Advanced Studies in Pure Mathematics 55, 2009
Noncommutativity and Singularities
pp. 249–262

The quantum Knizhnik–Zamolodchikov equation and non-symmetric Macdonald polynomials

Masahiro Kasatani and Yoshihiro Takeyama

Abstract.

We construct special solutions of the quantum Knizhnik–Zamolodchikov equation on the tensor product of the vector representation of the quantum algebra of type A_{N-1}. They are constructed from non-symmetric Macdonald polynomials through the action of the affine Hecke algebra.

§1. Introduction

In the present paper, we construct special solutions of the quantum Knizhnik–Zamolodchikov (qKZ) equation from non-symmetric Macdonald polynomials.

The qKZ equation, derived by Frenkel and Reshetikhin [FR], is the system of difference equations satisfied by matrix elements of the vertex operators in the representation theory of the quantum affine algebra. In this paper we consider the qKZ equation on the tensor product of the vector representation of the quantum algebra $U_q(sl_N)$:

$$\begin{aligned} & G(z_1, \ldots, pz_m, \ldots, z_n) \\ = \; & R_{m,m-1}(pz_m/z_{m-1}) \cdots R_{m,1}(pz_m/z_1) \times \left(\textstyle\prod_{j=1}^{N-1} \kappa_j^{h_j}\right)_m \\ & \times R_{n,m}(z_n/z_m)^{-1} \cdots R_{m+1,m}(z_{m+1}/z_m)^{-1} \\ & \times G(z_1, \ldots, z_m, \ldots, z_n). \end{aligned}$$

Here $G(z_1, \ldots, z_n)$ is an unknown function taking values in $V^{\otimes n}$, where $V \simeq \mathbb{C}^N$ is the vector representation. The operator $R(z)$ is the R-matrix (see (2.2) below), h_j $(j = 1, \ldots, N-1)$ is the basis of the Cartan subalgebra of sl_N, and $p, \kappa_1, \ldots, \kappa_{N-1}$ are parameters of the equation.

Received August 2, 2007.

2000 *Mathematics Subject Classification.* 39A13, 33C52, 81R50.

Key words and phrases. qKZ equation, affine Hecke algebra, non-symmetric Macdonald polynomial.

The indices of the operators in the right hand side signify the positions of the components in $V^{\otimes n}$ where the operator acts. The value ℓ determined by $p = q^{2(N+\ell)}$ is called a level of the qKZ equation.

There are some methods to construct solutions of the qKZ equation. One of them is to use multiple integrals of the hypergeometric type [Mi, VT, MTT]. This works for any parameters $p, \kappa_1, \ldots, \kappa_{N-1}$ such that the multiple integrals converge. Another method is the bosonization of vertex operators. For the integrable irreducible highest weight $U_q(\widehat{sl}_N)$-modules of level one, the bosonization is constructed by Koyama [Ko]. By using it, Nakayashiki calculated the matrix element of the vertex operators [N]. It gives by definition a solution to the qKZ equation of level one, where the parameters $\kappa_1, \ldots, \kappa_{N-1}$ are determined from the highest weight.

Recently Di Francesco and Zinn–Justin constructed a polynomial solution in the case of level one [DZ] by using the representation theory of the affine Hecke algebra (AHA). In a similar manner Kasatani and Pasquier obtained a solution of the qKZ equation of level $-1/2$ associated with $U_q(\widehat{sl}_2)$ [KP]. In this paper we generalize these results to the case of $U_q(\widehat{sl}_N)$ and other levels.

Let us give a sketch of our construction of solutions. We use the spin basis instead of the path basis in the construction of [DZ] and [KP]. Expand the unknown function $G(z_1, \ldots, z_n)$ into a linear combination of the tensor products $v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n}$, where $\{v_\epsilon\}_{\epsilon=0}^{N-1}$ is the standard basis of V. We consider the set of functions which appear in the expansion as coefficients. The qKZ equation can be described as a condition of constraint for the functions. In this paper we consider a stronger condition than the qKZ equation itself, and call a set of functions satisfying the condition a *qKZ family* (see Definition 3.3 below).

The defining condition of a qKZ family is described in terms of the action of the AHA on the space of functions. The generators of the AHA consist of two sets of elements $T_i \, (1 \le i \le n-1)$ and $Y_j \, (1 \le j \le n)$ satisfying some relations (see Definition 3.1 below). The actions of T_i and Y_j are given by the Demazure–Lusztig operator and the q-Dunkl operator, respectively. From a viewpoint of the representation theory, a qKZ family is a set of vectors which move to each other by the action of the generators of the AHA. Moreover, if one vector is known, then all the vectors are determined through the action of the AHA. Hence the linear span of the vectors of a qKZ family determines a cyclic module of the AHA.

