mathbb{K} \to \mathbb{H}$ is a collection of maps

$$\mathbb{K}(X \to Y) \xrightarrow{\gamma} \mathbb{H}(X \to Y)$$

consistent with the three operations.

In the particular cases of the maps $\mathrm{id} : X \to X$ and $X \to \mathrm{pt}$, one obtains the "classical" covariant and contravariant maps $\gamma^* : \mathbb{K}^* \to \mathbb{H}^*$ and $\gamma_* : \mathbb{K}_* \to \mathbb{H}_*$.

Here we give an example of application: the bivariant Chern classes.

4.3. Bivariant Chern classes

Fulton and MacPherson [FM] conjectured the following result:

Conjecture 4.1. *There is one unique Grothendieck transformation*

$$\gamma : \mathbb{F} \to \mathbb{H}$$

such that if X is smooth and f is the map $X \to \mathrm{pt}$, then

$$\gamma(\mathbf{1}_X) = c^*(TX) \cap [X] \in H_*(X; \mathbb{Z})$$

is the usual homology Chern class.

The existence of the above transformation $\gamma : \mathbb{F} \to \mathbb{H}$, called a *bivariant Chern class*, has been proved by J.-P. Brasselet in the case of so-called "cellular maps" [Br1] and by C. Sabbah in the case of one-dimensional target [Sab]. J. Zhou [Zho] proved that both constructions coincide in the case when the target of a morphism is a smooth curve.

The general uniqueness was still to be proved. Shoji Yokura [Y7] showed the following result:

Theorem 4.1. *If there exists a bivariant Chern classes $\gamma : \mathbb{F} \to \mathbb{H}$, then it is unique when restricted to morphisms with non singular target of any dimension. More precisely, let $f : X \to Y$ with Y nonsingular, then for every $\alpha \in \mathbb{F}(X \to Y)$, one has*

$$\gamma(\alpha) = f^* s(TY) \cap c_*(\alpha) \tag{8}$$

where $s(TY) = c(TY)^{-1}$ is the Segre class of the tangent bundle TY.

In fact the left side in (8) is an element $\gamma(\alpha) \in \mathbb{H}(X \to Y)$, the right side $f^* s(TY) \cap c_*(\alpha)$ is an element in $H_*(X)$, called the Ginsburg–Chern class of α. These two sides coincide via the following composition of isomorphisms

$$\mathbb{H}(X \to Y) \xrightarrow{\bullet [Y]} \mathbb{H}(X \to \mathrm{pt}) \xrightarrow{\text{Alexander}} H_*(X)$$

where the fundamental class $[Y] \in \mathbb{H}(Y \to \mathrm{pt})$ is a strong orientation (see [FM]).

Modulo Alexander isomorphism, one can write

$$\gamma(\alpha) \bullet [Y] = f^* s(TY) \cap c_*(\alpha)$$

The Yokura theorem has been generalized to the case when the target variety is a complex analytic oriented A-homology manifold, for some commutative Noetherian ring A [BSY2].

In [Y7] a new bivariant theory of constructible functions, denoted by $\overline{\mathbb{F}}(X \xrightarrow{f} Y)$ has been defined, satisfying suitable conditions. In particular with these groups, one proves that the transformation

$$\gamma^{Gin} : \overline{\mathbb{F}} \longrightarrow \mathbb{H}$$

is well-defined and becomes the unique Grothendieck transformation satisfying that γ^{Gin} for morphisms to a point is the MacPherson's Chern class transformation $c_* : \mathbb{F} \to H_*$. One remarks that, in order to prove that $\overline{\mathbb{F}}(X \to pt) = \mathbb{F}(X)$, we need the previous multiplicativity formula for the MacPherson's transformation c_* due to Kwieciński (formula 2).

The above result led to another *uniqueness theorem*, which in a sense gives a positive solution to the general uniqueness problem concerning Grothendieck transformations posed in [FM, §10 Open Problems]. In the paper [BSY3], we define suitable constructible function bivariant theory and homology theory for which uniqueness problem is solved. A key for the argument is the fact that $c_*(\alpha) = \gamma(\alpha) \bullet c_*(\mathbb{1}_Y)$. In a previous paper [Y6] Yokura posed the problem of whether or not there is a reasonable bivariant homology theory so that a kind of "quotient"

$$\frac{c_*(\alpha)}{c_*(\mathbb{1}_Y)}$$

is well-defined. The above theory is in a sense a positive answer to this problem.

§5. Motivic characteristic classes

5.1. The smooth case, Hirzebruch theory

Let M be a non-singular complex projective variety, the Euler–Poincaré characteristic of X:

$$\chi(M) = \sum_{i \geq 0} (-1)^i dim_{\mathbb{C}} H^i(M)$$

satisfies the Hirzebruch Riemann–Roch Theorem:

Theorem 5.1.

$$\chi(M) = T(M) := \int_M td^*(TM) \cap [M]\,. \tag{9}$$

where $td^*(TM)$ is the total *Todd class* of the tangent bundle TM:

$$td^*(TM) = \prod_{i=1}^{\dim M} \frac{\alpha_i}{1-e^{-\alpha_i}} \in H^{2*}(M;\mathbb{Q})$$

where α_i's are the Chern roots of TM .

