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A conjectural presentation of fusion algebras

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Dedicated to Prof. Masaki Kashiwara on his sixtieth birthday

§1. Introduction

Let G be a connected, simply-connected, simple algebraic group over $\mathbb{C}$. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. We denote their Lie algebras by $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ respectively. Let $P_+ \subset \mathfrak{h}^*$ be the set of dominant integral weights. For any $\lambda \in P_+$, let $V(\lambda)$ be the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ. We fix a positive integer ℓ and let $\mathcal{R}_\ell(\mathfrak{g})$ be the free $\mathbb{Z}$-module with basis $\{V(\lambda) : \lambda \in P_\ell\}$, where

$$P_\ell := \{\lambda \in P_+ : \lambda(\theta^\vee) \leq \ell\},$$

θ is the highest root of $\mathfrak{g}$ and $\theta^\vee$ is the associated coroot. There is a product structure on $\mathcal{R}_\ell(\mathfrak{g})$, called the *fusion product* (cf. Section 3), making it a commutative associative (unital) ring. In this paper, we consider its complexification $\mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g})$, called the *fusion algebra*, which is a finite dimensional (commutative and associative) algebra without nilpotents.

Let $\mathcal{R}(\mathfrak{g})$ be the Grothendieck ring of finite dimensional representations of $\mathfrak{g}$ and let $\mathcal{R}^{\mathbb{C}}(\mathfrak{g})$ be its complexification. As given in 3.6, there is a surjective ring homomorphism $\beta : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$. Let $\beta^{\mathbb{C}} : \mathcal{R}^{\mathbb{C}}(\mathfrak{g}) \to \mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g})$ be its complexification and let $I_\ell(\mathfrak{g})$ denote the kernel of $\beta^{\mathbb{C}}$. Since $\mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g})$ is an algebra without nilpotents, $I_\ell(\mathfrak{g})$ is a radical ideal.

The main aim of this note is to conjecturally describe this ideal $I_\ell(\mathfrak{g})$. Before we describe our result and conjecture, we briefly describe the known results in this direction. Identify the complexified representation ring $\mathcal{R}^{\mathbb{C}}(\mathfrak{g})$ with the polynomial ring $\mathbb{C}[\chi_1, \dots, \chi_r]$, where r is the rank of $\mathfrak{g}$ and χ_i denotes the character of $V(\omega_i)$, the i^{th} fundamental representation of $\mathfrak{g}$. It is generally believed (initiated by the physicists) that

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there exists an explicit potential function $F = F_\ell(\mathfrak{g})$ (depending upon $\mathfrak{g}$ and ℓ) in $\mathbb{Z}[\chi_1, \dots, \chi_r]$, coming from representation theory of $\mathfrak{g}$, with the property that the ideal generated over the integers by the gradient of F, i.e., $\langle \partial F/\partial\chi_1, \dots, \partial F/\partial\chi_r\rangle$, is precisely the kernel of β. This, in particular, would imply that the spec of $\mathcal{R}_\ell(\mathfrak{g})$ is a complete intersection over $\mathbb{Z}$ inside the spec of the representation ring $\mathbb{Z}[\chi_1, \dots, \chi_r]$. It may be remarked that, since $I_\ell(\mathfrak{g})$ is a radical ideal of $\mathbb{C}[\chi_1, \dots, \chi_r]$ with finite dimensional quotient, there exists an abstract potential function F such that the ideal generated by the gradient of F over the complex numbers coincides with $I_\ell(\mathfrak{g})$. However, an explicit construction of such an F attuned to representation theory is only known in the case of $\mathfrak{g} = sl_{r+1}$ and sp_{2r}.

We recall the following result (cf. [9], [10], [3] and [4]) obtained by first constructing explicitly a potential function. Let χ_λ denote the character of the irreducible representation $V(\lambda)$ of $\mathfrak{g}$.

1.1. Theorem. *(a) For $\mathfrak{g} = sl_{r+1}$,*

$$I_\ell(sl_{r+1}) = \langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \dots, \chi_{(\ell+r)\omega_1}\rangle.$$

(b) For $\mathfrak{g} = sp_{2r}$,

$$I_\ell(sp_{2r}) = \langle \chi_{\ell\omega_1+\omega_i};\ 1 \le i \le r\rangle.$$

In particular, the ideals on the right side of both of (a) and (b) are radical.

For the remaining simple Lie algebras, we have the following theorem and the conjecture.

For any ideal I in a ring R, we denote its radical ideal by $\sqrt{I}$. In the following theorem, for any $\mathfrak{g}$, we take any fundamental weight ω_d with minimum Dynkin index. (Recall from Section 2.2 that ω_d is unique up to a diagram automorphism except for $\mathfrak{g}$ of type B_3.) The list of such ω_d's as well as the dual Coxeter number $\check{h}(\mathfrak{g})$ of any $\mathfrak{g}$ is given in Table 1 (Section 2).

