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A table of θ-curves and handcuff graphs with up to seven crossings

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Abstract.

We enumerate all the θ-curves and handcuff graphs with up to seven crossings by using algebraic tangles and prime basic θ-polyhedra. Here, a θ-polyhedron is a connected graph embedded in a 2-sphere, whose two vertices are 3-valent, and the rest are 4-valent. There exist twenty-four prime basic θ-polyhedra with up to seven 4-valent vertices. We can obtain a θ-curve and handcuff graph diagram from a prime basic θ-polyhedron by substituting algebraic tangles for their 4-valent vertices.

§1. Introduction

This is a summary of author's papers [4]–[7]. A link is n circles $S^1 \cup \cdots \cup S^1$ embedded in a 3-sphere S^3. As a generalization of classical knot theory, we can consider other objects embedded in S^3. A spatial graph is a graph embedded in S^3, and two spatial graphs G, G' are *equivalent* if there is a homeomorphism $h : (S^3, G) \to (S^3, G')$. If the graph consists of two vertices and three edges such that each edge joins the vertices, then it is called a θ-curve. Moreover, if the graph consists of two loops and an edge jointing the vertices of each loop, then it is called a handcuff graph.

In knot theory, there exists the study of creating a prime knot table. Similarly, there exists the study of making a table of spatial graphs. There exist earlier studies on tabulation of spatial graphs as follows: In [8], J. Simon enumerated θ-curves with up to five crossings, and K_4 graphs with up to four crossings. In [3], R. Litherland announced a table of prime θ-curves with up to seven crossings. However, there was no published proof of the completeness of Litherland's table. Then we

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considered to complete the table of prime θ-curves with up to seven crossings.

We applied Conway's method to enumerate θ-curves. In [1], J. H. Conway made an enumeration of prime knots and links by introducing the concept of a *tangle* and a *basic polyhedron*. Here, a tangle is a disjoint union of two arcs and some or no loops properly embedded in a 3-ball B^3, and a basic polyhedron is the 4-regular planar graph which has no bigon. We can obtain knots from basic polyhedra by substituting tangles for their vertices.

In order to apply Conway's method, we need the following works. First, we enumerated algebraic tangles with up to seven crossings ([6]). Second, we constructed a prime basic θ-polyhedron to enumerate prime θ-curves. Here, a θ-polyhedron is a connected graph embedded in a 2-sphere, whose two vertices are 3-valent, and the rest are 4-valent. Then our θ-polyhedron is different from Conway's polyhedron. Since we considered to produce a prime θ-curve table, we omitted a non-prime θ-polyhedron. Then there exist twenty-four prime basic θ-polyhedra with up to seven 4-valent vertices. We can obtain θ-curves from prime basic θ-polyhedra by substituting *algebraic tangles* for their 4-valent vertices.

In [7], the author obtained all the prime θ-curves with up to seven crossings, which are the exactly same table as Litherland's. Litherland classified these θ-curves by constituent knots and the Alexander polynomial, but we classified them by the *Yamada polynomial*. Moreover, we first enumerated all the prime handcuff graphs with up to seven crossings in the same way ([4]). We also classified these handcuff graphs by using the Yamada polynomial.

This paper is organized as follows: In Section 2, we give main results. In Section 3, we introduce the concept of a prime basic θ-polyhedron. In Section 4, we mention the chirality.

§2. Main results

A *θ-curve* Θ is a graph embedded in S^3, which consists of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), such that each edge joins the vertices. A *constituent knot* Θ_{ij}, $1 \leq i < j \leq 3$, is a subgraph of Θ that consists of two vertices (v_1, v_2) and two edges (e_i, e_j). θ-curves are roughly classified by comparing the triples of constituent knots. A θ-curve is said to be *trivial* if it can be embedded in a 2-sphere in S^3.

A *handcuff graph* Φ is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2, respectively. A *constituent link* Φ_{12} is a subgraph of Φ that consists of two vertices

(v_1, v_2) and two edges (e_1, e_2). A handcuff graph is said to be *trivial* if it can be embedded in a 2-sphere in S^3.

