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# The space of triangle buildings

**Mikaël Pichot**

**Abstract.**

I report on recent work of Sylvain Barré and myself on the space of triangle buildings.

From a set-theoretic point of view the space of triangle buildings is the family of all triangle buildings (also called Bruhat–Tits buildings of type $\tilde{A}_2$) considered up to isomorphism. This is a continuum. We shall see that it provides new tools and a general framework for studying triangle buildings, which connects notably to foliation and lamination theory, quasi-periodicity of metric spaces, and noncommutative geometry.

This text is a general presentation of the subject and explains some of these connections. Several open problems are mentioned. The last sections set up the basis for an approach via $K$-theory.

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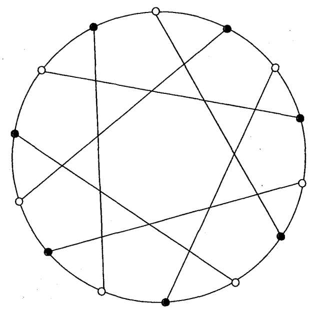
## §1. Triangle buildings

It is in the following graph, and more generally in the geometry of higher order *projective planes*, that the space of triangle buildings originates.



This graph (which has 14 vertices and 21 edges) is the incidence graph of the projective plane over the field $F_2$ of two elements, namely the Fano plane $P^2F_2$.

Let us start by recalling the following result of J. Tits (see [4] for a proof).

**Theorem 1.1.** *Let $\Delta$ be a simply connected simplicial complex of dimension 2 whose faces are equilateral triangles. Assume that for every vertex $z$ of $\Delta$ the link $L_z$ at $z$ (i.e. the simplicial sphere of radius 1 around $z$) is isomorphic to an incidence graph of a projective plane. Then $\Delta$ is a triangle building.*

A *triangle building* is a simply connected simplicial complex $\Delta$ of dimension 2, whose faces are Euclidean triangles, and which satisfies the following incidence and (weak) homogeneity axioms:
(1) For any two triangles in $\Delta$ there exists an isometric embedding of the tiling of the Euclidean plane $\mathbf{R}^2$ by equilateral triangles in $\Delta$ which contains these two triangles. The images of these embeddings are called flats, or apartments, and $\Delta$ is the union of its flats.
(2) For any two flats $\Pi, \Pi' \subset \Delta$ the intersection $\Pi \cap \Pi'$ is a convex set and there is a simplicial isometry $\rho : \Pi \to \Pi'$ fixing $\Pi \cap \Pi'$ pointwise (but in general $\rho$ does not extend to an isometry of $\Delta$).

In a triangle building the number of triangles incident to each edge is constant and denoted $q+1$. One calls $q$ the order. *We shall, from now on, fix $q$ to be a prime power $\geq 2$.* Recall that the existence of projective planes of order $q \neq p^n$, $p$ prime, $n \in \mathbf{N}^*$, is a long standing open question in projective geometry.

## §2. Classical triangle buildings

A *nonarchimedean local field* is a finite extension of the field of $p$-adic numbers $\mathbf{Q}_p$ in characteristic zero, or the field of formal Laurent series $\mathbf{F}_q((t))$ over a finite field $\mathbf{F}_q$ in characteristic $p$ (see [34]). To each such a field $K$ is associated a triangle building $\Delta_K$.

J. Tits introduced the structure of buildings while studying algebraic groups over arbitrary fields. He thereby discovered how to geometrize a large class of groups, namely all groups which have a so-called BN-pair. Here $\Delta_K$ plays the role of symmetric space for the special linear group $\mathrm{SL}_3(K)$.

The building $\Delta_K$ corresponds to a BN-pair in $\mathrm{SL}_3(K)$ of affine type. Tits' original BN-pair for $\mathrm{SL}_3(K)$, which can be constructed over any field $K$, is different and said to be of spherical type. Affine BN-pairs first appeared in well-known work of Iwahori–Matsumoto and Bruhat–Tits.

