Advanced Studies in Pure Mathematics 55, 2009
Noncommutativity and Singularities
pp. 263–268

Local Gromov–Witten invariants of cubic surfaces

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Abstract.

We compute local Gromov–Witten invariants of cubic surfaces via nef toric degeneration.

§1. Introduction

Let X be a smooth complex projective surface and K_X its canonical divisor. Assume that the anticanonical divisor $-K_X$ is nef. In this paper, such a surface X will be called a nef surface. Let $\overline{M}_{g,0}(X,\beta)$ (resp. $\overline{M}_{g,1}(X,\beta)$) be the moduli stack of stable maps to X of genus g without marked point (resp. with one marked point) and degree $\beta \in H_2(X,\mathbb{Z})$. Let $\pi : \overline{M}_{g,1}(X,\beta) \to \overline{M}_{g,0}(X,\beta)$ be the forgetful map and $ev : \overline{M}_{g,1}(X,\beta) \to X$ be the evaluation at the marked point. If β satisfies the condition

$$\int_\beta c_1(-K_X) > 0, \tag{1}$$

then the rank of $R^1\pi_* ev^* K_X$ is equal to the virtual dimension of the moduli space $\overline{M}_{g,0}(X,\beta)$, since $R^0\pi_* ev^* K_X = 0$. The rank is given by

$$(1-g)(\dim X - 3) + \int_\beta c_1(-K_X). \tag{2}$$

Definition 1.1. *For $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(X,\mathbb{Z})$ satisfying (1), the local Gromov–Witten (GW) invariant $N_{g,\beta}(K_X)$ of X of genus g and the homology class β is defined by*

$$N_{g,\beta}(K_X) = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} c_{top}(R^1\pi_* ev^* K_X),$$

Received August 2, 2007.
Revised March 29, 2008.
2000 *Mathematics Subject Classification.* 14N35, 53D45.
Key words and phrases. Gromov–Witten invariants.

where c_{top} denotes the Chern class of degree (2).

As long as the nef assumption and condition (1) are satisfied, the local GW invariants are deformation invariant.

Local GW invariants of del Pezzo surfaces S_d, i.e. surfaces whose anticanonical divisors $-K_{S_d}$ are ample (therefore nef) and $(-K_{S_d})^2 = d$ $(1 \le d \le 9)$, have been intensively studied in physics in relation to the non-critical string by mirror symmetry, Seiberg–Witten curve technique and the geometric transition. In the case of toric del Pezzo surfaces (i.e. $6 \le d \le 9$), an explicit formula for the generating function at all genera is known [2, 3, 1, 7, 12, 11]. For nontoric del Pezzo surfaces $(1 \le d \le 5)$, Diaconescu and Florea proposed a closed formula [5] by using the conjectural ruled vertex formalism [4].

For S_3, S_4, S_5 which admit smooth nef toric degenerations (nef toric surfaces which are deformation equivalent to them), the generating function of local GW invariants can be obtained from those of the toric degenerations. We will explain the deformation invariance of local GW invariants and a formula for the generating function for S_3. The materials of this article are based on the joint work [8] with Satoshi Minabe.

§2. Nef toric degeneration of S_3

Recall that a del Pezzo surface of degree 3, S_3, is realized as a blowup of $\mathbb{P}^2$ at six points in general position. Let $e_1, \cdots, e_6 \in \mathrm{Pic}(S_3)$ be the classes of the exceptional curves of the blowup and l be the pullback of the class of a line in $\mathbb{P}^2$. It admits a smooth deformation to a nef toric surface S_3^0 whose fan is shown in Figure 1. Let $C_i \cong \mathbb{P}^1$ $(1 \le i \le 9)$ be the toric divisors corresponding to the primitive generators v_i $(1 \le i \le 9)$ of the fan. The following gives an isomorphism between $\mathrm{Pic}(S_3^0)$ and $\mathrm{Pic}(S_3)$.

$$\begin{array}{lll} C_1 \mapsto e_2 - e_5, & C_2 \mapsto l - e_2 - e_3 - e_6, & C_3 \mapsto e_6, \\ C_4 \mapsto e_3 - e_6, & C_5 \mapsto l - e_1 - e_3 - e_4, & C_6 \mapsto e_4, \\ C_7 \mapsto e_1 - e_4, & C_8 \mapsto l - e_1 - e_2 - e_5, & C_9 \mapsto e_5. \end{array} \tag{3}$$

