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A note on asymptotic stability condition for delay difference equations

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Abstract.

In this paper, we obtain the necessary and sufficient condition for the asymptotic stability of the linear delay difference equation

$$x_{n+1} - x_{n-1} + p\sum_{j=1}^{N} x_{n-k+(j-1)l} = 0$$

where $n = 0, 1, 2, ...$, p is a real number ,and k, l, and N are positive integers such that $k > (N-1)l$.

§1. Introduction

In [5], the asymptotic stability condition for the linear delay difference equation

$$x_{n+1} - x_n + p\sum_{j=1}^{N} x_{n-k+(j-1)l} = 0 \tag{1}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, p is a real number and k, l, and N are positive integers with $k > (N-1)l$, is given as follows.

Theorem A. Let k, l, and N be positive integers with $k > (N-1)l$. Then the zero solution of (1.1) is asymptotically stable if and only if

$$0 < p < \frac{2\sin\left(\frac{\pi}{2M}\right)\sin\left(\frac{l\pi}{2M}\right)}{\sin\left(\frac{Nl\pi}{2M}\right)} \tag{2}$$

where $M = 2k + 1 - (N-1)l$.

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Theorem A. generalizes asymptotic stability conditions given in [1 p.87, 2-3, 5, 6 p.65]. Theorem A. is proved using the fact that the zero solution of a linear difference equation is asymptotically stable if and only if all the roots of its characteristic equation lie inside the unit disk. In [4], we give necessary and sufficient conditions for the asymptotic stability of the following linear difference equation

$$x_{n+1} - a^2 x_{n-1} + b x_{n-k} = 0.$$

Motivated by these results, we are interested in the asymptotic stability of the linear delay difference equation of higher order which is similar to (1.1) as follows:

$$x_{n+1} - x_{n-1} + p \sum_{j=1}^{N} x_{n-k+(j-1)l} = 0 \tag{3}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, p is a real number, and k, l, and N are positive integers with $k > (N-1)l$. These linear difference equations may be used as discrete models of population dynamics of Baleen whales, [2]. Our main theorem is the following.

Theorem 1.1. Let k, l, and N be positive integers with k odd, l even and $k > (N-1)l$. Then the zero solution of (1.3) is asymptotically stable if and only if

$$0 < p < \frac{2\sin\left(\frac{\pi}{M}\right)\sin\left(\frac{l\pi}{2M}\right)}{\sin\left(\frac{Nl\pi}{2M}\right)} \tag{4}$$

where $M = 2k - (N-1)l$.

Remark 1.1. For $p > 0$ and k is even, we have $F(-1) = pN > 0$ and $\lim_{z \to -\infty} F(z) = -\infty$; hence F has a root which lies outside the unit disk and the zero solution of (1.4) is not asymptotically stable.

§2. Proof of Theorem

The characteristic equation of (1.1) is given by

$$F(z) = z^{k+1} - z^{k-1} + p\left(z^{(N-1)l} + \cdots + z^l + 1\right) = 0. \tag{5}$$

For $p = 0$, $F(z)$ has simple roots at 1 and -1 and root at 0 of multiplicity $k-1$. We first consider the location of the roots of (2.1) as p varies. Throughout the paper, we denote the unit circle by C and let $M = 2k - (N-1)l$.

Proposition 2.1. Let z be a root of (2.1) which lies on C. Then the roots z and p are of the form

$$z = e^{w_m i}, \text{ and} \tag{6}$$

$$p = 2\,(-1)^m \frac{\sin w_m \sin \frac{l w_m}{2}}{\sin \frac{N l w_m}{2}} \equiv p_m \tag{7}$$

for some $m = 0, 1, ..., M-1$ where $w_m = \frac{2m+1}{M}\pi$. Conversely, if p is given by (2.3), then $z = e^{w_m i}$ is a root of (2.1).