We refer to [30] (for instance) for the construction of $\Delta_K$ from $\mathrm{SL}_3(K)$. By definition the set of vertices of $\Delta_K$ is the family of lattices in $K^3$ modulo homothety, and higher dimensional faces are defined via congruence relations between these lattices. The Lie group $\mathrm{SL}_3(K)$ over $K$ then acts strongly transitively on its symmetric space $\Delta_K$. This makes $\Delta_K$ highly homogeneous.

The subgroups $B$ and $N$ of $\mathrm{SL}_3(K)$ defining the affine BN-pair of $\mathrm{SL}_3(K)$ are respectively the stabilizer of a chamber of $\Delta_K$ (i.e. a triangle here) and the stabilizer of an apartment of $\Delta_K$ containing this chamber. In $\mathrm{SL}_3(K)$ they correspond to the group of upper triangular matrices and the group of matrices with exactly one non-zero entry in each row and column.

Tits' original BN-pair for $\mathrm{SL}_3(K)$ does not take into account the valuation of $K$, as the affine one did in the definition of the congruence relations between lattices in $K^3$ (which sets up the whole structure of $\Delta_K$). It gives rise to a different building associated to $\mathrm{SL}_3(K)$, of spherical type, which can be naturally identified with the boundary of $\Delta_K$ at infinity.

Note that there is a second spherical building associated to $K$ (or rather its residual field). It is finite and can be identified with the link at vertices in $\Delta_K$. The graph of Fig. 1 for instance, i.e. the unique link of order 2, appears (e.g.) in $\Delta_{\mathbf{Q}_2}$ over $K = \mathbf{Q}_2$.

Buildings of the form $\Delta_K$ are called *classical triangle buildings.* They form a countable family of buildings which has been extensively studied over the years, along with classical buildings of other types. In fact in most cases for an algebraic group $G$, the correspondence which maps $K$ to the affine building of $G(K)$ describes completely accurately (i.e., is surjective to) its associated class of buildings. This holds for any $G$ of rank $\geq 3$, by a fundamental theorem of Tits (see Theorem 5.1 below). Surjectivity, however, fails for $G = \mathrm{SL}_3$, to which are associated triangle buildings as described in Section 1.

## §3. The internal dynamic of triangle buildings

There are uncountably many triangle buildings [37, 31]. One can actually construct triangle buildings whose links exhaust any preassigned set of projective planes of fixed order (uncountability then follows by 'alternating' Desarguesian and non Desarguesian projective planes in the local choices, see [31]). Quoting [33], a "total freedom (or, shall we say, anarchy)" arises from these constructions, which are often called the *free constructions* of triangle buildings.

In fact one can show (see [6]) that given any *fixed Desarguesian projective plane* $P$, the family of triangle buildings whose links all come from $P$ is still uncountable. Thus the reason for the uncountability of the family of triangle buildings is intrinsic to the nature of triangle buildings themselves and is not merely a byproduct of the existence of exotic finite projective geometries.

These are first indications of the complexity of the family of triangle buildings.

Let us now introduce the following definition (see [6, 7]).

**Definition 3.1.** *The* space of triangle buildings *is the set $E$ of all triangle buildings up to isometric isomorphism.*

The above quoted results show that $E$ is a *continuum.* In fact, as we shall see below, $E$ is a *singular space* in the sense of Connes [11] (or a *noncommutative space* [12]). It exhibits an "internal dynamic" (see §6) that makes its structure similar to that of leaf spaces of foliations. The space $E$ can be studied via a simple operation, called desingularization [11], that we describe in the next paragraph. We write $E_q$ for the subset of $E$ consisting of buildings of order $q$.