§3. Deformation invariance of local GW invariants

Let $P_X := \mathbb{P}(K_X \oplus O_X)$ be the projective compactification of the canonical bundle and $\iota : X \hookrightarrow P_X$ be the inclusion as the zero section of $K_X \subset P_X$. By considering the $\mathbb{C}^*$-action on the fiber $\mathbb{P}^1$ of P_X and using the localization formula, we can show that the local GW invariant

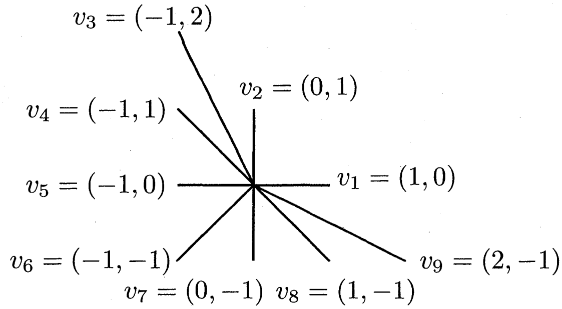

Fig. 1. The fan of the nef toric degeneration S_3^0

$N_{g,\beta}(K_X)$ of a nef surface X and $\beta \in H_2(X, \mathbb{Z})$ satisfying (1) is the same as the ordinary GW invariant of P_X:

$$N_{g,\iota_*\beta}(P_X) = \int_{[\overline{M}_{g,0}(P_X,\iota_*\beta)]^{vir}} 1 \,.$$

This shows that the deformation invariance of the local GW invariants in this case follows from that of the ordinary GW invariants [10]:

$$N_{g,\beta}(K_X) = N_{g,\iota_*\beta}(P_X) = N_{g,\iota'_*\beta'}(P_{X'}) = N_{g,\beta'}(K_{X'}).$$

§4. Equivariant local GW invariants of toric surfaces

In this section, we discuss equivariant local GW invariants of smooth toric surfaces.

4.1. Equivariant local GW invariants

Let X be a smooth toric surface. Let T denote the complex torus $(\mathbb{C}^*)^2$ acting on X. Let $v_1, \ldots, v_r$ be the primitive generators of the fan arranged couterclockwisely and let $C_1, \ldots, C_r$ be the corresponding toric divisors. Then the canonical divisor is written as $K_X^T = -\sum_{i=1}^r C_i$.

Definition 4.1. *For $\beta \in H_2(X, \mathbb{Z})$ and $g \in \mathbb{Z}_{\geq 0}$, the equivariant local GW invariant $N_{g,\beta}^T(K_X)$ is*

$$N_{g,\beta}^T(K_X) = \int_{[\overline{M}_{g,0}(X,\beta)^T]^{vir}} \frac{e_T(R^1\pi_*\mu^*K_X^T)}{e_T(R^0\pi_*\mu^*K_X^T)} \frac{1}{e_T(Norm)}$$

where $\overline{M}_{g,0}(X,\beta)^T$ is the torus fixed loci in $\overline{M}_{g,0}(X,\beta)$ and $e_T(Norm)$ is the equivariant Euler class of its virtual normal bundle.

We remark that a priori $N^T_{g,\beta}(K_X)$ is a rational function in the generators t_1, t_2 of the equivariant cohomology ring $H^*_T(pt) = \mathbb{Z}[t_1, t_2]$ of a point. However, it is actually a rational number.

By applying the virtual localization with respect to T-action, we have

Proposition 1. *If $-K_X$ is nef and β satisfies* (1), *then* $N_{g,\beta}(K_X) = N^T_{g,\beta}(K_X)$.

Remark 1. *In general, $N_{g,\beta}(K_X) = N^T_{g,\beta}(K_X)$ may not hold. For example, consider the Hirzebruch surface $\mathbb{F}_2$ of degree 2 and the class of the (-2)-curve $\beta_0 \in H_2(\mathbb{F}_2, \mathbb{Z})$ which does not satisfy the condition* (1). *For this β_0, $N_{0,\beta_0}(\mathbb{F}_2) = 0$ since the virtual dimension of $\overline{M}_{0,0}(\mathbb{F}_2, \beta_0)$ is negative whereas $N^T_{0,\beta_0}(\mathbb{F}_2) = -1$.*

Now consider the generating function of equivariant GW invariants summed over all genera and all nonzero second homology classes:

$$F^T_X(\lambda, Q) = \sum_{\beta \neq 0} \sum_{g \geq 0} N^T_{g,\beta}(K_X) \lambda^{2g-2} Q^\beta. \tag{4}$$

Here λ is the genus expansion parameter and Q^β denotes an element in the group ring of $H_2(X, \mathbb{Z})$.