1.2. Theorem. *(a) For $\mathfrak{g}$ of type $B_r (r \ge 3)$, $D_r (r \ge 4)$, E_6 or E_7,*

$$I_\ell(\mathfrak{g}) \supseteq \sqrt{\langle \chi_{(\ell+1)\omega_d}, \chi_{(\ell+2)\omega_d}, \dots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_d}\rangle}.$$

(For $\mathfrak{g}$ of type B_3, we must take $\omega_d = \omega_1$.)

(b) For $\mathfrak{g}$ of type G_2,

$$I_\ell(G_2) \supseteq \begin{cases} \sqrt{\langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \chi_{(\frac{\ell+1}{2})\omega_2}\rangle}, & \text{if } \ell \text{ is odd,} \\ \sqrt{\langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \chi_{\omega_1+\frac{\ell}{2}\omega_2}\rangle}, & \text{if } \ell \text{ is even.} \end{cases}$$

(c) For $\mathfrak{g}$ of type F_4,

$$I_\ell(F_4) \supseteq \sqrt{\langle \chi_{(\ell+1)\omega_4}, \chi_{(\ell+2)\omega_4}, \cdots, \chi_{(\ell+6)\omega_4} \rangle}.$$

(d) For $\mathfrak{g}$ of type E_8,

$$I_\ell(E_8) \supseteq \begin{cases} \sqrt{\langle \chi_{(\ell+2)\omega_8}, \chi_{(\ell+3)\omega_8}, \cdots, \chi_{(\ell+29)\omega_8} \rangle}, & \text{if } \ell \text{ is even,} \\ \sqrt{\langle \chi_{(\ell+2)\omega_8}, \chi_{(\ell+3)\omega_8}, \cdots, \chi_{(\ell+29)\omega_8}, \chi_{(\frac{\ell+1}{2})\omega_8} \rangle}, & \text{if } \ell \text{ is odd.} \end{cases}$$

1.3. Conjecture. All the inclusions in (a)-(b) of the above theorem are, in fact, equalities for $\mathfrak{g}$ of type $B_r (r \geq 3), D_r (r \geq 4)$ and G_2.

In addition, we conjecture the following.

(a) For $\mathfrak{g}$ of type $B_r (r \geq 3)$,

$$\sqrt{\langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \cdots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_1} \rangle} = \langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \cdots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_1}, \chi_{\ell\omega_1+\omega_r} \rangle.$$

(b) For $\mathfrak{g}$ of type $D_r (r \geq 4)$,

$$\sqrt{\langle \chi_{(\ell+1)\omega_d}, \chi_{(\ell+2)\omega_d}, \cdots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_d} \rangle} = \langle \chi_{(\ell+1)\omega_d}, \chi_{(\ell+2)\omega_d}, \cdots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_d}, \chi_{\ell\omega_1+\omega_{r-1}}, \chi_{\ell\omega_1+\omega_r} \rangle.$$

It is not clear to us (even conjecturally) how to explicitly construct potential functions in these cases.

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§2. Preliminaries and Notation

This section is devoted to setting up the notation and recalling some basic facts about affine Kac–Moody Lie algebras.

Let G be a connected, simply-connected, simple algebraic group over $\mathbb{C}$. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. Let $\mathfrak{h}$,$\mathfrak{b}$ and $\mathfrak{g}$ denote the Lie algebras of T, B and G respectively.

Let $R = R(\mathfrak{h}, \mathfrak{g}) \subset \mathfrak{h}^*$ be the root system; there is the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha)$, $\mathfrak{g}_\alpha$ being the root space corresponding to the root α. The choice of $\mathfrak{b}$ determines a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of R, where r is the rank of G. For each root α, denote by $\alpha^\vee$ the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(\alpha^\vee) = 2$; it is called the *coroot* associated to the root α. Let $\mathfrak{h}_\mathbb{R}$ denote the real span of the elements $\{\alpha_i^\vee, \alpha_i \in \Delta\}$.

Let $\{\omega_i\}_{1\leq i\leq r}$ be the set of fundamental weights, defined as the basis of $\mathfrak{h}^*$ dual to $\{\alpha_i^\vee\}_{1\leq i\leq r}$. We define the weight lattice $P = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}\ \forall \alpha_i \in \Delta\}$, and denote the set of dominant integral weights by P_+, i.e., $P_+ := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+\ \forall \alpha_i \in \Delta\}$, where $\mathbb{Z}_+$ is the set of nonnegative integers. The latter set parametrizes the set of isomorphism classes of all the finite dimensional irreducible representations of $\mathfrak{g}$. For $\lambda \in P_+$, let $V(\lambda)$ be the associated finite dimensional irreducible representations of $\mathfrak{g}$ with highest weight λ. Let ρ denote the sum of fundamental weights, and $\check{h} = \check{h}(\mathfrak{g}) := 1 + \rho(\theta^\vee)$ the *dual Coxeter number*, where θ is the highest root of $\mathfrak{g}$.