## §4. Desingularizing the space of triangle buildings

Let $X_q$ be the set of vertex-pointed triangle buildings of order $q$ up to pointed isomorphism, i.e. the set of couples $(\Delta, s)$ where $\Delta$ is a triangle building and $s$ is a vertex of $\Delta$, up to isomorphisms respecting to base point. We write $[\Delta, s]$ for the isomorphism class. Forgetting the base point we get a canonical surjective map

$$\pi_q : X_q \to E_q.$$

This map is called a desingularization [11]. The set $X_q$, unlike $E_q$, can be endowed with a locally compact *Hausdorff* topology: two points $[\Delta, s]$ and $[\Delta', s']$ of $X_q$ are said to be close if they coincide on large balls centered at their base points (the so-called pointed Gromov–Hausdorff topology). Define similarly $\pi : X \to E$ for the set $E$ of all triangle buildings, so that $X$ is locally compact Hausdorff, $X_q$ is a compact subset of $X$ and $\pi$ restricts to $\pi_q$ on $X_q$. The fibers of $\pi_q$ define an equivalence relation $R_q$ with countable classes on $X_q$ (and $R$ on $X$ so $R = \oplus_q R_q$). Thus two points $[\Delta, s]$ and $[\Delta', s']$ of $X_q$ are equivalent if and only if $\Delta$ and $\Delta'$ are isomorphic. There is a natural topology of étale principal $r$-discrete groupoid [11, 29] on $R_q$. (Note that there are more singular spaces which have no obvious desingularization, or no "classifying" $C^*$-algebra; we mentioned in [24] the "space of type $\mathrm{II}_1$ equivalence relations" as such an example—I thank E. Effros for interesting discussions on related subjects, see also [18].)

We also consider the space $\Lambda_q$ of pointed buildings $(\Delta, s)$ up to pointed isomorphism, where now $s$ is an arbitrary point of $\Delta$ (not necessarily a vertex). The corresponding surjection

$$\tilde{\pi}_q : \Lambda_q \to E_q$$

is another desingularization of $E_q$ (Morita equivalent to $\pi_q$) and its fibers define a partition of $\Lambda_q$ into "leaves". By construction these leaves are isomorphism classes of triangle buildings and $X_q$ is a transversal of $\Lambda_q$, i.e. it intersects every leaf of $\Lambda_q$ along a countable set. We call $X_q$ the transversal of vertices of $\Lambda_q$. (One defines $\Lambda$ corresponding to $E$ analogously.) The construction of the holonomy groupoid $G_q$ of $\Lambda_q$, based on $X_q$, carries out as in foliation theory and its principal subgroupoid is $R_q$. We shall now describe what the compact leaves of $\Lambda_q$ are.

## §5. The periodic case

Triangle buildings have rank 2, according to condition (1) in §1. Higher rank affine buildings can be defined in a similar fashion by using tilings of the higher dimensional Euclidean space $\mathbf{R}^r$ as apartments instead of $\mathbf{R}^2$ in (1), see [10, 30]. The fundamental result in rank $> 2$ is Tits' *classification of affine buildings theorem.* It can be expressed as follows (see [36, 38, 39, 30]).

**Theorem 5.1.** *All affine buildings of rank $> 2$ are classical, i.e., they all arise from algebraic groups over (nonarchimedean) local fields (see §2).*

In particular, the automorphism group an affine building of rank $> 2$ is always *strongly transitive* on the building.

Analogous properties fail dramatically in rank 2. For instance there are among triangle buildings so-called "exotic" buildings, which are cocompact but which are not related to algebraic groups. See [31, 42], and [5] for an explicit construction of a triangle building $\Delta$ having a finite quotient whose fundamental group is commensurable to the full automorphism group of $\Delta$.

Recent fundamental work on the *classification problem* for triangle buildings (and others) incorporates the description of all buildings satisfying additional properties, notably the *Moufang condition* at infinity. See [36, 39] for details and references. In rank $> 2$ the Moufang condition for the building at infinity is always satisfied.

In [6] we proved that generic triangle buildings have no symmetry:

**Theorem 5.2.** *In $E_q$ the automorphism group of a generic building is trivial. A generic leaf of $\Lambda_q$ is everywhere dense with no automorphism.*

The term *generic* in this theorem means saturated dense $G_\delta$ in $X_q$ (which makes a non trivial topological property of the 'topological singular space' $E_q$, although its quotient topology itself is very poor). This result is based on "large-scale" surgery and "local prescription" theorems for which we refer to [6] (see Theorem 2 of that paper). It follows that $E_q$ is non type I.

