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Lagrangian fibrations and theta functions

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Abstract.

It is known that holomorphic sections of an ample line bundle L (and its tensor power L^k) on an Abelian variety A are given by theta functions. Moreover, a natural basis of the space of holomorphic sections is related to a certain Lagrangian fibration of A. We study projective embeddings of A given by the basis for L^k, and show that moment maps of toric actions on the ambient projective spaces, restricted to A, approximate the Lagrangian fibration of A for large k. The case of Kummer variety is also discussed.

§1. Introduction

Let (X, ω) be a symplectic manifold of dimension $2n$. A Lagrangian fibration is a map $\pi : (X, \omega) \to B$ such that its general fiber $\pi^{-1}(b)$ is a Lagrangian submanifold (i.e. $\omega|_{\pi^{-1}(b)} = 0$ and $\dim \pi^{-1}(b) = n$). We allow Lagrangian fibrations to have degenerate fibers. Any Lagrangian fibration is locally given by a completely integrable system. In particular, if the fibers are compact, general fibers are Lagrangian tori.

For a polarized Kähler manifold (X, L), our interest is a relation between a Lagrangian fibration of X and a basis of the space $H^0(X, L)$ of holomorphic sections. A typical example is the case of toric varieties. For a polarized toric variety (X, L) of complex dimension n, $H^0(X, L^k)$ has a basis consisting of Laurent monomials $z^I = z_i^{i_1} \cdots z_n^{i_n}$. Let $\pi : X \to \mathbb{R}^n$ be the moment map of a natural torus action and denote the moment polytope of X by $\Delta = \pi(X) \subset \mathbb{R}^n$. Then each monomial corresponds to a lattice point in Δ:

$$I = (i_1, \ldots, i_n) \in k\Delta \cap \mathbb{Z}^n \longleftrightarrow z^I \in H^0(X, L^k).$$

This relation can be interpreted in terms of geometric quantization. Geometric quantization is a method to associate, for a symplectic manifold, a vector space (or a representation of a certain group or Lie algebra) which

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is called the space of wave functions. This vector space is constructed from a line bundle L with a unitary connection such that its curvature is proportional to the symplectic form, and an additional structure which is called a "polarization"[1]. A complex structure is an example of such polarizations, and the corresponding vector space is the space $H^0(X, L^k)$ of holomorphic sections. On the other hand, a Lagrangian fibration gives another polarization. If X is compact, then the vector space for this polarization is identified with the space formally spanned by Lagrangian fibers satisfying the *Bohr–Sommerfeld condition of level* k. We say that a fiber $\pi^{-1}(b)$ satisfies the Bohr–Sommerfeld condition of level k if the restriction $L^k|_{\pi^{-1}(b)}$ of L^k has trivial holonimies. In the toric case, it is easy to see that $\pi^{-1}(b)$ satisfies the Bohr–Sommerfeld condition of level k if and only if $b \in \Delta \cap \frac{1}{k}\mathbb{Z}^n$

A similar relation can be observed for Abelian varieties. Let $A = \mathbb{C}^n/\Omega\mathbb{Z}^n + \mathbb{Z}^n$ be an Abelian variety with a Kähler form

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum g_{ij} dz^i \wedge d\bar{z}^j = -\sum dx^i \wedge dy^i, \tag{1}$$

where Ω is an $n \times n$ symmetric matrix with positive definite imaginary part $\operatorname{Im}\Omega = (g_{ij})^{-1}$, and $z = \Omega x + y$. Then ω_0 is the first Chern form $c_1(L, h_0)$ of a Hermitian line bundle $(L, h_0) \to A$. In this case, holomorphic sections of L^k are given by *theta functions*. We consider the following Lagrangian fibration

$$\pi : A \longrightarrow T^n, \quad z = \Omega x + y \longmapsto y .$$

Then $H^0(A, L^k)$ has a basis $\{s_b\}_b$ indexed by k-torsion points $b \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$ of T^n (see [5]). This is also interpreted in terms of geometric quantization.