Proof. We consider roots of (2.1) which lie on C except the roots $z = 1$ and $z = -1$. Suppose that the value z satisfies $z^{Nl} = 1$ and $z^l \neq 1$. Then $z^{(N-1)l} + \cdots + z^l + 1 = 0$ and z is not a root of (2.1) which lies on C and we shall consider only the value z such that $z^{Nl} \neq 1$ or $z^l = 1$. Thus (2.1) can be written as

$$p = -\frac{z^{k-1}(z^2-1)}{z^{(N-1)l} + \cdots + z^l + 1}. \tag{8}$$

Since p is real, we have

$$\begin{aligned} p &= -\frac{\overline{z}^{k-1}(\overline{z}^2-1)}{\overline{z}^{(N-1)l} + \cdots + \overline{z}^l + 1} \\ &= -\frac{(z^2-1)z^{-k-1+(N-1)l}}{z^{(N-1)l} + \cdots + z^l + 1} \end{aligned} \tag{9}$$

where $\overline{z}$ denotes the conjugate of z. It follows from (2.4) and (2.5) that

$$z^{2k-(N-1)l} = -1$$

which implies that (2.2) is valid for $m = 0, 1, ..., M-1$ except for those integers m such that $e^{Nlw_m i} = 1$ and $e^{lw_m i} \neq 1$. We now show that p is of the form stated in (2.3). There are two cases to be considered as follows.

Case 1. z is of the form $e^{w_m i}$ for some $m = 1, 2, ..., M-1$ and $z^{Nl} \neq 1$.

From (2.4) we have

$$\begin{aligned}
p &= -\frac{z^{k-1}(z^2-1)\left(z^l-1\right)}{z^{Nl}-1} \\
&= -\frac{e^{(k-1)w_m i}\left(e^{2w_m i}-1\right)\left(e^{lw_m i}-1\right)}{e^{Nlw_m i}-1} \\
&= -\frac{e^{(k-(N-1)\frac{l}{2})w_m i}\left(e^{w_m i}-e^{-w_m i}\right)\left(e^{\frac{lw_m i}{2}}-e^{-\frac{lw_m i}{2}}\right)}{e^{\frac{Nlw_m i}{2}}-e^{-\frac{Nlw_m i}{2}}} \\
&= -e^{(k-(N-1)\frac{l}{2})w_m i}\frac{2i\sin(w_m)\sin\left(\frac{lw_m}{2}\right)}{\sin\left(\frac{Nlw_m}{2}\right)} \\
&= -e^{\frac{(2m+1)}{2}\pi i}\frac{2i\sin(w_m)\sin\left(\frac{lw_m}{2}\right)}{\sin\left(\frac{Nlw_m}{2}\right)} \\
&= 2(-1)^m\frac{\sin(w_m)\sin\left(\frac{lw_m}{2}\right)}{\sin\left(\frac{Nlw_m}{2}\right)} \equiv p_m.
\end{aligned}$$

Case 2. z is of the form $e^{w_m i}$ for some $m = 1, 2, ..., M-1$ and $z^{Nl} = 1$.

In this case, we have $lw_m = 2q\pi$ for some positive integer q. Then taking the limit as $lw_m \to 2q\pi$ we obtain

$$p = \frac{2(-1)^{m+q(N-1)}\sin(w_m)}{N}. \tag{10}$$

From these two cases, we conclude that p is of the form in (2.3) for $m = 1, 2, ..., M-1$ except for those m such that $e^{Nlw_m i} = 1$ and $e^{lw_m i} \neq 1$.

Conversely, if p is given by (2.3), then it is obvious that $z = e^{w_m i}$ is a root of (2.1). This completes the proof of the proposition. Q.E.D.