As we already mentioned Tits' foundational idea for the structure of building was the geometrization of semi-simple algebraic groups, and in particular, of the exceptional groups (see [36]). In some sense we will take the opposite point of view here: we use triangle buildings to construct interesting examples of groupoids and operator algebras. Classical and exotic triangle buildings (in the sense above) constitute the compact leaves of the lamination $\Lambda_q$ (the holonomy of which it might be interesting to describe).

Here is a basic question concerning compact leaves.

**Question 5.3** (Orbits counting problem)**.** *What is the behavior of the function $\#_q$ defined for integers $n$ by $\#_q(n) =$ the number of compact leaves with $n$ vertices in $\Lambda_q$ ?*

Very few results seems to be known in that direction.

## §6. Topological quasi-periodicity

We now enter the finer noncommutative structure of $E_q$. For technical reasons we make the further assumption that $q \neq 3, 4$ until the end of the text. We need the following definition (that we call "topological quasi-periodicity").

**Definition 6.1.** *A metric space* $\Delta$ *is said to be* quasi-periodic *if for any ball* $B \subset \Delta$ *there is a positive number* $\lambda > 0$ *such that any ball* $B' \subset \Delta$ *of radius* $\lambda$ *contains an isometric copy of* $B$.

For instance periodic buildings are quasi-periodic. The main result in [7] is as follows.

**Theorem 6.2.** *There exist infinitely many quasi-periodic triangle buildings of order* $q$ *which are not periodic.*

In fact we prove the following (stronger) statement.

**Theorem 6.3.** *There exist infinitely many minimal sublaminations of* $\Lambda_q$ *which are not reduced to compact leaves.*

Recall that *minimal* means that all leaves are dense. One of the point of the above 6.3 is to provide a way to *transfer properties of topological actions of* $\mathbf{Z}$ *and* $\mathbf{Z}^2$ *to the space of triangle buildings, by using the geometry of apartments.* (See [7].)

To slightly simplify the situation we shall now work in the following framework (in which leaves are buildings rather than quotients of buildings).

**Definition 6.4.** *By a* lamination by triangle buildings *we mean a triple* $(T, R, L)$ *where* $T$ *is a compact space,* $R$ *is an étale equivalence relation with countable classes on* $T$, *and* $L$ *is a lamination whose leaves are triangle buildings and whose transversal of vertices is* $T$ *with holonomy* $R$.

By universality of $E$ there is, for any lamination by triangle buildings $(T, R, L)$, a continuous map $L \to \Lambda$ which sends $T$ to a compact subset of $X$. "Trivial" examples of laminations by triangle buildings include diagonal quotients $(\Delta \times T)/\Gamma$ where $\Gamma$ is a countable group with a co-compact action on a building $\Delta$ and say, a minimal action on $T$. (These laminations all map to compact leaves of $\Lambda$.) Non trivial examples can be constructed from Theorem 6.3.

## §7. Measure-theoretic quasi-periodicity

From Theorem 6.3 and a variation on Garnett's harmonic measures theorem for compact foliated manifolds one can prove the following result (see [7]).

**Theorem 7.1.** *There exist infinitely many quasi-invariant diffuse harmonic measures on $X_q$ with disjoint support.*

This theorem readily implies for example that classification of triangle buildings is not possible, in the following elementary sense: one cannot find an *injective* Borel map $c : E_q \to V$ attaching an "invariant" $c(\Delta) \in V$ to each triangle building $\Delta$, where the value set $V$ is standard Borel (by a Borel map on $E_q$ we mean an invariant Borel map on $X_q$, see [11]).

It follows from this and the work of Van Maldeghem that *it is not possible to classify "planar ternary rings with valuations"*, which coordinatize the projective planes at infinity: see [42, 43].