Let Γ be a trivalent spatial graph such as a θ-curve Θ or a handcuff graph Φ and Σ a 2-sphere which decomposes S^3 into 3-balls B_1, B_2. If $\Sigma \cap \Gamma$ consists of a single point w, and neither $(\Gamma - w) \cap B_1$ nor $(\Gamma - w) \cap B_2$ is empty, then Σ is called an *admissible sphere of type* I for Γ. If $\Sigma \cap \Gamma$ consists of two points, and the annulus $A = \Sigma \setminus \mathrm{Int}N(\Gamma; S^3)$ is essential in $S^3 \setminus \mathrm{Int}N(\Gamma; S^3)$, then Σ is called an *admissible sphere of type* II for Γ; cf. [9]. If $\Sigma \cap \Gamma$ consists of three points, and neither $\Gamma \cap B_1$ nor $\Gamma \cap B_2$ is an unknotted bouquet (Fig. 1), then Σ is called an *admissible sphere of type* III for Γ; cf. [2]. By an *admissible sphere*, we mean either an admissible sphere of type I, II, or III.

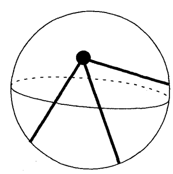


Fig. 1. An unknotted bouquet.

Definition 2.1. A trivalent spatial graph Γ is said to be *prime* if Γ is nontrivial and does not have an admissible sphere.

Note that the definition of primeness of Litherland ([3]) is different from ours. We give the following Main Theorems. The proofs of them are in [7]–[5].

Theorem 2.2. *Table* 1 *lists all the prime θ-curves with up to seven crossings. These θ-curves are mutually inequivalent.*

Theorem 2.3. *Table* 2 *lists all prime handcuff graphs with up to seven crossings. These handcuff graphs are mutually inequivalent.*

The θ-curves in Table 1 and the handcuff graphs in Table 2 are listed by considering their constituent knots and links.

§3. Prime basic θ-polyhedron

Let P_Θ be a connected planar graph. P_Θ is called a *θ-polyhedron* if its two vertices are 3-valent and the other vertices are 4-valent. A θ-polyhedron P_Θ is said to be *basic* if it contains no loop and no bigon. The primeness of a θ-polyhedron is defined in a similar way to the primeness of a trivalent spatial graph; cf. [7] and [4].

Table 1. Prime θ-curves with up to seven crossings.