Now we return to the description of our construction of solutions. From the definition, a qKZ family contains a joint eigenfunction of the

q-Dunkl operators Y_j. Moreover, it is also an eigenfunction of some of the Demazure–Lusztig operators T_i. We prove that such an eigenfunction, conversely, generates a qKZ family. Thus construction of a qKZ family is reduced to that of an eigenfunction of the q-Dunkl operators and the Demazure–Lusztig operators. As is well known, non-symmetric Macdonald polynomials [Ma] are such eigenfunctions. Therefore we can construct special solutions of the qKZ equation from non-symmetric Macdonald polynomials.

Cherednik [C] and Kato [Kat] unveiled the relation between the qKZ equation and the eigenvalue problem of the Macdonald type: a certain linear combination of the coefficients in a solution of the qKZ equation gives a symmetric joint eigenfunction of the q-Dunkl operators (see [Mi] for the explicit formula in the case of $n = N$). Our construction is consistent with this result because symmetric Macdonald polynomials can be obtained as linear combinations of non-symmetric ones.

The plan of this paper is as follows. First we recall the definition of the qKZ equation in Section 2. In Section 3 we give the definition of qKZ family, and prove that a qKZ family is constructed from a joint eigenfunction of the q-Dunkl operators and some of the Demazure–Lusztig operators. In Section 4 we give explicitly the construction explained above of solutions of the qKZ equation in the case where the level is generic or a value of the form $\frac{k+1}{r-1} - N$, where k and r are positive integers such that $1 \le k \le \min\{n-1, N\}$, $r \ge 2$, and $k+1$ and $r-1$ are coprime. Then the parameters $\kappa_1, \dots, \kappa_{N-1}$ are determined from the eigenvalues of the non-symmetric Macdonald polynomial for the q-Dunkl operators. Here it should be noted that in the latter case we need to specialize the two parameters in non-symmetric Macdonald polynomials, some of which are proved to be well-defined in [Kas].

In this paper, we focus on giving the statements and we do not give any proof. For more details, see [KT], on which most of this paper is based.

§2. The quantum Knizhnik–Zamolodchikov equation

Let $V = \oplus_{\epsilon=0}^{N-1} \mathbb{C}\, v_\epsilon$ be the N-dimensional vector space. We regard V as the vector representation of the quantum algebra $U_q(sl_N)$. Define the linear operator $\bar{R}(z)$ acting on $V^{\otimes 2}$ by

$$\bar{R}(z)\,(v_{\epsilon_1} \otimes v_{\epsilon_2}) = \sum_{\epsilon'_1, \epsilon'_2} \bar{R}(z)_{\epsilon'_1 \epsilon'_2}^{\epsilon_1 \epsilon_2}\, v_{\epsilon'_1} \otimes v_{\epsilon'_2},$$

where

$$\bar{R}(z)_{ii}^{ii} = 1,\ \bar{R}(z)_{ij}^{ij} = \frac{(1-z)q}{1-q^2z},\ \bar{R}(z)_{ij}^{ji} = \frac{1-q^2}{1-q^2z} z^{\theta(i>j)} \quad (i \neq j)$$

and $\bar{R}(z)_{i'j'}^{ij} = 0$ otherwise. Here

$$\theta(P) = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases} \tag{2.1}$$

Throughout this paper we assume that

$$0 < q < 1.$$

Then the R-matrix $R(z)$ is given as follows [DO]:

$$R(z) := r(z)\bar{R}(z). \tag{2.2}$$

Here $r(z)$ is the normalization factor

$$r(z) = q^{\frac{1}{N}-1} \frac{(q^2z;q^{2N})_\infty (q^{2N-2}z;q^{2N})_\infty}{(z;q^{2N})_\infty (q^{2N}z;q^{2N})_\infty},\ (z;x)_\infty := \prod_{j=0}^{\infty} (1-x^j z).$$

The matrix $R(z)$ is nothing but the image in $\mathrm{End}(V^{\otimes 2})$ of the universal R-matrix $\mathcal{R}'(z)$ of the quantum affine algebra $U_q(\widehat{sl}_N)$ in the sense of Appendix 1 in [IIJMNT].