Proposition 1.1 (Weitsman [14]). *A fiber $\pi^{-1}(b)$ of π satisfies the Bohr–Sommerfeld condition of level k if and only if $b \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$. In particular, the space of wave functions for the Lagrangian fibration π is isomorphic to $H^0(A, L^k)$.*

This relation can be also understood in terms of mirror symmetry for Abelian varieties. A mirror partner $\hat{A}$ of A is given by dualizing the torus fibers of $\pi : A \to T^n$. Since the dual torus parametrizes flat line bundles on a torus fiber of π, the line bundle L^k defines a Lagrangian

[1] In the context of geometric quantization, a polarization means a Lagrangian distribution on the complexified tangent bundle, not an ample line bundle.

section S_k of the dual torus fibration $\hat{\pi} : \hat{A} \to T^n$. We identify T^n with the zero section S_0 of $\hat{\pi}$. Then $\pi^{-1}(b)$ satisfies the Bohr–Sommerfeld condition of level k if and only if $b \in S_0 \cap S_k$. The following is a part of the mirror symmetry for Abelian varieties.

Theorem 1.2 (Polischuk–Zaslow [9], Fukaya [4]). *The Floer homology $HF(S_0, S_k)$ of S_0 and S_k is given by*

$$HF(S_0, S_k) = \bigoplus_{b \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n} \mathbb{C}[b],$$

and $s_b \mapsto [b]$ gives an isomorphism

$$H^0(A, L^k) = \operatorname{Hom}(L^0, L^k) \cong HF(S_0, S_k).$$

In these notes, we study the Lagrangian fibration of A using projective embeddings given by the basis $\{s_b\}_b$. Before that, we go back to the toric case. We consider the projective embedding $\iota_k : X \hookrightarrow \mathbb{CP}^{N_k} = \mathbb{P}H^0(X, L^k)^*$ defined by monomials. Let

$$\mu_k : \mathbb{CP}^{N_k} \longrightarrow \Delta_k \subset \operatorname{Lie}(T^{N_k})^*$$

be the moment map of a natural T^{N_k}-action, where Δ_k is the moment polytope of $\mathbb{CP}^{N_k}$. Since the embedding is torus equivariant, the restriction $\pi_k = \mu_k \circ \iota_k : X \to \Delta_k$ of μ_k to X is invariant under the T^n-action. In particular, $\pi_k : X \to \Delta_k$ is also a moment map of X. However, the situation is not so simple in the case of Abelian varieties. Let $\iota_k : A \to \mathbb{CP}^{k^n-1}$ be the projective embedding given by $\{s_b\}_b$, and consider the restriction π_k of the moment map of $\mathbb{CP}^{k^n-1}$. Then π_k is *not* a Lagrangian fibration, since the moment map image B_k of A is a $2n$-dimensional object. Nevertheless, π_k looks "close" to π for large k. In fact π_k is invariant under the translations

$$\Omega x + y \longmapsto \Omega(x + a) + y\,, \quad a \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n\,.$$

We see in this article that

Theorem 1.3. $\{\pi_k\}$ *converges to π as maps between compact metric spaces.*

The precise statement is given in the next section. The case of Kummer varieties is discussed in Section 3.

§2. The case of Abelian varieties

Let $A = \mathbb{C}^n/\Omega\mathbb{Z}^n + \mathbb{Z}^n$ be an n-dimensional Abelian variety, and $L \to A$ a principal polarization defined by

$$L = (\mathbb{C}^n \times \mathbb{C})/\sim,$$

where

$$(z, \zeta) \sim (z + \lambda, \zeta \exp\left(\pi {}^t\bar{\lambda}(\operatorname{Im}\Omega)^{-1}z + \frac{\pi}{2}{}^t\bar{\lambda}(\operatorname{Im}\Omega)^{-1}\lambda + \pi\sqrt{-1}{}^t\lambda_1\lambda_2\right))$$

for $\lambda = \Omega\lambda_1 + \lambda_2 \in \Omega\mathbb{Z}^n + \mathbb{Z}^n$. Then L is *symmetric*, i.e. $(-1)_A^* L \cong L$, where

$$(-1)_A : A \longrightarrow A, \quad z \longmapsto -z$$

is the inverse morphism.

Remark 2.1. The choice of L is not essential. In fact, any other principal polarization can be obtained as a pull-back of L by some translation on A. We remark that the symmetricity condition is necessary when we consider the case of Kummer varieties.