Here is a question in which we are presently interested.

**Question 7.2.** *Is there a non trivial (in the sense of the preceding section) lamination by triangle buildings which is minimal and admits an invariant probability measure ?*

The first motivation for this question is the following result of [26]. Let $(T, R, L)$ be a lamination by triangle buildings. Then $R$ has *measurewise property T*, in the sense that for any invariant mesure $\mu$ on $T$, the measured equivalence relation $(R, \mu)$ has Kazhdan's property T (in the sense of Zimmer, see [1, 26]). This result is a measure theoretic '$\lambda_1 > 1/2$ criterion', for which we refer to [26] and the references therein. A positive answer to question 7.2 would thus provide new examples of property T measured equivalence relation, where *no natural group with property T is involved* (compare the introduction of [1] for example). This also leads to the following problem (which presupposes the existence of invariant measures).

**Question 7.3.** *Is there a lamination by triangle buildings $(T, R, L)$ and an ergodic invariant probability measure $\mu$ on $T$ such that no countable group can act $\mu$-essentially freely on $T$ and generate $R$ ?*

Note that such a group would necessarily have property T by [26]. The question of the existence of measured equivalence relations that cannot be freely generated was a long standing problem ([13]) solved by Furman [14], who gave explicit examples using the superrigidity of lattices in higher rank Lie groups. In [28] Popa exhibited entirely new examples as a corollary of his cocycle superrigidity theorem. The higher rank nature of our examples and the absence of natural groups in the constructions would suggest a positive answer to 7.3. In fact given an ergodic invariant probability measure $\mu$ on $T$ we conjecture that the restriction $R_{|A}$ of $R$ to any Borel subset $A \subset T$ of measure $\mu(A) \neq 0$ cannot be produced by an essentially free action of a countable group.

In [27] we address the problem of showing that in a lamination by triangle buildings $(T, R, L)$, the equivalence relation $R$ *cannot be obtained from foliation theory.* Roughly speaking we showed that, assuming again the existence of an invariant probability measure $\mu$ on $T$, *there is no "finite energy" foliated map from* $(T, \mu)$ *to a lamination* $F$ *on a compact space* $M$ *whose leaves are simply connected Riemannian manifolds with non positive sectional curvature.* See [27] for precise assumptions and references. The proof is based on harmonic analysis and tools developed by Gromov in [16]. This theorem is stronger than the above asserting property T for $(R, \mu)$.

**Remark 7.4.** *Measure-theoretic rigidity statements of the above type, in general, requires the measure* $\mu$ *to be invariant to hold (compare however the precise assumptions in* [24, 26, 27]*). Would it turn out that some laminations by triangle buildings* $(T, R, L)$ *have no invariant probability measure, then this would produce type* III *dynamical systems (and type* III *von Neumann algebras) directly out of the geometrical input given by Tits' notion of building—which is further intimately related to projective planar geometry here. In particular this would define natural "time evolutions" (the flow of weights, see* [12]*) of the corresponding quasi-periodic triangle buildings.*

We now introduce the following definition (see [26] for the terminology, notations, and more on spectral theory for measured equivalence relations).

**Definition 7.5.** *One says that a Borel equivalence relation* $R$ *(with countable classes) has* uniform property T *(resp. is* uniformly strongly ergodic*) if there exists a generating Borel random walk* $\nu$ *on the orbit of* $R$ *and a constant* $\kappa_\nu < 1$ *such that for any invariant measure* $\mu$ *on* $X$ *and any unitary representation* $\pi$ *of* $(R, \mu)$ *on* $H_\pi$ *(resp. for the trivial representation of* $(R, \mu)$*), the spectrum of the diffusion operator* $D_{\nu,\pi}$ *(resp. the operator* $D_{\nu,\mathrm{triv}}$*) on* $L^2_\mu(H_\pi)$ *is included in* $[-1, \kappa_\nu] \cup \{1\}$.