The qKZ equation is the following system of difference equations for an unknown function $G(z_1, \dots, z_n)$ taking values in $V^{\otimes n}$:

$$\begin{aligned} (2.3)\quad & G(z_1, \dots, pz_m, \dots, z_n) \\ = \ & R_{m,m-1}(pz_m/z_{m-1}) \cdots R_{m,1}(pz_m/z_1) \times (\textstyle\prod_{j=1}^{N-1} \kappa_j^{h_j})_m \\ & \times R_{n,m}(z_n/z_m)^{-1} \cdots R_{m+1,m}(z_{m+1}/z_m)^{-1} G(z_1, \dots, z_m, \dots, z_n) \end{aligned}$$

for $m = 1, \dots, n$. Here $R_{m,l}(z)$ is the operator acting on the tensor product of the m-th and the l-th components in $V^{\otimes n}$ as the R-matrix $R(z)$ [1]. The operator $(\prod_j \kappa_j^{h_j})_m$ acts on the m-th component, where $h_j\ (j = 1, \dots, N-1)$ is the basis of the Cartan subalgebra of sl_N. The action of h_j on V is given by

$$h_j v_{j-1} = v_{j-1}, \quad h_j v_j = -v_j, \quad h_j v_i = 0\ (i \neq j-1, j).$$

[1] Note the order of indices: $R_{21}(z) = P\,R(z)\,P \neq R_{12}(z)$, where P is the transposition $P(u \otimes v) := v \otimes u$

The complex numbers $\kappa_1, \ldots, \kappa_{N-1}$ are parameters of the qKZ equation. For the sake of simplicity, hereafter we assume that the difference step p is a positive real number. When $p = q^{2(N+\ell)}$ the number ℓ is called a *level.*

§3. qKZ family

3.1. Affine Hecke algebra

Let us summarize the basic facts about the affine Hecke algebra. We use the notation in [MN].

Definition 3.1. *The affine Hecke algebra $\mathcal{H}_n^{\mathrm{aff}}$ of type GL_n is an associative $\mathbb{C}(t^{1/2})$-algebra generated by T_i $(i = 1, \ldots, n-1)$ and Y_j $(j = 1, \ldots, n)$ satisfying the following relations:*

$$\begin{aligned}
&(T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad (1 \le i \le n-1),\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \le i \le n-2),\\
&T_i T_j = T_j T_i \quad (|i-j| > 1), \qquad Y_i Y_j = Y_j Y_i \quad (1 \le i, j \le n),\\
&Y_i T_j = T_j Y_i \quad (j \ne i-1, i), \qquad T_i Y_{i+1} T_i = Y_i \quad (1 \le i \le n-1).
\end{aligned}$$

Define $\sigma \in \mathcal{H}_n^{\mathrm{aff}}$ by

$$\sigma := T_{n-1}^{-1} \cdots T_i^{-1} Y_i T_{i-1} \cdots T_1.$$

Note that the right hand side above does not depend on the value i. The algebra $\mathcal{H}_n^{\mathrm{aff}}$ is generated by T_i $(i = 1, \ldots, n-1)$ and σ.

Denote the Laurent polynomial ring with n variables by

$$P_n = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}].$$

Let $\widehat{T}_i$ $(i = 1, \ldots, n-1)$ and ω be the linear operators on P_n defined by

$$\widehat{T}_i := t^{1/2} \tau_i + \frac{t^{1/2} - t^{-1/2}}{z_i / z_{i+1} - 1} (\tau_i - 1), \tag{3.1}$$

$$(\omega f)(z_1, \ldots, z_n) := f(p z_n, z_1, \ldots, z_{n-1}). \tag{3.2}$$

Here τ_i is the permutation of the variables z_i and z_{i+1}, and p is a parameter. The operator $\widehat{T}_i$ is called the *Demazure–Lusztig operator.* We will identify the parameter p with the difference step p in the qKZ equation.

Proposition 3.2. *The linear map $\pi : \mathcal{H}_n^{\mathrm{aff}} \to \mathrm{End}(P_n)$ defined by $\pi(T_i) = \widehat{T}_i$ $(i = 1, \ldots, n-1)$ and $\pi(\sigma) = \omega$ gives a representation of $\mathcal{H}_n^{\mathrm{aff}}$.*

The operator

$$\widehat{Y}_j := \pi(Y_j) = \widehat{T}_j \cdots \widehat{T}_{n-1} \omega \widehat{T}_1^{-1} \cdots \widehat{T}_{j-1}^{-1}$$

is called *q-Dunkl operator.*

3.2. qKZ family

Hereafter we assume that

$$n \geq N \geq 2.$$

Let $d_0, \dots, d_{N-1}$ be positive integers satisfying $\sum_{j=0}^{N-1} d_j = n$. Denote by $I_{d_0,\dots,d_{N-1}}$ the set of n-tuples $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ satisfying

$$\#\{a \,|\, \epsilon_a = j\} = d_j \qquad (0 \leq j \leq N-1).$$

Now we give the definition of qKZ family:

Definition 3.3. *A set of Laurent polynomials*

$$\{f_{\epsilon_1,\dots,\epsilon_n} \in P_n \,|\, (\epsilon_1, \dots, \epsilon_n) \in I_{d_0,\dots,d_{N-1}}\}$$

is called a qKZ family of sign $(\pm)$ *with exponents* $(c_0, \dots, c_{N-1})$ *if it satisfies the following conditions:*

- *If* $\epsilon_i = \epsilon_{i+1}$*, then* $\widehat{T}_i f_{\dots,\epsilon_i,\epsilon_{i+1},\dots} = \pm t^{\pm 1/2} f_{\dots,\epsilon_i,\epsilon_{i+1},\dots}$.
- *If* $\epsilon_i > \epsilon_{i+1}$*, then* $\widehat{T}_i f_{\dots,\epsilon_i,\epsilon_{i+1},\dots} = f_{\dots,\epsilon_{i+1},\epsilon_i,\dots}$.
- $\omega f_{\epsilon_n,\epsilon_1,\dots,\epsilon_{n-1}} = c_{\epsilon_n} f_{\epsilon_1,\dots,\epsilon_n}$.

