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Masaki Kashiwara and representation theory

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It is a great honor for me to have an occasion to contribute an article on the work of Masaki Kashiwara in representation theory.

Let me start with my reminiscence of the period when I first met him. My main interest when I started my career as a mathematician around 1980 was on the infinite dimensional highest weight representations of semisimple Lie algebras. At that period the most important problem in this area was to determine the characters of irreducible highest weight modules. A precise conjecture had just been formulated by Kazhdan and Lusztig [36]. One day in 1981 I heard the news that the problem was solved by Brylinski–Kashiwara [5] and Beilinson–Bernstein [4], and I obtained a preprint of [5]. I was then a pure algebraist having no idea about D-modules, so I could not understand the language used in the preprint at all. But I gradually learned the theory of D-modules since then. Around 1982 I caught an idea that a problem in representation theory can be reformulated in the framework of D-modules, and wrote a letter to Kashiwara. Fortunately he was interested in the problem and it grew into my first work with him [26]. I still remember vividly my enthusiasm of the day when I got an unexpected phone call from the great mathematician mentioning his interest in the problem I posed.

A friend of mine once told me that the most important thing in achieving a success in collaboration with Kashiwara is to make him interested in the problem. I agree with this from my experiences including the above mentioned first one.

In this short manuscript I will describe some of the works of Kashiwara in the field of representation theory.

§1. Soliton equation

Methods from representation theory often shed light on problems in the theory of integrable models in mathematical physics. Conversely,

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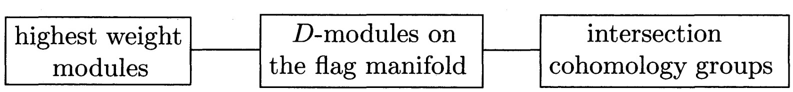
new progress is sometimes made in the representation theory by introducing physical ideas. The theory of soliton equations is one of the most striking successes of this type. The story begins with the work of M. Sato and Y. Sato finding the role of $GL(\infty)$ as the transformation group of the KP-hierarchy. Then Kashiwara and Miwa [22], Date, Kashiwara and Miwa [7], Date, Jimbo, Kashiwara and Miwa [8], [9] subsequently developed the theory of various soliton type equations for which affine Lie algebras act as the transformation groups of the solutions via the vertex operators. The work gave a great influence in underscoring the importance of affine Lie algebras both in mathematics and physics.

§2. Kazhdan–Lusztig conjecture

In the end of 1960s Verma proposed a problem of determining the characters of irreducible highest weight modules over semisimple Lie algebras. This problem was intensively investigated in 1970s by purely algebraic means by Bernstein–Gelfand–Gelfand, Jantzen and other people. A fantastic breakthrough to the problem was made around 1980. Kazhdan and Lusztig [36] introduced certain combinatorially defined polynomials, now called the Kazhdan–Lusztig polynomials, by which they formulated a conjecture giving the characters. They have also shown in [37] that the Kazhdan–Lusztig polynomials describe the local intersection cohomology groups of the Schubert varieties. Then Brylinski–Kashiwara [5] and Beilinson–Bernstein [4] independently settled the conjecture by employing D-modules on the flag manifold. The strategy of the proof can be illustrated in the following picture.



Namely, D-modules played the role of the missing link between highest weight modules and intersection cohomology groups. This is one of the monumental works in the history of representation theory, which changed the scenery of representation theory drastically.

Kashiwara also pursued the corresponding problem for Kac–Moody algebras with me in 1990s ([19], [27], [28], [28], [29], [30], [31], see also Casian [6]). In the Kac–Moody case one needs to deal with D-modules and intersection cohomology groups for infinite-dimensional varieties. A basis for those works was the scheme theoretical construction of the flag manifold for Kac–Moody algebras in [17]. The arguments are similar to those used for the original finite-dimensional version although some foundation for the theory of D-modules on the infinite dimensional manifold had to be prepared. In the case of affine Lie algebras character formulas for irreducible modules with positive and negative levels were obtained. The work for the negative level case together with Kazhdan–Lusztig [38], Lusztig [42], and Andersen–Jantzen–Soergel [3] contributed to the settlement of the Lusztig program on irreducible characters for reductive algebraic groups in positive characteristics. I finally point out that in the affine case there still remains an unsolved case for irreducible modules with critical highest weights.

§3. Representations of real semisimple Lie groups

In [11] Hotta and Kashiwara investigated the D-module corresponding to the system of differential equations satisfied by the characters of real semisimple Lie groups (more precisely its variant on the Lie algebras and its Fourier transform). The corresponding D-module was shown to be regular holonomic. Moreover, it was shown to be related via the Riemann–Hilbert correspondence to the Springer representations of Weyl groups. The Springer representations arose originally from the ordinary character theory of finite Chevalley groups. The work [11] reveals another role of the representations of Weyl groups in the character theory of real Lie groups. Later the version on the Lie groups (not on the Lie algebras) were treated in [15].