One then has:

**Proposition 7.6.** *Let* $(T, R, L)$ *be a lamination by triangle buildings. Then* $R$ *has uniform property* $T$ *and in particular is uniformly strongly ergodic. The set of ergodic invariant measures is a compact subset of the compact convex space of invariant measures on* $T$.

This follows from [26]. The last assertion should be compared to the case of Bernoulli system over groups in [15] (see also [21]); observe that $E_q$ is a 'universal system' as well, by definition.

We denote by $\widetilde{E}_q$ the space of ergodic invariant probability measure on $X_q$ and call it the *space of quasi-periodic triangle buildings.*

We thereby follow the *quasi-periodicity principle* of [24] that '(measure-theoretic) notions of quasi-periodicity' and 'ergodic measures' are ambivalent concepts (restricting to the measure-preserving case to start with). See [24] for more details. From this point of view a singular space (as $E$ for instance) generates a bunch of quasi-periodic concepts. Note that this 'family of quasi-periodic spaces' is always standard Borel by desintegration theory (see [25]). This would not be the case at the topological level. In the present situation however, it follows from 7.6 that $\widetilde{E}_q$ is a separable compact space, and in particular that in this sense *it is possible to classify quasi-periodic triangle buildings up to isomorphism at the topological level* (by continuous invariants). Note that the construction of "very exotic" periodic buildings, e.g. especially cocompact buildings whose quotients under their automorphism group can be arbitrary large (which would be useful in view of Proposition 7.6), seems to be a non trivial problem (compare Question 5.3).

## §8. On the $K$-theory of quasi-periodic triangle buildings

Let us come back to topology and consider the general question of computing the $K$-theory of $E_q$. In other word, letting $C_r^*(G_q)$ be the reduced $C^*$-algebra of the holonomy groupoid $G_q$ associated to $E_q$, the problem is to compute $K_*(C_r^*(G_q))$. To slightly simplify matters we keep working in the context of a lamination by triangle buildings, say $(T, R, L)$. Recall that there is a map

$$\mu : K_*^R(L) \to K_*(C_r^*(R))$$

from the equivariant $K$-homology of $L$ to the $K$-theory of $C_r^*(R)$ (note that $L$ is a classifying space for $R$). This map is called the index map (or analytic assembly map) and is conjectured by Baum and Connes to be an isomorphism in a variety of situations, see [9]. For its definition in our case, see [41, Section 5]. Due to limitations of length we only quote directly related papers (concerning groupoids) below, to which we shall refer for references and historical credits. See also [8].

In our case Kasparov's $\gamma$ element in equivariant $KK$-theory can be defined by fibrating the construction of Julg–Valette [19]. For the construction of $\gamma$ in the much more general framework of "bolic" (in the sense of Kasparov–Skandalis [20]) groupoids, see [41] (note that no continuity problem arises in our case). In particular the results of [41] apply to our setting and show that $\mu$ is injective.

Thus the problem is surjectivity. J.L. Tu proved (after the work of Higson and Kasparov [17]) the following theorem in [40]. *If* $\gamma = 1$ *in* $KK_R(C(T), C(T))$ *then* $\mu$ *is an isomorphism. Moreover, if* $R$ *has the*

*Haagerup property, then* $\gamma = 1$ *in* $KK_R(C(T), C(T))$. In the group case it is well-known that property T is an obstruction to the equality $\gamma = 1$ (roughly speaking $\gamma$ corresponds to the regular representation, 1 to the trivial representation, and equality in $KK$ involves a homotopy between the two, which is not compatible with property T). For $(T, R, L)$ however the issue of property T is unclear (see §7), and we actually don't know that indeed $\gamma \neq 1$ in $KK_R(C(T), C(T))$ for some minimal equivalence subrelation $(T, R)$ of $X_q$.

## §9. An approach via Banach $KK$-theory, after V. Lafforgue

This section is based on the work of Lafforgue [22, 23]: continuing the discussion of the previous section we are now looking for a homotopy between $\gamma$ and 1 in Banach $KK$-theory.