Let ω_0 be the standard Kähler metric defined in (1). Then ω_0 represents the first Chern class $c_1(L)$ of L. We fix a Hermitian metric h_0 on L such that $c_1(L, h_0) = \omega_0$ (such h_0 is unique up to constant multiples). Let T^f and T^b be n-dimensional tori $\mathbb{R}^n/\mathbb{Z}^n$, and identify A with $T^f \times T^b$ by $\Omega x + y \leftrightarrow (x, y)$. Then the projection

$$\pi : A \longrightarrow T^b, \quad \Omega x + y \longmapsto y$$

is a Lagrangian fibration with respect to ω_0.

For each $k \in \mathbb{N}$, we denote the subgroup of k-torsion points in T^b by

$$T_k^b = \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n = \{b_i\}_{i=1,\dots,k^n} \subset T^b.$$

Then the collection of

$$s_{b_i}(z) = Ck^{-\frac{n}{4}} \exp\left(\frac{\pi}{2}k{}^t z(\operatorname{Im}\Omega)^{-1}z\right) \vartheta\begin{bmatrix} 0 \\ -b_i \end{bmatrix}(k^{-1}\Omega, z), \quad b_i \in T_k^b,$$

gives an *orthonormal basis* of $H^0(A, L^k)$ with respect to the L^2-inner product, where

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\Omega, z) = \sum_{l \in \mathbb{Z}^n} \exp\Big(\pi\sqrt{-1}{}^t(l+a)\Omega(l+a) + 2\pi\sqrt{-1}{}^t(l+a)(z+b)\Big),$$

and C is a constant depending only on Ω (and h_0) (see [6] for a proof). An important property for our purpose is that s_{b_i} becomes concentrated on the fiber $\pi^{-1}(b_i)$ as $k \to \infty$. More precisely,

Lemma 2.2. *s_{b_j} has the following asymptotic behavior*

$$s_{b_j}(z) = Ck^{\frac{n}{4}} \exp\left(\frac{\pi k}{2} {}^t z (\operatorname{Im}\Omega)^{-1} z\right) \cdot \exp\left(\frac{\pi k}{\sqrt{-1}} {}^t(z - b_j)\Omega^{-1}(z - b_j)\right)(1 + \phi) \tag{2}$$

with

$$|\phi| = O\left(\frac{1}{\sqrt{k}}\right), \quad |d\phi| = O(1).$$

See Lemma 2.2 in [7] or the proof of Lemma 4.2 in [6].

We consider a projective embedding of A given by $\{s_b\}_b$:

$$\iota_k : A \hookrightarrow \mathbb{CP}^{k^n-1}, \; z \mapsto \left(\vartheta\begin{bmatrix} 0 \\ -b_1 \end{bmatrix}(k^{-1}\Omega, z) : \cdots : \vartheta\begin{bmatrix} 0 \\ -b_{k^n} \end{bmatrix}(k^{-1}\Omega, z)\right).$$

Recall that the moment map of a natural torus action on $\mathbb{CP}^{k^n-1}$ is given by

$$\mu_k : (Z^1 : \cdots : Z^{k^n}) \longmapsto \frac{1}{\sum |Z^i|^2}\left(|Z^1|^2, \ldots, |Z^{k^n}|^2\right),$$

where $(Z^1 : \cdots : Z^{k^n})$ is the homogeneous coordinate of $\mathbb{CP}^{k^n-1}$, and the dual $(\operatorname{Lie} T^{k^n-1})^*$ of the Lie algebra of T^{k^n-1} is identified with

$$\left\{(\xi_1, \ldots, \xi_{k^n}) \in \mathbb{R}^{k^n} \;\middle|\; \sum \xi_i = 1\right\}.$$

We set $B_k := \mu_k(\iota_k(A))$ and $\pi_k := \mu_k \circ \iota_k : A \to B_k$ as above. We denote the restriction of the normalized Fubini–Study metric to X by

$$\omega_k := \frac{1}{k}\iota_k^* \omega_{\mathrm{FS}}.$$

Then ω_k also represents $c_1(L)$.