We now consider p as a function of z:

$$p(z) = -\frac{z^{k-1}(z^2-1)}{z^{(N-1)l}+...+z^l+1}.$$

Then, we have

$$\begin{aligned}
\frac{dp(z)}{dz} = &- \frac{z^{k-2}\left(2z^2+(k-1)(z^2-1)\right)}{z^{(N-1)l}+...+z^l+1} \\
&+ \frac{z^{k-2}(z^2-1)\left\{(N-1)lz^{(N-1)l}+...+lz^l\right\}}{\left(z^{(N-1)l}+...+z^l+1\right)^2}.
\end{aligned} \tag{11}$$

From this we have

Lemma 2.1. $\left.\frac{dp}{dz}\right|_{z=e^{w_m i}} \neq 0$. In particular, the roots of (2.1) which lie on C are simple.

Proof. Suppose on the contrary that $\left.\frac{dp}{dz}\right|_{z=e^{w_m i}} = 0$. We divide (2.7) by $\frac{p(z)}{z}$ to obtain

(12)
$$\frac{2z^2+(k-1)(z^2-1)}{z^2-1} - \frac{l\left\{(N-1)z^{(N-1)l}+ \ldots + z^l\right\}}{z^{(N-1)l}+\ldots+z^l+1} = 0.$$

Substituting z by $\frac{1}{z}$ in (2.8) we obtain

(13)
$$\frac{2+(k-1)(1-z^2)}{1-z^2} - \frac{l\left\{(N-1)+(N-2)z^l+\ldots+z^{(N-2)l}\right\}}{z^{(N-1)l}+\ldots+z^l+1} = 0.$$

By adding (2.8) and (2.9), we obtain

$$2k-(N-1)l = 0$$

which contradicts $k \geq (N-1)l$. This completes the proof. Q.E.D.

From Lemma 2.1, there exists a neighborhood of $z = e^{w_m i}$ such that the mapping $p(z)$ is one-to-one and the inverse of $p(z)$ exists locally. Now, let z be expressed as $z = re^{i\theta}$. Then we have

$$\frac{dz}{dp} = \frac{z}{r}\left\{\frac{dr}{dp} + ir\frac{d\theta}{dp}\right\}$$

which implies that

$$\frac{dr}{dp} = Re\left\{\frac{r}{z}\frac{dz}{dp}\right\}$$

as p varies and remaining real. The following result describes the behavior of the roots of (2.1) as p varies.

Proposition 2.2. The moduli of the roots of (2.1) on C increases as $|p|$ increases.

Proof. Let r be the modulus of z. Let $z = e^{w_m i}$ be a root of C. To prove this proposition, it suffices to show that

(14)
$$\left.\frac{dr}{dp}\cdot p\right|_{z=e^{w_m i}} > 0.$$

There are two cases to be considered.

Case 1. $z^{Nl} \neq 1$. In this case we have

$$p(z) = -\frac{z^{k-1}(z^2-1)(z^l-1)}{z^{Nl}-1} = -\frac{z^{k-1}f(z)}{z^{Nl}-1}$$

where $f(z) = z\left(z^l - 1\right)$. Then

$$\frac{dp}{dz} = -\frac{z^{k-2}g(z)}{\left(z^{Nl} - 1\right)^2}$$

where $g(z) = ((k-1)f(z) + zf'(z))\left(z^{Nl} - 1\right) - Nlz^{Nl}f(z)$. Letting $w(z) = -\frac{\left(z^{Nl}-1\right)^2}{z^{k-1}g(z)}$, we obtain

$$\frac{dr}{dp} = Re\left(\frac{r}{z}\frac{dz}{dp}\right) = rRe(w).$$

We now compute $Re(w)$. We note that

$$\begin{aligned} f\left(\overline{z}\right) &= \frac{f(z)}{z^{l+2}} \text{ and} \\ f'\left(\overline{z}\right) &= \frac{h(z)}{z^{l+1}} \end{aligned}$$

where $h(z) = l(1-z^2)+2(1-z^l)$. From the above relation and $z^M = -1$, we have

$$\begin{aligned} \overline{z}^{k-1}g(\overline{z}) &= \frac{1}{z^{k-1}}\left\{\left((k-1)f\left(\overline{z}\right) + \frac{1}{z}f'\left(\overline{z}\right)\right)\left(\frac{1}{z^{Nl}} - 1\right) - \frac{Nl}{z^{Nl}}f\left(\overline{z}\right)\right\} \\ &= \frac{((k-1)f(z) + h(z))\left(1 - z^{Nl}\right) - Nlf(z)}{z^{Nl+l+1+k}} \\ &= -\frac{((k-1)f(z) + h(z))\left(1 - z^{Nl}\right) - Nlf(z)}{z^{2Nl-k+1}}. \end{aligned}$$