The Baum–Connes conjecture was proved for groups acting properly with compact quotient on triangle buildings (and many other cases) in the fundamental work of Lafforgue [22], who also showed that his techniques can be generalized to hyperbolic groupoids in [23]. His proof consists of two rather independent parts. The first part concerns Banach $KK$-theory and aims at proving (Lafforgue's versions of) the Baum–Connes conjecture for assembly maps

$$\mu_{\mathcal{A}} : K_*^R(L) \to K_*(\mathcal{A})$$

where $\mathcal{A}$ is an unconditional completion [22] of $C_c(R)$. Lafforgue established a descent principle in Banach $KK$-theory (i.e. available for every unconditional completion) allowing to deduce the bijectivity of $\mu_{\mathcal{A}}$ from the equality $\gamma = 1$ in (an asymptotic version $KK^{\mathrm{ban}}_{R,\, s\varphi}(C(T), C(T))$ of) $KK^{\mathrm{ban}}_R(C(T), C(T))$. See [23, Théorème 1.5.10] for the groupoid case. The second part of the proof concerns the question of the isomorphism (sujectivity) of the canonical map

$$K_*(\mathcal{A}) \to K_*(C_r^*(R))$$

for some well chosen unconditional completion $C_c(R) \subset \mathcal{A} \subset C_r^*(R)$.

Adapting this second part to laminations by triangle buildings seems to be a delicate problem and will not be addressed here (compare [23] and see also [8]). To adapt the Banach $KK$-theory part we need to show that $\gamma = 1$ in $KK^{\mathrm{ban}}$ and this can be done by imitating [22, Théorème 2.2.2] (the case of groups acting on bolics spaces) and [23, Proposition 2.0.11] (the case of laminations of compact manifolds whose leaves are smooth Riemannian manifolds with non positive sectional curvature). (See also [35] for a complete description in the case of triangle buildings.) Namely one proves the following:

**Theorem 9.1.** *Let $(T, R, L)$ be a lamination by triangle buildings and $\varphi$ be the length function on $R$ given by the 1-skeleton of $L$. Then for every $s > 0$ the images of $\gamma$ and 1 in $KK^{\mathrm{ban}}_{R,\, s\varphi}(C(T), C(T))$ coincide.*

Then a direct application of [23, Théorème 1.5.10] shows that *the map*

$$\mu_{\mathcal{A}} : K^R_*(L) \to K_*(\mathcal{A})$$

*is an isomorphism for any unconditional completion $\mathcal{A}$ of $C_c(R)$*, in particular for the algebra $\ell^1(R) \subset C^*_r(R)$ of uniformly $r,s$-summable functions on $R$ ([29]) or for the "maximal unconditional" completion of $C_c(R)$ in $C^*_r(R)$ relative to the norm $\|f\| = \|(x, y) \mapsto |f(x, y)|\,\|_{C^*_r(R)}$ ([22]).

## §10. Final remark—"spaces of spaces"

More details, proofs and elaborations of the results presented here will appear in [8]. A basic recipe for producing singular spaces (and thus $C^*$-algebras, etc.) *without using any infinite group in the constructions* is to start with some elementary geometrical data $S$ and to consider the set $E_S$ of all spaces (in some fixed category, e.g. metric, polyhedral, CAT(0)) which have $S$ as "local data", up to isomorphism in the given category. For instance $S$ can be some fixed finite metric graph, as we did for $E_2$ and the graph $P^2F_2$, some finite family of compatible shapes (e.g. triangles, squares,...) out of which a specific class of polyhedra $E_S$ arises, or (more classically) some set of equations defining Riemannian local prescriptions. The properties of the resulting universal $E_S$ should depend in some interesting way on $S$ and the chosen category.

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*Institut des Hautes Études Scientifiques*
*35 Route de Chartres*
*F-91440 Bures-sur-Yvette*
*France*

*E-mail address*: `pichot@ihes.fr`