Advanced Studies in Pure Mathematics 56, 2009
Singularities — Niigata–Toyama 2007
pp. 35–38

Appendix to "A divisor on the moduli space of curves associated to the signature of fibered surfaces" by T. Ashikaga and K.-I. Yoshikawa

Kazuhiro Konno

Abstract.

The Horikawa index and the local signature are introduced for relatively minimal, Eisenbud–Harris general fibred algebraic surfaces of genus 4.

§1. Introduction

Let S (resp. B) be a non-singular projective surface (resp. curve) defined over the complex number field $\mathbb{C}$ and $f : S \to B$ a relatively minimal fibration whose general fibre F is a non-hyperelliptic curve of genus 4. According to [2], we say that f is Eisenbud–Harris general (E-H general for short) if F has two distinct g^1_3's, or equivalently, if F, regarded as a canonical curve, lies on a quadric surface of rank 4 in $\mathbb{P}^3$.

When $f : S \to B$ is a (semi-)*stable* E-H general fibration of genus 4, two important local invariants, the local signature and the Horikawa index, are successfully introduced in [2]. The purpose of this short note is to show that an analogous result also holds even if we drop the assumption on stability. Namely, we show the following:

Theorem 1. *Let $f : S \to B$ be a relatively minimal, E-H general fibration of genus 4. Then the Horikawa index $\mathrm{Ind}(f^{-1}b) \in \mathbb{Q}_{\geq 0}$ is defined for $b \in B$ so that the slope equality*

$$K^2_{S/B} = \frac{7}{2}\chi_f + \sum_{b \in B} \mathrm{Ind}(f^{-1}b) \tag{1}$$

Received April 28, 2008.
Revised May 19, 2008.
2000 *Mathematics Subject Classification.* 14D06.
Key words and phrases. Slope, fibred surface.

holds. Furthermore, the local signature $\sigma(f^{-1}b) \in \mathbb{Q}$ *is defined and the signature of* S *can be localized as*

$$\text{Sign } S = \sum_{b \in B} \sigma(f^{-1}b). \tag{2}$$

Precise definitions of the Horikawa index and the local signature are given in (8), (9) below. We adopt the notation and terminology in [1] without any further comments.

§2. Proof of theorem

Let $f : S \to B$ be a relatively minimal, E-H general fibration of genus 4. For such fibrations, we have the slope inequality

$$K_{S/B}^2 \geq \frac{7}{2}\chi_f, \tag{3}$$

which was shown independently in [4] and [3], where $K_{S/B} = K_S - f^*K_B$ and $\chi_f = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_F)\chi(\mathcal{O}_B)$. The common idea underlying their proofs was to use the relative canonical algebra multiplications. More precisely, they used the exact sequence of sheaves on B,

$$0 \to \mathcal{L} \to \text{Sym}^2(f_*\omega_{S/B}) \to f_*(\omega_{S/B}^{\otimes 2}) \to \mathcal{T} \to 0, \tag{4}$$

induced by the multiplication map $\text{Sym}^2 H^0(F, K_F) \to H^0(F, 2K_F)$ (see [5]). Since F is a non-hyperelliptic curve of genus 4, $\mathcal{L}$ is an invertible sheaf and $\mathcal{T}$ a torsion sheaf by Max Noether's theorem. We have $\deg f_*\omega_{S/B} = \chi_f$ and $\deg f_*(\omega_{S/B}^{\otimes 2}) = K_{S/B}^2 + \chi_f$. Therefore, the equality $\deg(\text{Sym}^2(f_*\omega_{S/B})) - \deg \mathcal{L} = \deg f_*(\omega_{S/B}^{\otimes 2}) - \text{length}(\mathcal{T})$ from (4) yields