Hirzebruch [Hir] defined:

Definition 5.1.

$$\begin{aligned}\chi_y(M) : &= \sum_{p\geq 0}\left(\sum_{q\geq 0}(-1)^q dim_{\mathbf{C}} H^q(M,\Lambda^p T^*M)\right) y^p \\ &= \sum_{p\geq 0}\chi(M,\Lambda^p T^*M) y^p\end{aligned}$$

where T^*M is the dual of the tangent bundle TM, i.e., the cotangent bundle of M.

$$T_y(M) := \int_M \widetilde{td}_{(y)}(TM) \cap [M],$$

$$\widetilde{td_{(y)}}(TM) := \prod_{i=1}^{\dim M}\left(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} - \alpha_i y\right),$$

where α_i's are the Chern roots of TM.

F. Hirzebruch [Hir, §21.3] showed the following theorem:

Theorem 5.2. *(Generalized Hirzebruch–Riemann–Roch)*

$$\chi_y(M) = T_y(M). \tag{10}$$

That means that the modified Todd class $\widetilde{td}_{(y)}$ defined above unifies the following three important characteristic cohomology classes:

(y = -1) the total Chern class

$$\widetilde{td}_{(-1)}(TM) = c^*(TM),$$

(y = 0) the total Todd class

$$\widetilde{td}_{(0)}(TM) = td^*(TM),$$

(y = 1) the total Thom–Hirzebruch L-class

$$\widetilde{td}_{(1)}(TM) = L^*(TM).$$

5.2. The singular case, motivic classes

In the case of a possibly singular compact complex algebraic variety X, the three characteristic classes have been respectively generalized as Schwartz–MacPherson classes (Definition 2.1):

$$c_* : \mathbb{F}(X) \to H_{2*}(X;\mathbb{Z})$$

Baum–Fulton–MacPherson Todd classes [BFM]

$$td_* : \mathbf{G}_0(X) \to H_{2*}(X;\mathbb{Q})$$

where $\mathbf{G}_0(X)$ denotes the Grothendieck group of algebraic coherent sheaves on X, and Cappell–Shanneson L-classes [CS]

$$L_* : \Omega(X) \to H_*(X;\mathbb{Q})$$

where $\Omega(X)$ denotes the abelian group of cobordism classes of selfdual constructible complexes.

So, one can ask for a generalized Hirzebruch–Riemann–Roch type theorem unifying these three transformations.

The solution uses the so-called *relative Grothendieck ring of complex algebraic varieties over* X, denoted by $K_0(\mathcal{V}/X)$. That is the quotient of the free abelian group of isomorphism classes of morphisms to X, denoted by $[Y \xrightarrow{h} X])$, modulo the following *additivity* relation:

$$[Y \xrightarrow{h} X] = [Z \hookrightarrow Y \xrightarrow{h} X] + [Y \setminus Z \hookrightarrow Y \xrightarrow{h} X]$$

for $Z \subset Y$ a closed subvariety of Y.

In fact, one shows the following [BSY1]:

Theorem 5.3. *Let $K_0(\mathcal{V}/X)$ be the Grothendieck group of complex algebraic varieties over X. Then there exists a unique natural transformation (with respect to proper maps)*

$$T_y : K_0(\mathcal{V}/\quad) \to H_{2*}^{BM}(\quad) \otimes \mathbb{Q}[y]$$

(Borel–Moore homology) such that for X non-singular

$$T_y([X \xrightarrow{id} X]) = \widetilde{td}_{(y)}(TX) \cap [X].$$

One proves the unification result:

Theorem 5.4. • (y = -1) *There exists a unique natural transformation*

$$\varepsilon : K_0(\mathcal{V}/\quad) \to F(\quad)$$

such that for X nonsingular $\varepsilon([X \xrightarrow{id} X]) = 1_X$ and such that the following diagram commutes

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{\varepsilon} & F(X) \\ & T_{-1} \searrow \quad \swarrow c_* & \\ & H_{2*}(X) \otimes \mathbb{Q}\,. & \end{array}$$

• (y = 0) *There exists a unique natural transformation*

$$\gamma : K_0(\mathcal{V}/\quad) \to \mathbf{G}_0(\quad)$$

such that for X nonsingular $\gamma([X \xrightarrow{id} X]) = [\mathcal{O}_X]$ (structural sheaf on X) and such that the following diagram commutes

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{\gamma} & \mathbf{G}_0(X) \\ & T_0 \searrow \quad \swarrow td_* & \\ & H_{2*}(X) \otimes \mathbb{Q}\,. & \end{array}$$

• (y = 1) *There exists a unique natural transformation*

$$\omega : K_0(\mathcal{V}/\quad) \to \Omega(\quad)$$

such that for X nonsingular $\omega([X \xrightarrow{id} X]) = [\mathbb{Q}_X[\dim X]]$ and such that the following diagram commutes

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{\omega} & \Omega(X) \\ & T_1 \searrow \quad \swarrow L_* & \\ & H_{2*}(X) \otimes \mathbb{Q}\,. & \end{array}$$

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