Here the operators $\widehat{T}_i\,(i = 1, \dots, n-1)$ *and* ω *are defined by* (3.1) *and* (3.2)*, respectively.*

In the rest of this subsection we show that a solution of the qKZ equation can be constructed from a qKZ family.

Let $\mathbf{f} = \{f_{\epsilon_1,\dots,\epsilon_n}\}$ be a qKZ family with exponents $(c_0, \dots, c_{N-1})$. Now we determine two parameters α and β, and a function $h(z)$ according to the sign of $\mathbf{f}$ as follows. If the sign of $\mathbf{f}$ is plus, we define α, β by

$$p^{\alpha} = (\prod_{j=0}^{N-1} c_j)^{-1/N} q^{-(n+1)(1/N-1)}, \qquad p^{\beta} = q^{2(1/N-1)},$$

and take a solution $h(z)$ of the difference equation

$$\frac{h(p^{-1}z)}{h(z)} = \frac{(z; q^{2N})_{\infty} (q^{2N}z; q^{2N})_{\infty}}{(q^2 z; q^{2N})_{\infty} (q^{2N-2}z; q^{2N})_{\infty}}.$$

Similarly, in the case where the sign of $\mathbf{f}$ is minus, we determine α, β and $h(z)$ by the following formulas:

$$(3.3)\quad p^{\alpha} = (-1)^{n-1}\Big(\prod_{j=0}^{N-1} c_j\Big)^{-1/N} q^{-(n+1)(1+1/N)}, \qquad p^{\beta} = q^{2(1+1/N)},$$

$$(3.4)\quad \frac{h(p^{-1}z)}{h(z)} = \frac{(z;q^{2N})_\infty (q^{2N}z;q^{2N})_\infty}{(q^{2(N+1)}z;q^{2N})_\infty (q^{-2}z;q^{2N})_\infty}.$$

Now let us construct a solution of the qKZ equation. Define the function $K(z_1,\ldots,z_n)$ by

$$(3.5)\qquad K(z_1,\ldots,z_n) := \prod_{a=1}^{n} z_a^{\alpha+\beta a} \prod_{1\le a<b\le n} h(z_b/z_a).$$

and the $V^{\otimes n}$-valued function $F(z_1,\ldots,z_n)$ by

$$F(z_1,\ldots,z_n) := \sum_{(\epsilon_1,\ldots,\epsilon_n)\in I_{d_0,\ldots,d_{N-1}}} f_{\epsilon_1,\ldots,\epsilon_n}(z_1,\ldots,z_n)\, v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_n}.$$

Set

$$(3.6)\qquad G(z_1,\ldots,z_n) := K(z_1,\ldots,z_n)F(z_1,\ldots,z_n).$$

Proposition 3.4. *Let $\mathbf{f} = \{f_{\epsilon_1,\ldots,\epsilon_n}\}$ be a qKZ family of sign $(\pm)$ with exponents $(c_0,\ldots,c_{N-1})$. Then $G(z_1,\ldots,z_n)$ is a solution of the qKZ equation whose parameters q and κ_j $(j=1,\ldots,N-1)$ are determined by $q = \pm t^{\pm 1/2}$ and*

$$(3.7)\qquad \kappa_j = \prod_{l=0}^{j-1} c_l \cdot \Big(\prod_{l=0}^{N-1} c_l\Big)^{-j/N}.$$

3.3. Equivalence to the eigenvalue problem

We can construct a qKZ family from a joint eigenfunction of q-Dunkl operators $\widehat{Y}_i$ and some Demazure–Lusztig operators $\widehat{T}_i$.

Let us introduce some notation. We often use the short notation $\epsilon = (\epsilon_1,\ldots,\epsilon_n)$ to specify an element of $I_{d_0,\ldots,d_{N-1}}$. An element $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$ is called *dominant* (or *anti-dominant*) if $\lambda_1 \ge \cdots \ge \lambda_n$ (or $\lambda_1 \le \cdots \le \lambda_n$, resp.). The symmetric group S_n acts on $\mathbb{Z}^n$ by $\sigma\lambda := (\lambda_{\sigma^{-1}(1)},\ldots,\lambda_{\sigma^{-1}(n)})$. We denote the orbit of $\lambda\in\mathbb{Z}^n$ by $S_n\lambda$.