As we remarked in the previous section, we have $\dim B_k = \dim_{\mathbb{R}} A = 2n$, while $\dim T^n = n$. We thus compare $\pi : (A, \omega_0) \to T^b$ and $\pi_k : (A, \omega_k) \to B_k$ as maps between metric spaces. For that purpose, we need to define distances on T^b and B_k. We define a metric on T^b in such a way that $\pi : (A, \omega_0) \to T^b$ is a Riemannian submersion. The distance on B_k is induced from a metric on the moment polytope Δ_k. The metric on Δ_k is also defined in such a way that

$$\mu_k : \left(\mathbb{CP}^{N_k}, \frac{1}{k}\omega_{\mathrm{FS}}\right) \longrightarrow \Delta_k$$

is a Riemannian submersion in the interior of Δ_k.

Now we can state our main theorem.

Theorem 2.3 ([6]). *The sequence of maps $\pi_k : (A, \omega_k) \to B_k$ converges to $\pi : (A, \omega_0) \to T^b$ in the following sense.*

(i) *$\{\omega_k\}$ converges to ω_0 in the C^∞-topology as $k \to \infty$. In particular, the sequence $\{(A, \omega_k)\}$ of Riemannian manifolds converges to (A, ω_0) with respect to the Gromov–Hausdorff distance.*

(ii) *$\{B_k\}$ converge to T^b as $k \to \infty$ with respect to the Gromov–Hausdorff distance.*

(iii) *$\{\pi_k\}$ converges to π as maps between metric spaces.*

Before proceeding to the proof, we recall definitions of the Gromov–Hausdorff convergence and the convergence of maps. First we recall a definition of the *Hausdorff distance.* Let Z be a metric space and $X, Y \subset Z$ two subsets. We denote the ε-neighborhood of X in Z by $B(X, \varepsilon)$. Then the Hausdorff distance between X and Y is given by

$$d_{\mathrm{H}}^{Z}(X, Y) = \inf\left\{\varepsilon > 0 \mid X \subset B(Y, \varepsilon),\ Y \subset B(X, \varepsilon)\right\}.$$

For metric spaces X and Y, the *Gromov–Hausdorff distance* between X and Y is defined by

$$d_{\mathrm{GH}}(X, Y) = \inf\{d_{\mathrm{H}}^{Z}(X, Y) \mid X, Y \hookrightarrow Z \text{ are isometric embeddings}\}.$$

Next we recall the notion of convergence of maps (see also [8]). Let $f_k : X_k \to Y_k$ and $f : X \to Y$ be maps between metric spaces. Suppose that X_k and Y_k converge to X and Y respectively with respect to the Gromov–Hausdorff distance. Then, by definition, there exist isometric embeddings X, $X_k \hookrightarrow Z$ and Y, $Y_k \hookrightarrow W$ into some metric spaces Z and W such that X_k (resp. Y_k) converges to X (resp. Y) with respect to the Hausdorff topology in Z (resp. W). We say that $\{f_k\}$ converges to f if, for every sequence $x_k \in X_k$ converging to $x \in X$ in Z, $f_k(x_k)$ converges to $f(x)$ in W.

Outline of the proof. (i) is a direct consequence of the following theorem.

Theorem 2.4 (Ruan [10], Zelditch [15]). *Let (X,ω) be a compact Kähler manifold and $(L,h)\to X$ a Hermitian line bundle such that $\omega = c_1(L,h)$. For each $k \gg 1$, we take a basis $\{s_0,\dots,s_{N_k}\}$ of $H^0(X,L^k)$ and consider a projective embedding $\iota_k : X \hookrightarrow \mathbb{CP}^{N_k}$ given by $\{s_i\}$. Set $\omega_k = (1/k)\iota_k^*\omega_{\mathrm{FS}}$. If the basis is orthonormal with respect to the L^2-inner product for each $k \gg 1$, then, for each q, there exists a constant $C_q > 0$ independent of k such that*

$$\|\omega - \omega_k\|_{C^q} \le \frac{C_q}{k}.$$

For the proof of (ii), we decompose $T\mathbb{CP}^{N_k}$ into the horizontal and vertical components:

$$\begin{array}{cccccc} (3) & T_p\mathbb{CP}^{N_k} & = & T_{\mathbb{CP}^{N_k}/\Delta_k,p} & \oplus & (T_{\mathbb{CP}^{N_k}/\Delta_k,p})^{\perp} \\ & \xi & = & \xi^V & + & \xi^H, \end{array}$$