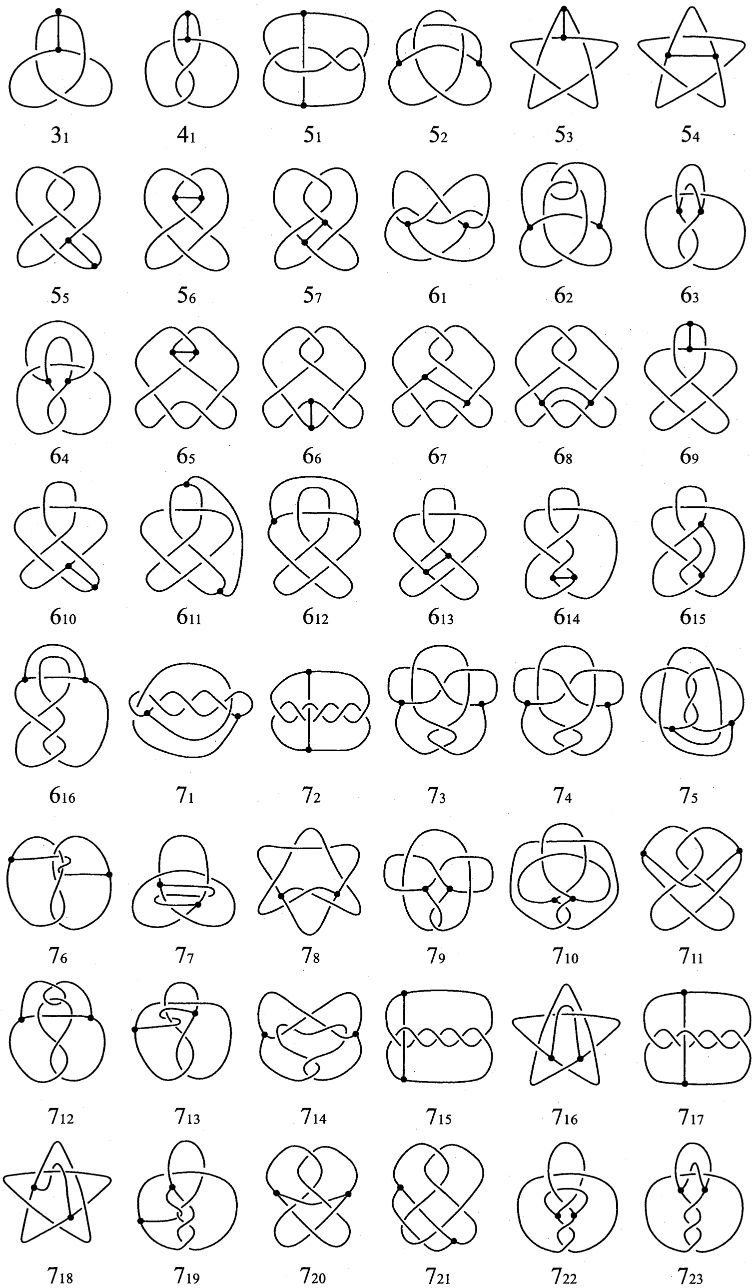
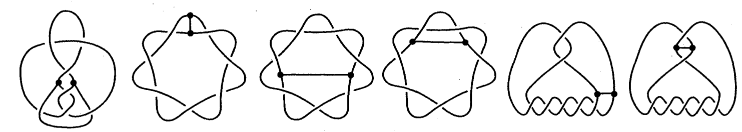
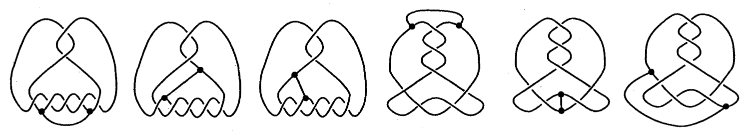


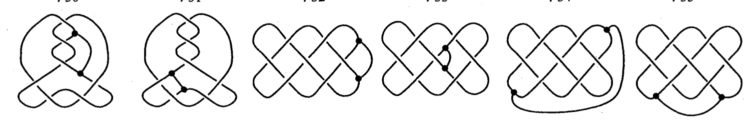
Table 1. Prime θ-curves with up to seven crossings (continued).



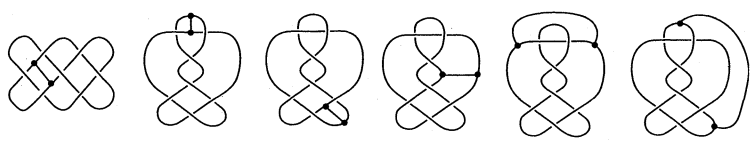
7_{24} 7_{25} 7_{26} 7_{27} 7_{28} 7_{29}



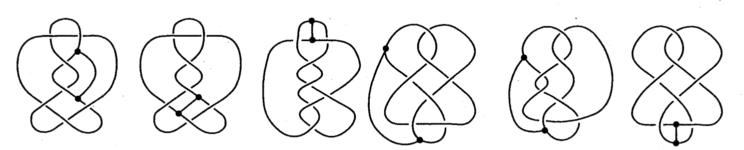
7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35}



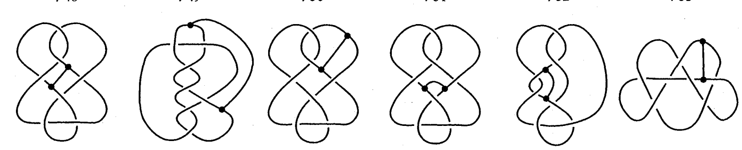
7_{36} 7_{37} 7_{38} 7_{39} 7_{40} 7_{41}