Let $(\ |\ )$ denote the Killing form on $\mathfrak{g}$ normalized such that $(\theta^\vee|\theta^\vee) = 2$. We will use the same notation for the restricted form on $\mathfrak{h}$, and the induced form on $\mathfrak{h}^*$. Let $W := N_G(T)/T$ be the Weyl group of G, where $N_G(T)$ is the normalizer of T in G. Let w_o denote the longest element in the Weyl group. For any $\lambda \in P$, the dual of λ, denoted by λ^*, is defined to be $-w_o\lambda$. For $\lambda \in P_+, \lambda^*$ is again in P_+ and, moreover, $V(\lambda)^* \simeq V(\lambda^*)$.

2.1. Dynkin index. We recall the following definition from [6], §2.

2.2. Definition. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two (finite dimensional) simple Lie algebras and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. There exists a unique number $m_\varphi \in \mathbb{C}$, called the *Dynkin index* of the homomorphism φ, satisfying

$$(\varphi(x)|\varphi(y)) = m_\varphi(x|y), \text{ for all } x, y \in \mathfrak{g}_1,$$

where $(\ |\ )$ is the normalized Killing form on $\mathfrak{g}_1$ (and $\mathfrak{g}_2$) as defined above. Then, as proved by Dynkin [6], Theorem 2.2, the Dynkin index is always a nonnegative integer.

For a Lie algebra $\mathfrak{g}_1$ as above and a finite dimensional representation V of $\mathfrak{g}_1$, by the *Dynkin index* m_V of V, we mean the Dynkin index of the Lie algebra homomorphism $\rho_V : \mathfrak{g}_1 \to sl(V)$, where $sl(V)$ is the Lie algebra of traceless endomorphisms of V.

For any simple Lie algebra $\mathfrak{g}$, there is a fundamental weight ω_d such that $m_{V(\omega_d)}$ divides m_V for every finite dimensional representation V. Moreover, such a ω_d is unique up to diagram automorphisms (except for $\mathfrak{g}$ of type B_3). The following table gives the list of all such ω_d's together with their indices $m_{V(\omega_d)}$ and also the dual Coxeter number $\check{h}(\mathfrak{g})$. The list of ω_d and $m_{V(\omega_d)}$ can be obtained from [6], Table 5 (see also [5], Proposition 2.3). We follow the indexing convention as in [1].

Type of G	ω_d	$m_{V(\omega_d)}$	$\check{h}(\mathfrak{g})$
A_r $(r \geq 1)$	ω_1, ω_r	1	$r+1$
C_r $(r \geq 2)$	ω_1	1	$r+1$
B_r $(r \geq 3)$	ω_1	2	$2r-1$
D_r $(r \geq 4)$	ω_1	2	$2r-2$
G_2	ω_1	2	4
F_4	ω_4	6	9
E_6	ω_1, ω_6	6	12
E_7	ω_7	12	18
E_8	ω_8	60	30.

Table 1.

We remark that for B_3, ω_3 also satisfies $m_{V(\omega_3)} = 2$; for D_4, ω_3 and ω_4 both have $m_{V(\omega_3)} = m_{V(\omega_4)} = 2$.

2.3. Affine Kac–Moody Lie algebras. Let $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}K$ denote the (untwisted) affine Lie algebra associated to $\mathfrak{g}$ (where $\mathbb{C}((z))$ denotes the field of Laurent series in one variable z), with the Lie bracket

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x|y) Res_{z=0}(gdf) \cdot K \text{ and } [\tilde{\mathfrak{g}}, K] = 0,$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}((z))$.