Definition 3.5. *For $\lambda \in \mathbb{Z}^n$, we denote by λ^+ (λ^-) the unique dominant (anti-dominant) element in $S_n\lambda$, respectively. We denote by w_λ^+ (w_λ^-) the shortest element in S_n such that $w_\lambda^+\lambda^+ = \lambda$ $(w_\lambda^-\lambda = \lambda^-)$, respectively.*

For $w \in S_n$ we denote its length by $\ell(w)$. Let $w = s_{i_1} \cdots s_{i_m}$ be a reduced expression, where s_i is the transposition $s_i = (i, i+1)$. Then we set $\widehat{T}_w := \widehat{T}_{i_1} \cdots \widehat{T}_{i_m}$. This does not depend on the choice of reduced expression of w.

Now we are in position to give the main theorem which plays a key role in the next section:

Theorem 3.6. *Fix positive integers $d_0, \ldots, d_{N-1}$ satisfying $d_0 + \ldots + d_{N-1} = n$ and set $\delta = (0^{d_0}, 1^{d_1}, \cdots, (N-1)^{d_{N-1}})$. Suppose that $E = E(z_1, \ldots, z_n)$ is a solution to the following eigenvalue problem:*

$$\widehat{Y}_j E = \chi_j E \quad (1 \le \forall j \le n) \tag{3.8}$$

$$\widehat{T}_i E = \pm t^{\pm 1/2} E \quad \text{if} \quad \delta_i = \delta_{i+1}. \tag{3.9}$$

Here the sign in the right hand side (3.9) *should be independent on i. Set $f_\epsilon := (\widehat{T}_{w_\epsilon^-})^{-1} E$ for $\epsilon \in I_{d_0, \ldots, d_{N-1}}$. Then $\{f_\epsilon\}$ is a qKZ family of sign $(\pm)$ with exponents $c_i = \chi_{d_0 + \cdots + d_i} (\pm t^{\pm 1/2})^{d_i - 1}$ $(0 \le i \le N-1)$.*

Remark 3.7. *The consistency of the eigenvalue problem* (3.8) *and* (3.9) *implies that the eigenvalues χ_i should satisfy $\chi_i = t^{\pm 1} \chi_{i+1}$ if $\delta_i = \delta_{i+1}$. Hence all the eigenvalues χ_i can be recovered from the exponents c_i.*

Remark 3.8. *For any qKZ family $\{f_\epsilon\}$ of sign $(\pm)$, f_δ is a solution to* (3.8) *and* (3.9). *Namely, the problem of finding a qKZ family is equivalent to the eigenvalue problem.*

§4. Construction of special solutions

From the result in the foregoing sections we can construct special solutions of the qKZ equation as follows. Find a solution E to the eigenvalue problem (3.8) and (3.9). Setting $f_\epsilon = (\widehat{T}_{w_\epsilon^-})^{-1} E$, we obtain a qKZ family $\mathbf{f} = \{f_\epsilon\}$ of sign $(\pm)$ according to the sign $\pm$ in the right hand side of (3.9). Define the parameters α, β and take a function $h(z)$ as explained in Section 3.2. Using these ingredients above we define $G(z_1, \ldots, z_n)$ by the formula (3.6). Then from Proposition 3.4 G is a solution of the qKZ equation with parameter $q = \pm t^{\pm 1/2}$. Thus the first step of our construction is to solve the eigenvalue problem (3.8) and (3.9), and we can find a solution in terms of non-symmetric Macdonald polynomials.

In the following we use the wording "the eigenvalue problem of sign $(\pm)$" to refer the eigenvalue problem (3.8) and (3.9) where the sign in the right hand side is $\pm$, respectively.

4.1. Non-symmetric Macdonald polynomials

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, we set $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$. We introduce the dominance order $\geq$ on the set $\mathbb{Z}^n$:

$$\lambda \geq \mu \quad \underset{\text{def}}{\Leftrightarrow} \quad \textstyle\sum_{i=1}^j \lambda_i \geq \sum_{i=1}^j \mu_i \text{ for any } 1 \leq j \leq n,$$

and a partial order $\succ$:

$$\lambda \succ \mu \quad \underset{\text{def}}{\Leftrightarrow} \quad \lambda^+ > \mu^+ \quad \text{or} \quad \text{“}\lambda^+ = \mu^+ \text{ and } \lambda > \mu\text{”}.$$

Definition 4.1. *For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, the non-symmetric Macdonald polynomial $E_\lambda = E_\lambda(z_1, \dots, z_n; t, p)$ with two parameters t and p is a Laurent polynomial satisfying*

$$\begin{aligned} \widehat{Y}_i E_\lambda &= t^{\rho(\lambda)_i} p^{\lambda_i} E_\lambda \\ E_\lambda &= z^\lambda + \sum_{\mu \prec \lambda} c_\mu z^\mu \end{aligned} \tag{4.1}$$

where $\rho(\lambda) := w_\lambda^+ \rho$, $\rho := (\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2})$.