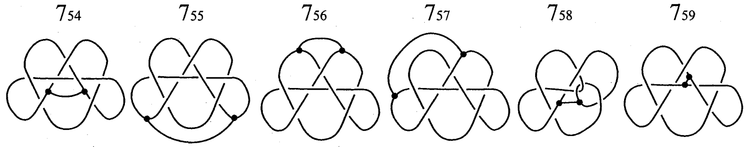
7_{42} 7_{43} 7_{44} 7_{45} 7_{46} 7_{47}



7_{48} 7_{49} 7_{50} 7_{51} 7_{52} 7_{53}

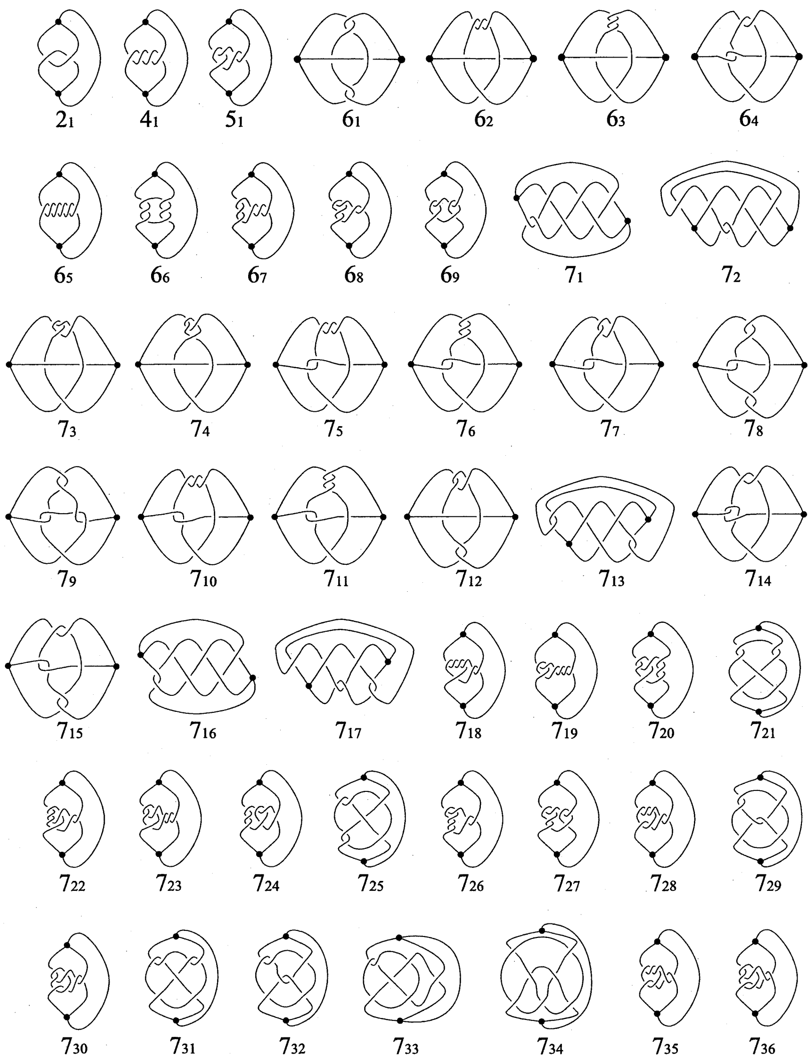


7_{54} 7_{55} 7_{56} 7_{57} 7_{58} 7_{59}



7_{60} 7_{61} 7_{62} 7_{63} 7_{64} 7_{65}

Table 2. Prime handcuff graphs with up to seven crossings.



Prime basic θ-polyhedra are classified into two types, according as if their 3-valent vertices are adjacent or not. We call the former *type-*$\times$ *prime basic θ-polyhedra*, and the latter *type-$*$ prime basic θ-polyhedra.* We give prime basic θ-polyhedra with up to seven 4-valent vertices.