The Lie algebra $\mathfrak{g}$ sits as a Lie subalgebra of $\tilde{\mathfrak{g}}$ as $\mathfrak{g} \otimes z^0$. The Lie algebra $\tilde{\mathfrak{g}}$ admits a distinguished 'parabolic' subalgebra

$$\tilde{\mathfrak{p}} := \mathfrak{g} \otimes \mathbb{C}[[z]] \oplus \mathbb{C}K.$$

We also define its 'nil-radical' $\tilde{\mathfrak{u}}$ (which is an ideal of $\tilde{\mathfrak{p}}$) by

$$\tilde{\mathfrak{u}} := \mathfrak{g} \otimes z\mathbb{C}[[z]],$$

and its 'Levi component' (which is a Lie subalgebra of $\tilde{\mathfrak{p}}$)

$$\tilde{\mathfrak{p}}^o := \mathfrak{g} \otimes z^0 \oplus \mathbb{C}K.$$

Clearly (as a vector space)

$$\tilde{\mathfrak{p}} = \tilde{\mathfrak{u}} \oplus \tilde{\mathfrak{p}}^o.$$

2.4. Irreducible representations of $\tilde{\mathfrak{g}}$. Fix an irreducible (finite dimensional) representation $V = V(\lambda)$ of $\mathfrak{g}$ and a positive integer ℓ. We define the associated *generalized Verma module* for $\tilde{\mathfrak{g}}$ as

$$M(V, \ell) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{p}})} Y_\ell(V),$$

where the $\tilde{\mathfrak{p}}$-module $Y_\ell(V)$ has the underlying vector space same as V on which $\tilde{\mathfrak{u}}$ acts trivially, the central element K acts via the scalar ℓ and the action of $\mathfrak{g} = \mathfrak{g} \otimes z^0$ is via the $\mathfrak{g}$-module structure on V. Then, $M(V, \ell)$ has a unique irreducible quotient denoted $L(V, \ell)$. Observe that K acts by the constant ℓ on $M(V, \ell)$ and hence also on $L(V, \ell)$. The constant by which K acts on the representation is called the *central charge* of the representation. Thus, $L(V, \ell)$ has central charge ℓ.

2.5. Definition. Consider the Lie subalgebra $\mathfrak{r}^o$ of $\tilde{\mathfrak{g}}$ spanned by $\{X_{-\theta} \otimes z, K - \theta^\vee \otimes 1, X_\theta \otimes z^{-1}\}$, where $X_{-\theta}$ (resp. X_θ) is a non-zero root vector of $\mathfrak{g}$ corresponding to the root $-\theta$ (resp. θ). Then, the Lie algebra $\mathfrak{r}^o$ is isomorphic with $sl(2)$.

A $\tilde{\mathfrak{g}}$-module L is said to be *integrable* if every vector $v \in L$ is contained in a finite dimensional $\mathfrak{g}$-submodule of L and also v is contained in a finite dimensional $\mathfrak{r}^o$-submodule of L.

Then, it follows easily from the $sl(2)$-theory that the irreducible module $L(V(\lambda), \ell)$ is integrable if and only if $\lambda \in P_\ell := \{\lambda \in P_+ : \lambda(\theta^\vee) \leq \ell\}$.

§3. An Introduction to the Fusion Ring

Let $\mathcal{R}(\mathfrak{g})$ denote the Grothendieck ring of finite dimensional representations of $\mathfrak{g}$. It is a free $\mathbb{Z}$-module with basis $\{V(\lambda) : \lambda \in P_+\}$, with the product structure induced from the tensor product of representations as follows:

$$V(\lambda) \otimes V(\mu) = \sum_{\nu \in P_+} m^\nu_{\lambda,\mu} V(\nu),$$

with $m^\nu_{\lambda,\mu} = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu) \otimes V(\nu^*), \mathbb{C})$.

The following is the most standard definition of the fusion ring.

3.1. First definition of the fusion ring. Fix a positive integer ℓ. The *fusion ring* $\mathcal{R}_\ell(\mathfrak{g})$ (associated to $\mathfrak{g}$ and the positive integer ℓ) is a free $\mathbb{Z}$-module with basis $\{V(\lambda) : \lambda \in P_\ell\}$. The product structure, called the *fusion product*, is defined as follows:

$$V(\lambda) \otimes^F V(\mu) := \bigoplus_{\nu \in P_\ell} n^\nu_{\lambda,\mu}(\ell) V(\nu),$$

where $n^\nu_{\lambda,\mu}(\ell)$ is the dimension of the space $V^\dagger_{\mathbb{P}^1}(\lambda, \mu, \nu^*)$ of *conformal blocks* on $\mathbb{P}^1$ with three distinct marked points $\{x_0, x_1, x_2\}$ and the weights λ, μ, ν^* attached to them with central charge ℓ. Recall that

$$V^\dagger_{\mathbb{P}^1}(\lambda, \mu, \nu^*) := \operatorname{Hom}_{\mathfrak{g} \otimes \mathcal{O}[U]}\big(L(V(\lambda), \ell) \otimes L(V(\mu), \ell) \otimes L(V(\nu^*), \ell), \mathbb{C}\big),$$