It follows that

$$\begin{aligned}
Re(w) &= \frac{w+\overline{w}}{2} \\
&= -\frac{1}{2}\left\{\frac{\left(z^{Nl}-1\right)^2}{z^{k-1}g(z)}+\frac{\left(\overline{z}^{Nl}-1\right)^2}{\overline{z}^{k-1}g(\overline{z})}\right\} \\
&= -\frac{1}{2}\left\{\frac{\overline{z}^{k-1}g(\overline{z})\left(z^{Nl}-1\right)^2+z^{k-1}g(z)\left(\overline{z}^{Nl}-1\right)^2}{|g(z)|^2}\right\} \\
&= -\frac{1}{2|g(z)|^2}\left\{\begin{array}{l} \frac{((k-1)f(z)+h(z))\left(z^{Nl}-1\right)+Nlf(z)}{z^{2Nl-k+1}}\cdot\left(z^{Nl-1}\right)^2+ \\ z^{k-1}\left(((k-1)f(z)+zf'(z))\left(z^{Nl}-1\right)\right. \\ \left.-Nlz^{Nl}f(z)\right)\left(\frac{1}{z^{Nl}}-1\right)^2 \end{array}\right\} \\
&= -\frac{\left(z^{Nl}-1\right)^2 z^{k-1}}{2z^{2Nl}|g(z)|^2}\left\{\begin{array}{l} ((k-1)f(z)+h(z))\left(z^{Nl}-1\right) \\ +Nlf(z)+(((k-1)f(z)+zf'(z)) \\ \left(z^{Nl}-1\right))-Nlz^{Nl}f(z) \end{array}\right\} \\
&= -\frac{\left(z^{Nl}-1\right)^3 z^{k-1}}{2z^{2Nl}|g(z)|^2}\left\{h(z)+zf'(z)+(2(k-1)-Nl)f(z)\right\}.
\end{aligned}$$

Since

$$h(z)+zf'(z)+(2(k-1)-Nl)f(z)=Mf(z)$$

we obtain

$$Re(w)=\frac{\left(z^{Nl}-1\right)^4 M}{2z^{2Nl}|g(z)|^2}\cdot\frac{-z^{k-1}f(z)}{z^{Nl}-1}=\frac{\left(z^{Nl}-1\right)^4 Mp}{2z^{2Nl}|g(z)|^2}.$$

The value of $Re(w)$ at $z=e^{w_m i}$ is

$$\begin{aligned}
Re(w) &= \frac{\left(z^{Nl}-1\right)^4}{z^{2Nl}}\cdot\frac{Mp}{2|g(z)|^2} \\
&= (2\cos Nlw_m-2)^2\cdot\frac{Mp}{2|g(z)|^2}.
\end{aligned}$$

Therefore,

$$\frac{dr}{dp}=\frac{2r\left(\cos Nlw_m-1\right)^2 Mp}{|g(z)|^2}>0$$

and it follows that (2.10) holds at $z=e^{w_m i}$.

Case 2. $z^l=1$. With an argument similar to Case 1., we obtain

$$\frac{dr}{dp}=\frac{2rN^2Mp}{|(M+1)z-M+1|^2}$$

which implies that (2.10) is valid for $z = e^{w_m i}$.

This completes the proof. □

We now determine the minimum of the absolute values of p_m given by (2.3). We have the following result.