Theorem 3.1. *There exist seven type-$\times$ prime basic θ-polyhedra with up to seven 4-valent vertices as in Fig. 2.*

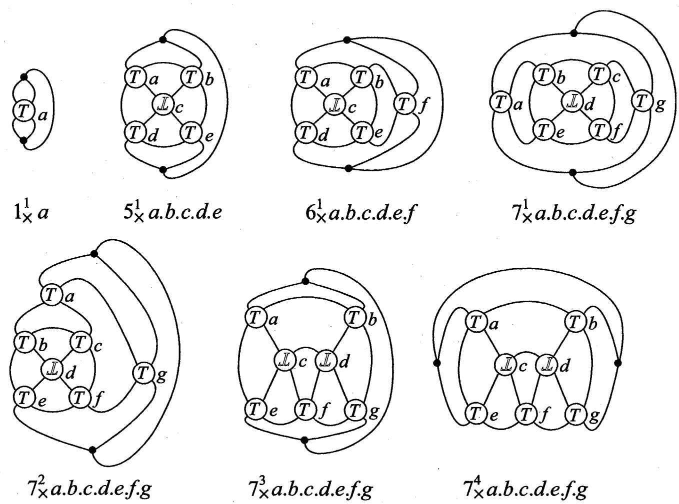


Fig. 2. Type-$\times$ prime basic θ-polyhedra.

Theorem 3.2. *There exist seventeen type-$*$ prime basic θ-polyhedra with up to seven 4-valent vertices as in Fig. 3.*

The proofs of Theorems 3.1 and 3.2 are in [5]. We can obtain θ-curves and handcuff graphs from prime basic θ-polyhedra by substituting *algebraic tangles* for their 4-valent vertices. For the definition of an algebraic tangle we refer to [1]. From Theorems 3.1 and 3.2, we can obtain all the prime θ-curves with up to seven crossings, which are the exactly same ones as in Litherland's; see Table 1. We show that these θ-curves are mutually distinct by investigating the *Yamada polynomial*, which is defined in [10].

In the same way, we also obtain all the prime handcuff graphs with up to seven crossings. We can show that handcuff graphs in Table 2 are mutually distinct by investigating their constituent links and the Yamada polynomial.

§4. Chirality

Finally, we mention the chirality of θ-curves and handcuff graphs.