where $U := \mathbb{P}^1 \setminus \{x_0, x_1, x_2\}$ and $\mathcal{O}[U]$ denotes the ring of $\mathbb{C}$-valued regular functions on the affine curve U. The action of $\mathfrak{g} \otimes \mathcal{O}[U]$ on $L(V(\lambda), \ell) \otimes L(V(\mu), \ell) \otimes L(V(\nu^*), \ell)$ is given by

$$(x \otimes f)(v_1 \otimes v_2 \otimes v_3) := ((x \otimes (f)_{x_0})v_1) \otimes v_2 \otimes v_3 + v_1 \otimes ((x \otimes (f)_{x_1})v_2) \otimes v_3 \\ + v_1 \otimes v_2 \otimes ((x \otimes (f)_{x_2})v_3),$$

for $x \in \mathfrak{g}$ and $f \in \mathcal{O}[U]$, where $(f)_{x_i} \in \mathbb{C}((z))$ denotes the Laurent series expansion of f at x_i. The numbers $n^\nu_{\lambda,\mu}(\ell)$ are given by the celebrated Verlinde formula [17] (for a justification, see [8] together with [11], (13.8.9)).

It is clear from its definition that the product $\otimes^F$ is commutative. It is also associative as a result of the 'factorization rules' of [15].

We also remark that the canonical map

$$p : V^\dagger_{\mathbb{P}^1}(\lambda, \mu, \nu^*) \to \mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu) \otimes V(\nu^*), \mathbb{C})$$

induced from the natural inclusion $V(\lambda) \otimes V(\mu) \otimes V(\nu^*) \hookrightarrow L(V(\lambda), \ell) \otimes L(V(\mu), \ell) \otimes L(V(\nu^*), \ell)$ is an injection [16], Theorem 4.40. In particular, the inequality $n^\nu_{\lambda,\mu}(\ell) \le m^\nu_{\lambda,\mu}$ holds.

3.2. Second definition. Let $\tilde{\mathcal{G}}$ be the affine Kac–Moody group associated to the Lie algebra $\tilde{\mathfrak{g}}$ and $\tilde{\mathcal{P}}$ its parabolic subgroup (corresponding to the Lie subalgebra $\tilde{\mathfrak{p}}$) (cf. [13], Chapter 13). Then, $X = \tilde{\mathcal{G}}/\tilde{\mathcal{P}}$ is a projective ind-variety. Now, given a finite dimensional algebraic representation V of $\tilde{\mathcal{P}}$, we can consider the associated homogeneous vector bundle $\mathcal{V}$ on X and the corresponding Euler–Poincaré characteristic (which is a virtual $\tilde{\mathcal{G}}$-module)

$$\mathcal{X}(X, \mathcal{V}) := \sum_i (-1)^i H^i(X, \mathcal{V}).$$

Recall that $H^i(X, \mathcal{V})$ is determined in [13], Chapter 8.

For any positive integer ℓ and $\lambda, \mu \in P_\ell$, define

$$[L(V(\lambda), \ell) \otimes^\bullet L(V(\mu), \ell)]^* \cong \mathcal{X}(X, \mathcal{V}),$$

as virtual $\tilde{\mathcal{G}}$-modules, where the $\tilde{\mathcal{P}}$-module $V := (Y_\ell(V(\lambda) \otimes V(\mu)))^*$ (cf. §2.4 for the notation Y_ℓ). Writing

$$\mathcal{X}(X, \mathcal{V}) \cong \oplus_{\nu \in P_\ell} d^\nu_{\lambda,\mu} L(V(\nu), \ell)^*,$$

we get another definition of the fusion product as follows:

$$V(\lambda) \otimes^\bullet V(\mu) := \bigoplus_{\nu \in P_\ell} d^\nu_{\lambda,\mu} V(\nu).$$

By a result of Faltings [7], Appendix (see also Kumar [12], Theorem 4.2), the two products $\otimes^F$ and $\otimes^\bullet$ coincide for G of type A_r, B_r, C_r, D_r and G_2. In fact, these products coincide for any G as proved by Teleman (cf. [14], Theorem 0 together with [12], Lemma 4.1).

3.3. Third definition. Consider the Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ spanned by $X_\theta, X_{-\theta}$ and $\theta^\vee$, where $X_{\pm\theta}$ are the root vectors corresponding to the roots $\pm\theta$ respectively, choosen so that $(X_\theta | X_{-\theta}) = 1$. Then, $\mathfrak{s}$ is isomorphic with sl_2 under the isomorphism $\phi : sl_2 \to \mathfrak{s}$ taking $X \mapsto X_\theta, Y \mapsto X_{-\theta}, H \mapsto \theta^\vee$, where X, Y, H is the standard basis of sl_2. For any finite dimensional $\mathfrak{s}$-module V and nonnegative integer d, let $V^{(d)}$ denote the isotypical component of the $\mathfrak{s}$-module V corresponding to the highest weight d (where we follow the convention that the highest weight of an irreducible sl_2-module is one less than its dimension).