Proposition 2.3. $p_0 = \min\{|p_m| : m = 0, 1, ..., M-1\}$

To prove Proposition 2.3, we need the following lemmas.

Lemma 2.2. [5] Let N be a positive integer, then

$$\left|\frac{\sin Nt}{\sin t}\right| \leq N$$

holds for all $t \in \mathbb{R}$.

Lemma 2.3. [5] Let $0 < \theta < \frac{\pi}{2}$, then the inequality

$$\sin x\theta \, \sin y\theta \leq \sin\theta \, \sin xy\theta$$

holds for all $x, y \in \left(1, \frac{\pi}{2\theta}\right)$.

Proof of Proposition 2.3. By assumption, l is even which implies that M is also even. It is clear that $p_0 > 0$. Since each p_m is corresponded to $e^{w_m i}$ and its conjugate $\overline{e^{w_m i}}$, it is sufficient to consider p_m for $m = 0, 1, ..., \left[\frac{M-1}{2}\right] = \frac{M}{2} - 1$. To this end, we consider the following three cases.

Case I. $N = 1$. In this case, we have

$$p_m = 2(-1)^m \sin\frac{(2m+1)\pi}{2k}.$$

It follows immediately that $p_m \geq p_0$.

Case II. $N = 2$. It suffices to show that $\frac{1}{p_m} \leq \frac{1}{p_0}$ for $m = 1, 2, ..., \frac{M}{2} - 1$. Since $z^l = -z^{2k}$ and $z = e^{w_m i}$, we get

$$\begin{aligned} p_m &= -\frac{z^{k-1}(z^2-1)\left(-z^{2k}-1\right)}{z^{4k}-1} \\ &= \frac{z^{k-1}\left(z^2-1\right)}{z^{2k}-1} \\ &= \frac{z-z^{-1}}{z^k-z^{-k}} \\ &= \frac{e^{w_m i}-e^{-w_m i}}{e^{kw_m i}-e^{-kw_m i}} \\ &= \frac{\sin w_m}{\sin kw_m}. \end{aligned}$$

We observe that

$$\begin{aligned} p_{\frac{M}{2}-i} &= \frac{\sin \frac{2(\frac{M}{2}-i)+1}{M}\pi}{\sin \frac{2(\frac{M}{2}-i)+1}{M}k\pi} \\ &= \frac{\sin \frac{M-(2i-1)}{M}\pi}{\sin \frac{M-(2i-1)}{M}k\pi} \\ &= \frac{\sin\left(\pi - \frac{(2i-1)}{M}\pi\right)}{\sin\left(k\pi - \frac{(2i-1)}{M}k\pi\right)} \\ &= \frac{\sin \frac{(2i-1)}{M}\pi}{\sin \frac{(2i-1)}{M}k\pi} \\ &= p_{i-1}. \end{aligned}$$

Therefore, it suffices to show that

$$\frac{1}{p_m} \leq \frac{1}{p_0} \tag{15}$$

for $m = 1, 2, ..., \left[\frac{M}{4} - \frac{1}{2}\right]$. Note that when $M = 4j$ then $\left[\frac{M}{4} - \frac{1}{2}\right] = \frac{M}{4}$ and when $M = 2j$ for an odd number j, then $\left[\frac{M}{4} - \frac{1}{2}\right] = \frac{M}{4} - \frac{1}{2}$. Let $\theta = \frac{\pi}{M}$. Then we have

$$\frac{1}{p_0} = \frac{\sin k\theta}{\sin\theta} \text{ and } \frac{1}{p_m} = \frac{\sin k(2m+1)\theta}{\sin(2m+1)\theta}.$$

Note that $0 < \theta < \frac{\pi}{2}$ and

$$1 \leq M - k \leq \frac{\pi}{2\theta},\ 1 \leq 2m+1 \leq \frac{\pi}{2\theta},$$

since $k > l$. It follows from Lemma 2.2 that

$$\sin(M-k)\,\theta\,\sin(2m+1)\,\theta \geq \sin\theta\,\sin(M-k)\,(2m+1)\,\theta.$$

Taking into account that $(M-k)\,\theta = \pi - k\theta$, we obtain (2.8) for $m = 1, 2, ..., \left[\frac{M}{4} - \frac{1}{2}\right]$.