Let us recall the action of $\widehat{T}_i$ on E_λ following [Kas]. Put

$$f_i(\lambda) := t^{\rho(\lambda)_{i+1} - \rho(\lambda)_i} p^{\lambda_{i+1} - \lambda_i}.$$

If $\lambda_i < \lambda_{i+1}$, then

$$\widehat{T}_i E_\lambda = t^{1/2} E_{s_i \lambda} - \frac{t^{1/2} - t^{-1/2}}{f_i(\lambda) - 1} E_\lambda.$$

If $\lambda_i = \lambda_{i+1}$, then

$$\widehat{T}_i E_\lambda = t^{1/2} E_\lambda. \tag{4.2}$$

If $\lambda_i > \lambda_{i+1}$, then

(4.3)
$$\widehat{T}_i E_\lambda = t^{-1/2} \frac{(t f_i(\lambda) - 1)(t^{-1} f_i(\lambda) - 1)}{(f_i(\lambda) - 1)^2} E_{s_i \lambda} - \frac{t^{1/2} - t^{-1/2}}{f_i(\lambda) - 1} E_\lambda.$$

The parameters t and p are called *generic* if

$$t^l p^m \neq 1 \quad \text{for any } 0 \leq l \leq n \text{ and } 0 \leq m.$$

For generic parameters, E_λ is well-defined for any $\lambda \in \mathbb{Z}^n$.

4.2. Generic case

First we consider the case where the parameters t and p are generic. From the properties (4.1) and (4.2), the non-symmetric polynomials give solutions of the eigenvalue problem of sign $(+)$. Hence we can get solutions of the qKZ equation:

Proposition 4.2. *Suppose that the parameters t and p are generic. Let $d_0, \ldots, d_{N-1}$ be positive integers satisfying $\sum_{j=0}^{N-1} d_j = n$ and set $\delta = (0^{d_0}, 1^{d_1}, \cdots, (N-1)^{d_{N-1}})$. Take $\lambda \in \mathbb{Z}^n$ such that $\lambda_i = \lambda_{i+1}$ if $\delta_i = \delta_{i+1}$. Then the non-symmetric Macdonald polynomial E_λ is a solution of the eigenvalue problem of sign $(+)$, and we obtain a solution of the qKZ equation from it by setting $t^{1/2} = q$. The parameters $\kappa_1, \ldots, \kappa_{N-1}$ in the qKZ equation are determined by (3.7) from the exponents*

$$c_i = q^{d_i - 1 + 2\rho(\lambda)_{d_0 + \cdots + d_i}} p^{\lambda_{d_0 + \cdots + d_i}}.$$

We note that in the case where $d_0 = \cdots = d_{N-1} = 1$ the requirement (3.9) becomes empty. Hence any non-symmetric Macdonald polynomial is also a solution to the eigenvalue problem of sign $(-)$ in this special case, and we obtain the following proposition:

Proposition 4.3. *Suppose that the parameters t and p are generic. In the case where $d_0 = \cdots = d_{N-1} = 1$, and hence $n = N$, any non-symmetric Macdonald polynomial E_λ creates a solution of the qKZ equation. The parameters $\kappa_1, \ldots, \kappa_{N-1}$ are determined by (3.7), where $c_i = (-1)^{n-1} q^{-2\rho(\lambda)_{i+1}} p^{\lambda_{i+1}}$.*

Remark 4.4. *When we determine κ_j's and α by (3.7) and (3.3) in practice, the branch of $(\prod_i c_i)^{1/N}$ should be chosen suitably. In the situation described in Proposition 4.3 the exponent c_i is a value of the form $(-1)^{n-1}\tilde{c}_i$, where $\tilde{c}_i$ is a positive real number. Then we set $(\prod_i c_i)^{1/N} = (-1)^{n-1}(\prod_i \tilde{c}_i)^{1/N}$ and determine κ_j's and α. In Theorem 4.6 below the situation is the same, and we take the same branch.*

4.3. Specialized case

In Proposition 4.3 we saw that any non-symmetric Macdonald polynomial gives a solution to the eigenvalue problem of sign $(-)$, but this is a very special case. In order to solve this problem in general, we need to find an eigenfunction of the Demazure–Lusztig operator with the eigenvalue $-t^{-1/2}$, and this is not the situation in (4.2). However, if $f_i(\lambda) = t$ in (4.3), then E_λ becomes such an eigenfunction. It should be noted that the relation $f_i(\lambda) = t$ implies that the parameters t and p are not generic. In the rest of this paper we consider this kind of case.