The following proposition can be found in [2], Proposition 7.2.

3.4. Proposition. *For any $\lambda, \mu \in P_\ell$, the fusion product $V(\lambda) \otimes^F V(\mu)$ is the isomorphism class of the ($\mathfrak{g}$-module) quotient of $V(\lambda) \otimes V(\mu)$ by the $\mathfrak{g}$-submodule generated by the $\mathfrak{s}$-module*

$$\bigoplus_{p+q+d>2\ell} \left(V(\lambda)^{(p)} \otimes V(\mu)^{(q)}\right)^{(d)}.$$

3.5. The homomorphism $\beta : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$. We first introduce the affine Weyl group W_ℓ. As in Section 2, let W be the (finite) Weyl group of G which acts naturally on P and hence also on $P_\mathbb{R} := P \otimes_\mathbb{Z} \mathbb{R}$. Let W_ℓ be the group of affine transformations of $P_\mathbb{R}$ generated by W and the translation $\lambda \mapsto \lambda + (\ell + \check{h})\theta$. Then, W_ℓ is the semi-direct product of W by the lattice $(\ell + \check{h})Q^{\text{long}}$, where Q^{long} is the sublattice of P generated by the long roots. For any root $\alpha \in R$ and $n \in \mathbb{Z}$, define the *affine wall*

$$H_{\alpha,n} = \{\lambda \in P_\mathbb{R} : (\lambda | \alpha) = n(\ell + \check{h})\}.$$

The closures of the connected components of $P_\mathbb{R} \setminus \left(\cup_{\alpha \in R, n \in \mathbb{Z}} H_{\alpha,n}\right)$ are called the *alcoves*. Then, any alcove is a fundamental domain for the action of W_ℓ. The *fundamental alcove* is by definition

$$A^o = \{\lambda \in P_\mathbb{R} : \lambda(\alpha_i^\vee) \geq 0 \, \forall \, \alpha_i \in \Delta, \text{ and } \lambda(\theta^\vee) \leq \ell + \check{h}\}.$$

3.6. Definition. Define the $\mathbb{Z}$-linear map $\beta : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$ as follows. Let $\lambda \in P_+$. If $\lambda + \rho$ lies on an affine wall, then $\beta(V(\lambda)) = 0$. Otherwise, there is a unique $\mu \in P_\ell$ and $w \in W_\ell$ such that $\lambda + \rho = w(\mu + \rho)$. In this case, define $\beta(V(\lambda)) = \varepsilon(w)V(\mu)$, where $\varepsilon(w)$ is the sign of the affine Weyl group element w.

The following theorem follows from the equivalence of the first two definitions of the fusion product given above together with [12], Lemma 3.3 and the Kac–Moody analogue of the Borel–Weil–Bott theorem (cf. [13], Corollary 8.3.12).

3.7. Theorem. *The $\mathbb{Z}$-linear map β defined above is, in fact, an algebra homomorphism with respect to the fusion product on $R_\ell(G)$.*

§4. A Conjectural Presentation of the Fusion Algebra

We consider the complexification $\mathcal{R}^{\mathbb{C}}(\mathfrak{g})$ of $\mathcal{R}(\mathfrak{g})$, and similarly consider the complexification $\mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g})$ of $\mathcal{R}_\ell(\mathfrak{g})$. We will denote the character of the i^{th} fundamental representation of $\mathfrak{g}$ by χ_i (instead of χ_{ω_i}). As is well known, $\mathcal{R}(\mathfrak{g})$ is identified with the polynomial ring $\mathbb{Z}[\chi_1, \dots, \chi_r]$. Moreover, the $\mathbb{C}$-algebra $\mathcal{R}^{\mathbb{C}}(\mathfrak{g})$ is identified with the affine coordinate ring $\mathbb{C}[T/W]$ of the quotient T/W. This identification is obtained by taking the character of any virtual representation in $\mathcal{R}(\mathfrak{g})$. Similarly, the finite dimensional algebra $\mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g})$ is identified with the affine coordinate ring $\mathbb{C}[T_\ell^{\mathrm{reg}}/W]$ of the reduced subscheme T_ℓ^{reg}/W of T/W by taking the character values on the finite set T_ℓ^{reg}/W (cf. [2], §9), where

$$T_\ell := \{t \in T : e^\alpha(t) = 1 \, \forall \alpha \in (\ell + \check{h})Q^{\mathrm{long}}\},$$