Case III. $N \geq 3$. We will show that

$$|p_m| \geq p_0 \tag{16}$$

for $m = 0, 1, ..., \left[\frac{M-1}{2}\right]$. With the same argument as in Case II, it suffices to show (2.9) for $m = 0, 1, ..., \left[\frac{M}{4} - \frac{1}{2}\right]$. Let $\theta = \frac{\pi}{M}$. Then

$0 < (2m+1)\,\theta \le \frac{\pi}{2}$ and

$$|p_m| = 2\sin(2m+1)\,\theta \left| \frac{\sin\frac{(2m+1)l\theta}{2}}{\sin\frac{(2m+1)Nl\theta}{2}} \right|.$$

By Lemma 2.3 and Jordan's inequality, namely $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ for $0 \le \theta \le \frac{\pi}{2}$, we obtain

$$|p_m| \ge 2 \cdot \frac{2}{\pi}\,(2m+1)\,\theta \cdot \frac{1}{N} = \frac{4\,(2m+1)\,\theta}{\pi N}. \tag{17}$$

We will show that (2.13) holds in the following three subcases:

Subcase (IIIa): $\frac{Nl\theta}{2} \le \frac{\pi}{2}$. In this subcase we have

$$p_0 = \frac{2\sin\theta\,\sin\frac{l\theta}{2}}{\sin\frac{Nl\theta}{2}} \le \frac{2\cdot\theta\cdot\frac{l\theta}{2}}{\frac{2}{\pi}\cdot\frac{Nl\theta}{2}} = \frac{\pi\theta}{N}. \tag{18}$$

Inequalities (2.14) and (2.15) imply that (2.13) holds for $m = 0, 1, ...,$ $\left[\frac{M}{4} - \frac{1}{2}\right]$.

Subcase (IIIb): $\frac{Nl\theta}{2} > \frac{\pi}{2}$. In this subcase we have

$$\frac{Nl\theta}{2} = \frac{Nl\pi}{2M} < \frac{\pi}{2}\cdot\frac{Nl}{(N-1)\,l} = \frac{\pi}{2}\cdot\frac{N}{(N-1)}$$

since $k > (N-1)l$ and $M = 2k - (N-1)\,l > 2(N-1)l - (N-1)l = (N-1)l$. By using $\sin\frac{Nl\theta}{2} = \sin\left(\pi - \frac{Nl\theta}{2}\right)$, we get

$$p_0 = \frac{2\sin\theta\,\sin\frac{l\theta}{2}}{\sin\frac{Nl\theta}{2}} \le \frac{2\cdot\theta\cdot\frac{l\theta}{2}}{\frac{2}{\pi}\cdot\left(\pi - \frac{Nl\theta}{2}\right)} = \frac{\pi l\theta^2}{2\pi - Nl\theta}.$$

It follows from (2.14), (2.15), and (2.16) that

$$\begin{aligned} \frac{|p_m|}{p_0} &\ge \frac{4(2m+1)\theta}{\pi N}\cdot\frac{2\pi - Nl\theta}{\pi l\theta^2} \\ &= \frac{4(2m+1)}{\pi^2}\left(\frac{2\pi}{Nl\theta} - 1\right) \\ &> \frac{4(2m+1)}{\pi^2}\left(\frac{2\,(N-1)}{N} - 1\right) \\ &= \frac{4(2m+1)}{\pi^2}\left(1 - \frac{2}{N}\right). \end{aligned}$$

From the above we have the following:

(i) If $N \ge 12$ and $m \ge 1$, then (2.13) holds.