where $T_{\mathbb{CP}^{N_k}/\Delta_k,p} = \ker d\mu_k$ is the tangent space to the fiber of μ_k and $(T_{\mathbb{CP}^{N_k}/\Delta_k,p})^{\perp}$ is its orthogonal complement with respect to the Fubini–Study metric. Similarly we decompose the tangent space of A:

$$T_zA = T_{A/T^b,z} \oplus (T_{A/T^b,z})^{\perp}, \tag{4}$$

where $(T_{A/T^b,z})^{\perp}$ is the orthogonal complement of $T_{A/T^b,z} = \ker d\pi$ with respect to ω_0. Then the metrics on Δ_k and T^b are the restrictions of ω_k and ω_0 on the horizontal subspaces, respectively. Since we know from (i) that ω_0 and ω_k are "close" for large k , it suffices to show that also the decompositions (3) and (4) are "close".

Lemma 2.5. (i) *If $\xi \in T_{A/T^b,z}$, then*

$$\left|d\iota_k(\xi)^H\right| \le \frac{C}{\sqrt{k}}|\xi|.$$

(ii) *If $\eta \in (T_{A/T^b,z})^{\perp}$, then*

$$\left|d\iota_k(\eta)^V\right| \le \frac{C}{\sqrt{k}}|\eta|.$$

This lemma follows from the asymptotic behavior (2) of theta functions. By using the above estimates, we have

$$d_{\mathrm{GH}}(T^b, B_k) \le \frac{C}{\sqrt{k}}.$$

In fact, we can show that the composition

$$\varphi_k = \pi_k \circ \sigma_0 : T^b \longrightarrow B_k$$

of the zero section $\sigma_0 : T^b \to A$ and π_k is "almost isometric" (a $(C/\sqrt{k})$-Hausdorff approximation (see [3] for the definition)).

(iii) easily follows from the proof of (ii).

§3. The case of Kummer varieties

Let (A, L) be a polarized Abelian variety as above. The Kummer variety of A is an orbifold defined by

$$X = A/(-1)_A .$$

Since L is symmetric, there is a line bundle $M \to X$ satisfying

$$p^*M \cong L^2 ,$$

where $p : A \to X$ is the natural projection. From the fact that $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(A)$ is injective, we have $p^*M^k \cong L^{2k}$. It is easy to see that $p^* : H^0(X, M^k) \to H^0(A, L^{2k})$ is injective and the image is spanned by

$$s_{b_i} + s_{-b_i} , \quad b_i \in T^b_{2k}$$

(see [1] and [11]). We identify $H^0(X, M^k)$ with its image in $H^0(A, L^{2k})$. Note that

$$N_k + 1 := \dim H^0(X, M^k) = 2^{n-1}(k^n + 1) .$$

Let ω be an orbifold Kähler metric induced from the flat metric $2\omega_0$ on A. Then $[\omega] = c_1(M)$. Moreover we have a Lagrangian fibration

$$\pi : (X, \omega) \to B = T^b/(-1)$$

induced by $\pi : A \to T^b$. We set

$$t_i = \begin{cases} \dfrac{1}{\sqrt{2^n}}(s_{b_i} + s_{-b_i}) , & \text{if } b_i \in T^b_{2k} \backslash T^b_2, \\ \dfrac{1}{\sqrt{2^{n-1}}} s_{b_i} , & \text{if } b_i \in T^b_2 . \end{cases}$$

Then $\{t_i\}$ is an orthonormal basis of $H^0(X, M^k)$.

We denote by $\iota_k : X \to \mathbb{CP}^{N_k}$ a projective embedding given by $\{t_i\}$, $\pi_k : X \to B'_k$ the restriction of the moment map of $\mathbb{CP}^{N_k}$, and $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$. Then the same theorem holds for X.