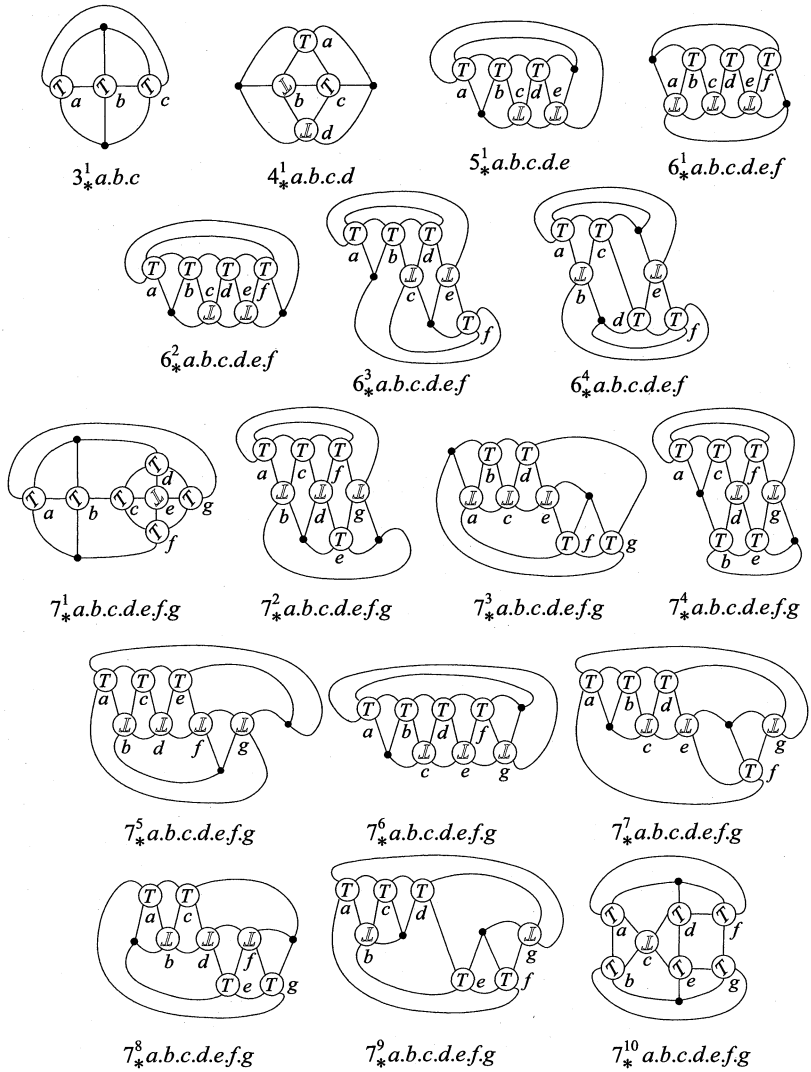


Fig. 3. Type-$*$ prime basic θ-polyhedra.

We prove that the handcuff graphs 2_1, 6_1, and 7_{17} are *achiral* (isotopic to its mirror image) by deformations as in Fig. 4. On the other hand, we can conclude all the θ-curves in Table 1 and almost all handcuff graphs in Table 2 are chiral (not achiral) by computing the Yamada polynomial; cf. [7] and [4].

Therefore we obtain the following:

Theorem 4.1. *There exist only three achiral handcuff graphs in Table* 2 : 2_1, 6_1, *and* 7_{17}.

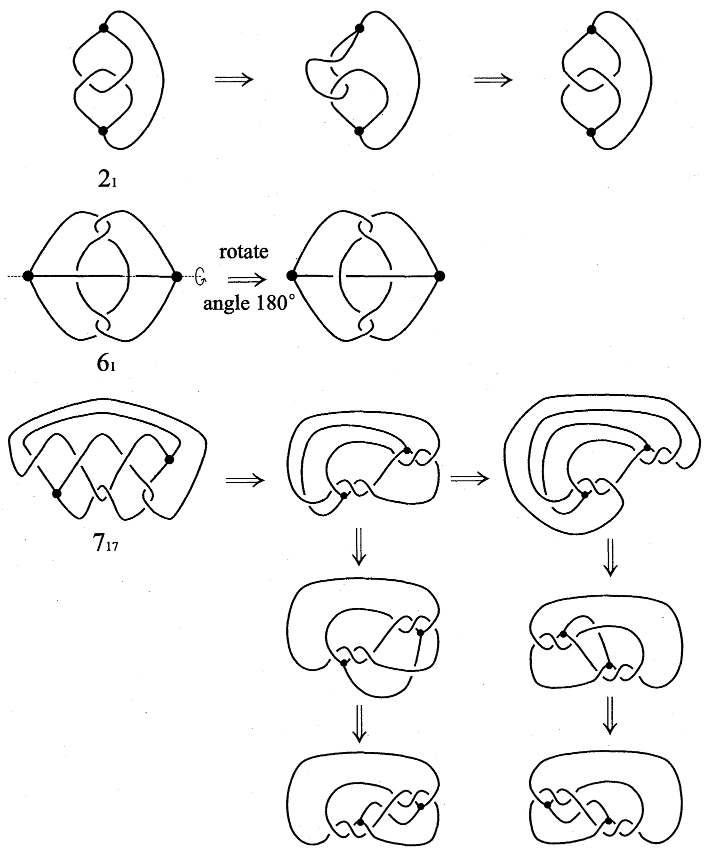


Fig. 4. The handcuff graphs 2_1, 6_1, and 7_{17} are achiral.

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