and T_ℓ^{reg} is the subset of regular elements of T_ℓ. Moreover, the homomorphism $\beta^{\mathbb{C}} : \mathcal{R}^{\mathbb{C}}(\mathfrak{g}) \to \mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g})$ obtained from the complexification of β defined in Definition 3.6 (under the above identifications) corresponds to the inclusion of $T_\ell^{\mathrm{reg}}/W \hookrightarrow T/W$. Let $I_\ell(\mathfrak{g})$ denote the kernel of $\beta^{\mathbb{C}}$. Since T_ℓ^{reg}/W is reduced, $I_\ell(\mathfrak{g})$ is a radical ideal. Identifying $\mathcal{R}^{\mathbb{C}}(\mathfrak{g})$ with $\mathbb{C}[\chi_1, \dots, \chi_r]$ as above, we can think of $I_\ell(\mathfrak{g})$ as an ideal inside $\mathbb{C}[\chi_1, \dots, \chi_r]$. Thus,

$$\mathcal{R}_\ell^{\mathbb{C}}(\mathfrak{g}) \simeq \mathbb{C}[\chi_1, \dots, \chi_r]/I_\ell(\mathfrak{g}).$$

We begin by recalling the following result on the presentation of fusion algebras. The (a)-part of the result is due to Gepner [9] and for the (b)-part, see [10] and [3]. In fact, both the parts are obtained by first constructing explicitly a potential function. Moreover, the following presentations are valid even over $\mathbb{Z}$ (cf. [4]).

4.1. Theorem. *(a) For $\mathfrak{g} = sl_{r+1}$,*

$$I_\ell(sl_{r+1}) = \langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \dots, \chi_{(\ell+r)\omega_1} \rangle.$$

(Recall that for $sl_{r+1}, \check{h} = r + 1$.)

(b) For $\mathfrak{g} = sp_{2r}$,

$$I_\ell(sp_{2r}) = \langle \chi_{\ell\omega_1+\omega_i};\ 1 \le i \le r\rangle.$$

In particular, the ideals on the right side of both of (a) and (b) are radical.

For any ideal I in a ring R, we denote its radical ideal by $\sqrt{I}$. Recall that the complete list of the fundamental weights ω_d such that the Dynkin index $m_{V(\omega_d)}$ is minimum (as well as the dual Coxeter numbers $\check{h}$ of any $\mathfrak{g}$) is given in Table 1. In the following theorem, for any $\mathfrak{g}$, we take any one ω_d with minimum Dynkin index. (Recall from Section 2.2 that ω_d is unique up to a diagram automorphism except for $\mathfrak{g}$ of type B_3.)

4.2. Theorem. *Let* ℓ *be any positive integer.*
(a) For $\mathfrak{g}$ *of type* $B_r (r \ge 3)$, $D_r (r \ge 4)$, E_6 *or* E_7,

$$I_\ell(\mathfrak{g}) \supseteq \sqrt{\langle \chi_{(\ell+1)\omega_d}, \chi_{(\ell+2)\omega_d}, \dots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_d}\rangle}.$$

(For $\mathfrak{g}$ *of type* B_3, *we must take* $\omega_d = \omega_1$.*)*
(b) For $\mathfrak{g}$ *of type* G_2,

$$I_\ell(G_2) \supseteq \begin{cases} \sqrt{\langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \chi_{(\frac{\ell+1}{2})\omega_2}\rangle}, & \text{if } \ell \text{ is odd,} \\ \sqrt{\langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \chi_{\omega_1+\frac{\ell}{2}\omega_2}\rangle}, & \text{if } \ell \text{ is even.} \end{cases}$$

(c) For $\mathfrak{g}$ *of type* F_4,

$$I_\ell(F_4) \supseteq \sqrt{\langle \chi_{(\ell+1)\omega_4}, \chi_{(\ell+2)\omega_4}, \dots, \chi_{(\ell+6)\omega_4}\rangle}.$$

(d) For $\mathfrak{g}$ *of type* E_8,

$$I_\ell(E_8) \supseteq \begin{cases} \sqrt{\langle \chi_{(\ell+2)\omega_8}, \chi_{(\ell+3)\omega_8}, \dots, \chi_{(\ell+29)\omega_8}\rangle}, & \text{if } \ell \text{ is even,} \\ \sqrt{\langle \chi_{(\ell+2)\omega_8}, \chi_{(\ell+3)\omega_8}, \dots, \chi_{(\ell+29)\omega_8}, \chi_{(\frac{\ell+1}{2})\omega_8}\rangle}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Proof. Using the tables in Bourbaki [1], in the cases (a), it is easy to verify that for $\lambda = (\ell+m)\omega_d$, $1 \le m \le \check{h}(\mathfrak{g}) - 1$, there exists an affine wall $H_{\mu,n}$ determined by a positive root μ of $\mathfrak{g}$ such that $\lambda + \rho$ lies on it. Then, by the definition of β, $\beta(V(\lambda))$ is trivial.