Let k and r be integers such that $1 \leq k \leq \min\{n-1, N\}$, $r \geq 2$, and $k+1$ and $r-1$ are coprime. We assume that t, p are not roots of unity and take a specialization $t^{k+1}p^{r-1} = 1$. To be more precise we specialize t and p as follows:

$$t = u^{r-1}, \qquad p = u^{-(k+1)} \tag{4.4}$$

where u is not a root of unity. We will set $q = -t^{-1/2}$ and take $u = q^{-\frac{2}{r-1}}$. Then we have $p = q^{\frac{2(k+1)}{r-1}}$ and the level of the qKZ equation is equal to $\frac{k+1}{r-1} - N$.

We call $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ *admissible* if

$$\begin{aligned} \lambda^+_i - \lambda^+_{i+k} &\geq r-1 \quad \text{for any } 1 \leq i \leq n-k, \text{ and} \\ \lambda^+_i - \lambda^+_{i+k} &= r-1 \quad \text{only if } w^+_\lambda(i) < w^+_\lambda(i+k). \end{aligned}$$

The following statement is a corollary of Theorem 3.11 in [Kas]:

Lemma 4.5. *For any admissible $\lambda \in \mathbb{Z}^n$, the non-symmetric Macdonald polynomial E_λ is well-defined under the specialization* (4.4). *If $\lambda \in \mathbb{Z}^n$ is admissible and $s_i\lambda$ is not admissible, then $\widehat{T}_i E_\lambda = -t^{-1/2} E_\lambda$.*

Let m and l be integers satisfying $n = km + l$ and $0 \leq l \leq k-1$. Let $(d^{(0)}, \dots, d^{(k-1)})$ be a permutation of $((m+1)^l, m^{k-l})$. Note that $\sum_{j=0}^{k-1} d^{(j)} = n$. Take a dominant element $a = (a_1, \dots, a_k) \in \mathbb{Z}^k$ satisfying

$$a_1 - a_k \leq r-1 \quad \text{and} \quad w^-_a((m+1)^l, m^{k-l}) = (d^{(0)}, \dots, d^{(k-1)}), \tag{4.5}$$

where $w^-_a \in S_k$ (see Definition 3.5). Now define $\lambda \in \mathbb{Z}^n$ by

$$\begin{aligned} \lambda_i &= a_i \quad \text{for} \quad 1 \leq i \leq k, \quad \text{and} \\ \lambda_i - \lambda_{i+k} &= r-1 \quad \text{for} \quad 1 \leq i \leq n-k. \end{aligned}$$

Then λ is admissible. For simplicity, we write $w = w^-_a$ and define $\mu \in S_n\lambda$ by

$$\begin{aligned} \mu \;=\; & (\lambda_{w^{-1}(1)}, \lambda_{w^{-1}(1)+k}, \lambda_{w^{-1}(1)+2k}, \dots, \lambda_{w^{-1}(1)+b_1 k}, \\ & \lambda_{w^{-1}(2)}, \lambda_{w^{-1}(2)+k}, \lambda_{w^{-1}(2)+2k}, \dots, \lambda_{w^{-1}(2)+b_2 k}, \\ & \dots, \\ & \lambda_{w^{-1}(k)}, \lambda_{w^{-1}(k)+k}, \lambda_{w^{-1}(k)+2k}, \dots, \lambda_{w^{-1}(k)+b_k k}), \end{aligned}$$

where $b_j := m - \theta(w^{-1}(j) > l)$ (see (2.1) for the definition of $\theta(P)$).

Example. Set $n = 13$ and $k = 5$, and consider the case of

$$(d^{(0)}, d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}) = (3, 2, 2, 3, 3).$$

Then the condition (4.5) for a dominant $a = (a_1, \ldots, a_5) \in \mathbb{Z}^5$ implies that $a_1 \geq a_2 > a_3 = a_4 = a_5$. Now suppose that $r = 6$ and take $a = (13, 10, 9, 9, 9)$. Then λ and μ are given by

$$\lambda = (13, 10, 9, 9, 9, 8, 5, 4, 4, 4, 3, 0, -1),$$
$$\mu = (9, 4, -1, 9, 4, 9, 4, 10, 5, 0, 13, 8, 3).$$

Now let $(d_0, \ldots, d_{N-1})$ be a subdivision of $(d^{(0)}, \ldots, d^{(k-1)})$, that is, $d_i > 0$ and $d_{i_j} + \cdots + d_{i_{j+1}-1} = d^{(j)}$ for some $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = N$. It is easy to see that μ is also admissible and $s_i\mu$ is not admissible if $\delta_i = \delta_{i+1}$, where $\delta = (0^{d_0}, 1^{d_1}, \cdots, (N-1)^{d_{N-1}})$. From Lemma 4.5, E_μ is a solution of the eigenvalue problem of sign $(-)$. Therefore we get the following theorem (see Remark 4.4).