$$K_{S/B}^2 = 4\chi_f + \text{length}(\mathcal{T}) - \deg \mathcal{L}. \tag{5}$$

We are going to "localize" the term $\deg \mathcal{L}$ with the quantity naturally associated to f. Via the injection $\mathcal{O}_B \hookrightarrow \text{Sym}^2(f_*\omega_{S/B}) \otimes \mathcal{L}^{-1}$ obtained by (4), we get the canonical section $q \in H^0(B, \text{Sym}^2(f_*\omega_{S/B}) \otimes \mathcal{L}^{-1})$ as the image of $1 \in H^0(B, \mathcal{O}_B)$. The geometric meaning of q is as follows. Consider the relative canonical map $\Phi_f : S \cdots \to \mathbb{P}(f_*\omega_{S/B})$, that is, the rational map induced by the sheaf homomorphism $f^*f_*\omega_{S/B} \to \omega_{S/B}$. Leray spectral sequence gives us an isomorphism $H^0(\mathbb{P}(f_*\omega_{S/B}), 2H - \pi^*\mathcal{L}) \simeq H^0(B, \text{Sym}^2(f_*\omega_{S/B}) \otimes \mathcal{L}^{-1})$, where H denotes a tautological divisor on $\mathbb{P}(f_*\omega_{S/B})$ and π the projection map $\mathbb{P}(f_*\omega_{S/B}) \to B$. Then q, regarded as an element of $H^0(\mathbb{P}(f_*\omega_{S/B}), 2H - \pi^*\mathcal{L})$, defines the relative hyperquadric Q through $\Phi_f(S)$.

Since f is E-H general, any general fibre of $\pi|_Q : Q \to B$ is a quadric of rank four. Therefore, the determinant of $q : f_*\omega_{S/B}^\vee \to f_*\omega_{S/B} \otimes \mathcal{L}^{-1}$ defines a non-zero element $\det(q) \in H^0(B, \det(f_*\omega_{S/B})^2 \otimes \mathcal{L}^{-4})$ whose zero divisor, which we denote by $\mathrm{Discr}(Q)$, is nothing more than the discriminant locus of $Q \to B$. We have

$$(6)\quad \deg(\mathrm{Discr}(Q)) = \deg(\det(f_*\omega_{S/B})^2 \otimes \mathcal{L}^{-4}) = 2\chi_f - 4\deg\mathcal{L}.$$

In particular, we get $\deg\mathcal{L} \leq \chi_f/2$ because $\mathrm{Discr}(Q)$ is an effective divisor, and (3) immediately follows from (5).

Now, by eliminating $\deg\mathcal{L}$ from (5) using (6), one obtains

$$(7)\qquad K_{S/B}^2 = \frac{7}{2}\chi_f + \frac{1}{4}\deg\mathrm{Discr}(Q) + \mathrm{length}(\mathcal{T}).$$

If we put

$$(8)\qquad \mathrm{Ind}(f^{-1}b) = \frac{1}{4}\mathrm{mult}_b(\mathrm{Discr}(Q)) + \mathrm{length}(\mathcal{T}_b) \quad \text{for } b \in B,$$

then it is a well-defined non-negative rational number. Furthermore, we have $\mathrm{Ind}(f^{-1}b) = 0$ when $f^{-1}b$ is an E-H general curve of genus 4. So, $\mathrm{Ind}(f^{-1}b)$ deserves the name of the Horikawa index (see, [1, §2]) and we get (1) from (7). Once the Horikawa index is introduced, we can define the local signature by the formula [1, (2.1.3)]:

$$(9)\qquad \sigma(f^{-1}b) = \frac{8}{17}\mathrm{Ind}(f^{-1}b) - \frac{9}{17}e_f(f^{-1}b),$$

where $e_f(f^{-1}b) = e(f^{-1}b) + 6$ is the Euler contribution at $b \in B$, and we get the localization of the signature of S as in (2).

References

[1] T. Ashikaga and K. Konno, Global and local properties of pencils of algebraic curves, In: Algebraic Geometry 2000 Azumino, (eds. S. Usui et. al.), Adv. Stud. Pure Math., **36**, Math. Soc. Japan, 2002, pp. 1–49.

[2] T. Ashikaga and K.-I. Yoshikawa, A divisor on the moduli space of curves associated to the signature of fibered surfaces, this volume.

[3] Z. Chen, On the lower bound of the slope of a non-hyperelliptic fibration of genus 4, Intern. J. Math., **4** (1993), 367–378.

[4] K. Konno, Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **20** (1993), 575–595.

[5] M. Reid, Problems in pencils of small genus, unpublished note, 1990.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka 560-0043
Japan

E-mail address: `konno@math.sci.osaka-u.ac.jp`