Let us describe another important work. Let $G_{\mathbb{R}}$ be a real semisimple Lie group with finite center, $K_{\mathbb{R}}$ its maximal compact subgroup, and K the complexification of $K_{\mathbb{R}}$. Denote by $\mathfrak{g}$ the complexification of the Lie algebra of $G_{\mathbb{R}}$. By a work of Harish-Chandra a representation of the real Lie group $G_{\mathbb{R}}$ is in a sense governed by the corresponding $(\mathfrak{g}, K)$-module which is a complex algebraic (rather than real analytic) object. Correspondingly, we meet two geometric situations (complex and real) concerning the flag manifold X for $\mathfrak{g}$. Namely, $G_{\mathbb{R}}$-equivariant sheaves on X are related to representations of $G_{\mathbb{R}}$ in the original sense, while K-equivariant sheaves are related to $(\mathfrak{g}, K)$-modules. In [16] Kashiwara formulated several conjectures concerning the relation between the two geometric objects. The most fundamental one was solved by Mirković, Uzawa and Vilonen [43], and the proof for the remaining more involved parts was announced in Kashiwara and Schmid [25].

§4. Quantized enveloping algebras

The most important contribution of Kashiwara in the area of quantized enveloping algebras is the theory of crystal bases. It arose from his attempt to construct a theory explaining the observation that in the representation theory of quantized enveloping algebras things become simpler at $q = 0$. In [18] he conjectured the existence of a basis of any integrable highest weight module at $q = 0$ characterized by certain standard properties, and proved it for type A, B, C, D. He named it a crystal basis. Around the same period Lusztig [39] independently constructed a basis of the negative part of the quantum groups for type A, D, E, and called it a canonical basis. Lusztig also showed that it induces a basis of any integrable highest weight module which can be specialized at $q = 0$. Subsequently Kashiwara [20] lifted the notion of the crystal basis to the global basis of the negative part of quantized enveloping algebra, and proved the existence of the crystal and the global basis for general quantized enveloping algebras by a huge induction. On the other hand Lusztig [41] extended the notion of the canonical basis to any quantized enveloping algebras. The coincidence of the global basis with the canonical basis was proved later in [40] and [10]. The theory of the (global) crystal basis and the canonical basis played very important roles in the subsequent history of representations of the quantized enveloping algebras.

Let me describe shortly other works of Kashiwara and his collaborators on quantized enveloping algebras that are more or less related to crystal basis theory. Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki [13] constructed the theory of perfect crystals, which played a decisive role in the analysis of the solvable lattice models in mathematical physics. Kashiwara and Nakashima [23] gave a parametrization of the crystal basis for A, B, C, D. Kashiwara and Saito [25] gave a geometric theory of crystals. Akasaka and Kashiwara [1] investigated finite-dimensional representations of quantum affine algebras.

§5. Other works

As for the works mentioned below I can only give brief description since they were accomplished before I started my career as a mathematician and I am not so familiar with them.

Kashiwara's first paper on representation theory [14] deals with the restriction of characters of representations of Lie groups to its maximal compact subgroups from the viewpoint of hyperfunctions. It shows that his interest in representation theory already started at a very early stage.

Kashiwara made essential contribution to the microlocal calculus for the relative invariants of prehomogeneous vector spaces. Results about b-functions appeared in [35]; however, there still remain unpublished results on the Fourier transforms.

Kashiwara and Vergne [33] constructed new series of unitary representations of the metaplectic group $Mp(n)$ and the indefinite unitary group $U(p,q)$ by decomposing the tensor product of the Weil representations via analysis of certain harmonic polynomials. The work made important contribution to the (still unsolved) problem of understanding all unitary representations of real semisimple Lie groups. A closely related result was also obtained by Howe [12] by another method.

Motivated by a problem in representation theory Kashiwara and Vergne [34] proposed a conjecture related to the Campbell–Hausdorff formula and proved it for solvable Lie algebras. Many people were fascinated and discussed the conjecture and its variants from various points of view. The conjecture itself in its full generality was settled recently by Alekseev and Meinrenken [2].

Kashiwara, Kowata, Minemura, Okamoto, Oshima, Tanaka [21] proved the so called Helgason conjecture on the relation between eigenfunctions of invariant differential operators on a symmetric space and hyperfunctions on the boundary. The theory of hyperfunctions and the notion of the regular singularity for systems of differential equation which was developed in Kashiwara and Oshima [24] were crucially used in the proof.

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