For a sequence of integers $(c_1, \ldots, c_r)$, define

$$\mathcal{Z}_{c_1,\ldots,c_r}(q, t_1, \ldots, t_r) = \Big[\sum_{\nu^1,\ldots,\nu^r} \prod_{i=1}^{r} ((-1)^{c_i} t_i)^{|\nu^i|} q^{\frac{c_i \kappa(\nu^i)}{2}} W_{\nu^i,\nu^{i+1}}(q) \Big].$$

Here the summation is over r partitions ν^i $(1 \leq i \leq r)$, $\kappa(\mu) = \sum_i \mu_i(\mu_i - 2i + 1)$ for a partition $\mu = (\mu_1, \mu_2, \ldots)$, $W_{\mu,\nu}(q) = s_\mu(q^\rho) s_\nu(q^{\mu+\rho}) \in \mathbb{Q}(q^{\frac{1}{2}})$, $q^{\mu+\rho} = (q^{\mu_i - i + \frac{1}{2}})_{i \geq 1}$, and s_μ is the Schur function.

Proposition 2.

$$F^T_X(\lambda, Q) = \log \Big[\mathcal{Z}_{C_1^2,\ldots,C_r^2}(e^{\sqrt{-1}\lambda}, Q^{[C_1]}, \ldots, Q^{[C_r]}) \Big],$$

where C_i^2 is the self-intersection number of the torus-invariant curve C_i.

This result is due to [7, 12, 11]. Although only the case of Fano toric surfaces are treated in [12], the calculation works for any smooth toric surface if local GW invariants are replaced with equivariant ones. This is why we introduced the notion of equivariant local GW invariants.

4.2. Local GW invariants of S_3

We define the generating function $F_X(\lambda, Q)$ of local GW invariants of a nef surface X by the formula same as (4) except for the summation over $H_2(X, \mathbb{Z})$ is restricted to β satisfying (1).

Given Proposition 1, if X is a Fano toric surface, then $F_X(\lambda, Q) = F_X^T(\lambda, Q)$. On the other hand, if X is nef but not Fano, then the generating function of local GW invariants can be obtained by subtracting the contribution of $\beta \in H_2(X, \mathbb{Z})$ which do not satisfy (1).

For S_3^0, such effective T-invariant cycles are supported on the three chains of (-2)-curves, $C_1 + C_2$, $C_4 + C_5$, $C_7 + C_8$. Their contribution turns out to be

$$\log \prod_{i=1,4,7} Z_{(-2)}\left(\lambda, Q^{[C_i]}\right) Z_{(-2)}\left(\lambda, Q^{[C_{i+1}]}\right) Z_{(-2)}\left(\lambda, Q^{[C_i]+[C_{i+1}]}\right).$$

where

$$Z_{(-2)}(\lambda, t) = \exp\left[-\sum_{j \geq 1} \frac{1}{j} \left(2 \sin \frac{j\lambda}{2}\right)^{-2} t^j \right].$$

Finally, the generating function of local GW invariants of S_3 is obtained by applying the deformation invariance to S_3 and S_3^0.

Theorem 1. *The generating function $F_{S_3}(\lambda, Q)$ of local GW invariants of S_3 is given by*

$$\exp[F_{S_3}(\lambda, Q)] = \frac{\mathcal{Z}_{\vec{c}}(e^{\sqrt{-1}\lambda}, t_1, \ldots, t_9)}{\prod_{i=1,4,7} Z_{(-2)}(\lambda, t_i) Z_{(-2)}(\lambda, t_{i+1}) Z_{(-2)}(\lambda, t_i t_{i+1})}$$

with $\vec{c} = (-2, -2, -1, -2, -2, -1, -2, -2, -1)$ and the identification

$$t_1 = Q^{e_2 - e_5}, \quad t_2 = Q^{l - e_2 - e_3 - e_6}, \quad t_3 = Q^{e_6},$$

$$t_4 = Q^{e_3 - e_6}, \quad t_5 = Q^{l - e_1 - e_3 - e_4}, \quad t_6 = Q^{e_4},$$

$$t_7 = Q^{e_1 - e_4}, \quad t_8 = Q^{l - e_1 - e_2 - e_5}, \quad t_9 = Q^{e_5}$$

Acknowledgement. The author thanks the referee for many suggestions for the improvement of the manuscript.

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