Theorem 3.1 ([7]). *The sequence of maps $\pi_k : (X, \omega_k) \to B'_k$ converges to $\pi : (X, \omega) \to B$ as $k \to \infty$ in the following sense.*

(i) *$\{(X, \omega_k)\}$ converges to (X, ω) as $k \to \infty$ with respect to the Gromov–Hausdorff distance.*

(ii) *$\{B'_k\}$ converge to B with respect to the Gromov–Hausdorff distance.*

(iii) *$\{\pi_k\}$ converges to π as maps between metric spaces.*

Outline of the proof. (i) follows from an orbifold version of Theorem 2.4:

Theorem 3.2 (Dai–Liu–Ma [2]). *Let (X, ω) be a compact Kähler orbifold of dimension $n \geq 2$ and $(M, h) \to X$ an orbifold Hermitian line bundle with $c_1(M, h) = \omega$. For $k \gg 1$, we consider a projective embedding $\iota_k : X \to \mathbb{CP}^{N_k}$ given by an orthonormal basis of $H^0(X, M^k)$, and set $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ as above. Then there exist constants C_q, $\delta > 0$ such that*

$$\|\omega - \omega_k\|_{C^q, z} \leq C_q \left(\frac{1}{k} + k^{\frac{q}{2}} e^{-k\delta r(z)^2} \right),$$

where $\|\cdot\|_{C^q, z}$ is the C^q-norm at $z \in X$, and $r(z)$ is the distance between z and the singular set $\mathrm{Sing}\,(X)$ *of X.*

(ii) Note that each singular fiber is isomorphic to $T^n/(-1)$ and appears on the singular set $\mathrm{Sing}(B) = T_2^b/(-1)$ of B. For each $b \in \mathrm{Sing}(B)$, we denote the $\sqrt{(1/\delta k)\log k}$-neighborhood of the singular fiber $\pi^{-1}(b)$ by

$$N_{b,k} = \left\{ z \in X \;\middle|\; d(z, \pi^{-1}(b)) \leq \sqrt{\frac{\log k}{\delta k}} \right\},$$

where δ is the constant in Theorem 3.2, and set

$$X(k) = X \setminus \bigcup_{b \in \mathrm{Sing(B)}} N_{b,k}\,.$$

Then we can show that $\pi(N_{b,k})$ and $\pi_k(N_{b,k})$ are small for large k (their diameters can be bounded by $O\left(\sqrt{(1/k)\log k}\right)$). Thus the neighborhoods of singular fibers do not affect the Gromov–Hausdorff convergence. On the other hand, we have the same estimates as in Lemma 2.5 on $X(k)$. Hence we can apply the arguments for A to this situation.

References

[1] Th. Bauer, Projective images of Kummer surfaces, Math. Ann., **299** (1994), 155–170.

[2] X. Dai, K. Liu and X. Ma, On the asymptotic expansion of Bergman kernel, C. R. Math. Acad. Sci. Paris, **339** (2004), 193–198.

[3] K. Fukaya, Hausdorff convergence of Riemannian manifolds and its applications, In: Recent Topics in Differential and Analytic Geometry, Adv. Stud. Pure Math., **18-**I, Academic Press, Boston, MA, 1990, pp. 143–238.

[4] K. Fukaya, Mirror symmetry of Abelian varieties and multi-theta functions, J. Alg. Geom., **11** (2002), 393–512.

[5] D. Mumford, Tata lectures on theta. I, Progress in Math., **28**, Birkhäuser, 1983.

[6] Y. Nohara, Projective embeddings and Lagrangian fibrations of Abelian varieties, Math. Ann., **333** (2005), 741–757.

[7] Y. Nohara, Projective embeddings and Lagrangian fibrations of Kummer varieties, Internat. J. Math., to appear.

[8] P. Petersen, Riemannian Geometry, Graduate Texts in Math., **171**, Springer-Verlag, New York, 1998.

[9] A. Polishchuk and E. Zaslow, Categorical mirror symmetry: the elliptic curve, Adv. Theor. Math. Phys., **2** (1998), 443–470.

[10] W.-D. Ruan, Canonical coordinates and Bergman metrics, Comm. Anal. Geom., **6** (1998), 589–631.

[11] R. Sasaki, Bounds on the degree of the equations defining Kummer varieties, J. Math. Soc. Japan, **33** (1981), 323–333.

[12] A. Strominger, S. Yau and E. Zaslow, Mirror symmetry is T-duality, Nuclear Phys. B, **479** (1996), 243–259.

[13] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom., **32** (1990), 99–130.

[14] J. Weitsman, Quantization via real polarization of the moduli space of flat connections and Chern–Simons gauge theory in genus one, Comm. Math. Phys., **137** (1991), 175–190.

[15] S. Zelditch, Szegö kernels and a theorem of Tian, Internat. Math. Res. Notices, **6** (1998), 317–331.

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