We give below some of the details.

$\underline{B_r}$: In this case, we have $\omega_d = \omega_1$ and $\check{h} = 2r-1$. Observe that $((\ell+m)\omega_1 + \rho|\mu) = \ell + 2r - 1$, where

$$\begin{cases} \mu = \sum\limits_{1\le k\le m} \alpha_k + \sum\limits_{1+m\le k\le r} 2\alpha_k\,, & \text{if } 1\le m\le r-1, \\ \mu = \sum\limits_{1\le k<2r-m} \alpha_k\,, & \text{if } r\le m\le 2r-2. \end{cases}$$

Thus, $(\ell+m)\omega_1+\rho$ lies on the affine wall determined by μ as above.

Similarly, $\ell\omega_1+\omega_r+\rho$ lies on the wall determined by the highest root θ of B_r (see the next conjecture, part (a)).

$\underline{D_r}$: In this case, we take $\omega_d = \omega_1$ and we have $\check{h} = 2r-2$. Observe that $((\ell+m)\omega_1+\rho|\mu) = \ell+2r-2$, where

$$\begin{cases} \mu = \sum\limits_{1\le k\le m} \alpha_k + \Big(\sum\limits_{m+1\le k<r-1} 2\alpha_k\Big) + \alpha_{r-1} + \alpha_r\,, & \text{if } 1\le m\le r-2, \\ \mu = \sum\limits_{1\le k<2r-1-m} \alpha_k\,, & \text{if } r-1\le m\le 2r-3. \end{cases}$$

Since any ω_d is obtained from ω_1 by a diagram automorphism in this case, the result holds for any ω_d.

A similar calculation shows that both $\ell\omega_1+\omega_{r-1}+\rho$ and $\ell\omega_1+\omega_r+\rho$ lie on the affine wall determined by the highest root θ of D_r (see the next conjecture, part (b)).

$\underline{G_2}$: In this case, we have $\omega_d = \omega_1$ and $\check{h} = 4$. Observe that $(\ell+1)\omega_1+\rho$ (resp. $(\ell+2)\omega_1+\rho$) lies on the affine wall determined by $\theta = 3\alpha_1+2\alpha_2$ (resp. $3\alpha_1+\alpha_2$). Moreover, $\frac{\ell+1}{2}\omega_2+\rho$ (for odd ℓ) and $\omega_1+\frac{\ell}{2}\omega_2+\rho$ (for even ℓ) lie on the affine wall determined by the root θ.

$\underline{F_4}$: In this case, we have $\omega_d = \omega_4$ and $\check{h} = 9$. Observe that $(\ell+1)\omega_4+\rho, \ldots, (\ell+6)\omega_4+\rho$ lie on the affine wall determined by the roots $\theta = 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$, $\theta-\alpha_1$, $\theta-\alpha_1-\alpha_2$, $\theta-\alpha_1-\alpha_2-2\alpha_3$, $\theta-\alpha_1-2\alpha_2-2\alpha_3$, $\theta-2\alpha_1-2\alpha_2-2\alpha_3$ respectively.

The corresponding calculation for the E series is more involved and is left to the reader. This proves the theorem. Q.E.D.

We would like to make the following conjecture.

4.3. Conjecture. All the inclusions in (a)-(b) of the above theorem are, in fact, equalities for $\mathfrak{g}$ of type $B_r(r\ge 3), D_r(r\ge 4)$ and G_2.

In addition, we conjecture the following:

(a) For $\mathfrak{g}$ of type $B_r (r \geq 3)$,

$$\sqrt{\langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \dots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_1} \rangle} = \langle \chi_{(\ell+1)\omega_1}, \chi_{(\ell+2)\omega_1}, \dots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_1}, \chi_{\ell\omega_1+\omega_r} \rangle.$$

(b) For $\mathfrak{g}$ of type $D_r (r \geq 4)$:

$$\sqrt{\langle \chi_{(\ell+1)\omega_d}, \chi_{(\ell+2)\omega_d}, \dots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_d} \rangle} = \langle \chi_{(\ell+1)\omega_d}, \chi_{(\ell+2)\omega_d}, \dots, \chi_{(\ell+\check{h}(\mathfrak{g})-1)\omega_d}, \chi_{\ell\omega_1+\omega_{r-1}}, \chi_{\ell\omega_1+\omega_r} \rangle.$$

4.4. Question. One may ask if the inclusions in the above theorem 4.2 for $\mathfrak{g}$ of types F_4, E_6, E_7 and E_8 are equalities as well.

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