Theorem 4.6. *The non-symmetric Macdonald polynomial E_μ with the specialization* (4.4) *and $t^{1/2} = -q^{-1}$ creates a solution of the qKZ equation of level $\frac{k+1}{r-1} - N$. The parameters $\kappa_1, \ldots, \kappa_{N-1}$ are determined by* (3.7) *from the exponents $c_i = (-1)^{n-1}q^{A_i}$, where*

$$A_i := d_i - 1 - 2\rho(\mu)_{d_0+\cdots+d_i} + \frac{2(k+1)}{r-1}\mu_{d_0+\cdots+d_i}.$$

Acknowledgements. Research of YT is supported by Grant-in-Aid for Young Scientists (B) No. 17740089. Research of MK is supported by Grant-in-Aid for JSPS Fellows No. 17-2106. We are deeply grateful to Saburo Kakei and Yoshihisa Saito for valuable discussions. MK also thanks Vincent Pasquier for his interest. YT thanks Takeshi Ikeda and Hiroshi Naruse for discussions.

References

[C] I. Cherednik, Double affine Hecke algebras, Knizhnik–Zamolodchikov equations, and Macdonald's operators, Internat. Math. Res. Notices, **1992** (1992), 171–180.

[DO] E. Date and M. Okado, Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type $A_n^{(1)}$, Internat. J. Modern Phys. A, **9** (1994), 399–417.

[DZ] P. Di Francesco and P. Zinn-Justin, Quantum Knizhnik–Zamolodchikov equation, generalized Razumov–Stroganov sum rules and extended Joseph polynomials, J. Phys. A, **38** (2005), L815–L822.

[FR] I. B. Frenkel and N. Yu. Reshetikhin, Quantum affine algebras and holonomic difference equations, Comm. Math. Phys., **146** (1992), 1–60.

[IIJMNT] M. Idzumi, T. Tokihiro, K. Iohara, M. Jimbo, T. Miwa and T. Nakashima, Quantum affine symmetry in vertex models, Internat. J. Modern Phys. A, **8** (1993), 1479–1511.

[Kas] M. Kasatani, Subrepresentations in the polynomial representation of the double affine Hecke algebra of type GL_n at $t^{k+1}q^{r-1} = 1$, Int. Math. Res. Not., **2005** (2005), 1717–1742.

[KP] M. Kasatani and V. Pasquier, On polynomials interpolating between the stationary state of a $O(n)$ model and a Q.H.E. ground state, Comm. Math. Phys., **276** (2007), 397–435.

[KT] M. Kasatani and Y. Takeyama, The quantum Knizhnik–Zamolodchikov equation and non-symmetric Macdonald polynomials, Funkcial. Ekvac., **50** (2007), 491–509.

[Kat] S. Kato, R-matrix arising from affine Hecke algebras and its application to Macdonald's difference operators, Comm. Math. Phys., **165** (1994), 533–553.

[Ko] Y. Koyama, Staggered polarization of vertex models with $U_q(\widehat{sl(n)})$-symmetry, Comm. Math. Phys., **164** (1994), 277–291.

[Ma] I. G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Astérisque, **237** (1996), 189–207.

[Mi] K. Mimachi, A solution to quantum Knizhnik–Zamolodchikov equations and its application to eigenvalue problems of the Macdonald type, Duke Math. J., **85** (1996), 635–658.

[MN] K. Mimachi and M. Noumi, A reproducing kernel for nonsymmetric Macdonald polynomials, Duke Math. J., **91** (1998), 621–634.

[MTT] T. Miwa, Y. Takeyama and V. Tarasov, Determinant formula for solutions of the quantum Knizhnik–Zamolodchikov equation associated with $U_q(sl_n)$ at $|q| = 1$, Publ. Res. Inst. Math. Sci., **35** (1999), 871–892.

[N] A. Nakayashiki, Trace construction of a basis for the solution space of sl_N qKZ equation, Comm. Math. Phys., **212** (2000), 29–61.

[VT] A. N. Varchenko and V. O.Tarasov, Jackson integral representations for solutions of the quantized Knizhnik–Zamolodchikov equation, Algebra i Analiz, **6** (1994), 90–137; translation in St. Petersburg Math. J., **6** (1995), 275–313.

Masahiro Kasatani
Graduate School of Mathematical Sciences
The University of Tokyo
Tokyo 153-8914
Japan

Yoshihiro Takeyama
Department of Mathematics
Graduate School of Pure and Applied Sciences
Tsukuba University
Tsukuba, Ibaraki 305-8571
Japan

E-mail address: `kasatani@ms.u-tokyo.ac.jp`
`takeyama@math.tsukuba.ac.jp`