(ii) If $N \geq 4$ and $m \geq 2$, then (2.13) holds.

(iii) If $N = 3$ and $m \geq 4$, then (2.13) holds.

We now consider the remaining cases.

(iv) $N \geq 4$ and $m = 1$. In this case it follows from (2.15) that $l\theta < \frac{\pi}{3}$ which implies that

$$|p_1| = \left| \frac{2\sin 3\theta \sin\frac{3l\theta}{2}}{\sin\frac{3Nl\theta}{2}} \right| \geq 2 \cdot \frac{2}{\pi} \cdot 3\theta \cdot \frac{2}{\pi} \cdot \frac{3l\theta}{2} = \frac{36 l\theta^2}{\pi^2}. \tag{19}$$

It follows from (2.15), (2.16), and (2.17) that

$$\begin{aligned} \frac{|p_1|}{p_0} &\geq \frac{36l\theta^2}{\pi^2} \cdot \frac{2\pi - Nl\theta}{\pi l \theta^2} > \frac{36}{\pi^3}(2\pi - Nl\theta) \\ &> \frac{36}{\pi^3}\left(2\pi - \frac{\pi N}{N-1}\right) = \frac{72}{\pi^3}\left(\pi - \frac{\pi N}{2(N-1)}\right) \\ &\geq \frac{24}{\pi^2} > 1. \end{aligned}$$

(v) $N = 3$ and $1 \leq m \leq 3$. By (2.15) and the assumption of Subcase (IIIb) it follows that $\frac{\pi}{6} < \frac{l\theta}{2} < \frac{\pi}{4}$ and we have

$$\frac{|p_m|}{p_0} = \left| \frac{\sin(2m+1)\theta \sin\frac{3l\theta}{2}}{\sin\frac{3(2m+1)l\theta}{2} \sin\frac{l\theta}{2}} \right| \frac{\sin\frac{(2m+1)l\theta}{2}}{\sin\theta}. \tag{20}$$

By Lemma 2.3, we get

$$\left| \frac{\sin(2m+1)\theta \sin\frac{3l\theta}{2}}{\sin\frac{3(2m+1)l\theta}{2} \sin\frac{l\theta}{2}} \right| \geq \frac{1}{3}\left| \frac{\sin\frac{3l\theta}{2}}{\sin\frac{l\theta}{2}} \right| = \frac{1}{3}\left| 3 - 4\sin^2\frac{l\theta}{2} \right| > \frac{1}{3}.$$

By Jordan's inequality we have

$$\frac{\sin\frac{(2m+1)l\theta}{2}}{\sin\theta} > \frac{\frac{2}{\pi}\cdot(2m+1)\theta}{\theta} = \frac{2(2m+1)}{\pi}.$$

Therefore,

$$\frac{|p_m|}{p_0} > \frac{2(2m+1)}{3\pi} > 1 \qquad \text{for } m = 2, 3.$$

If $m = 1$ and $p_1 > 0$, using $\frac{\pi}{6} < \frac{l\theta}{2} < \frac{\pi}{4}$, we obtain

$$\frac{p_1}{p_0} = \frac{3 - 4\sin^2\frac{l\theta}{2}}{4\sin^3\frac{3l\theta}{2} - 3} \frac{\sin 3\theta}{\sin\theta} > 1 \cdot \frac{1}{\theta} \cdot \frac{2}{\pi} \cdot 3\theta = \frac{6}{\pi} > 1.$$

This completes the proof of Proposition 2.3. Q.E.D.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Note that when $p < 0$ we have $F(1) = -pN < 0$ and $\lim_{z \to +\infty} F(z) = +\infty$. Thus F has a root which lies outside the unit disk. For $p = 0$, $F(z)$ has simple roots at 1 and -1 and root at 0 of multiplicity $k-1$. Let $z_1(p)$ be the branch of the root of (2.1) with $z_1(0) = 1$. Then it follows from (2.7) that

$$\left.\frac{dz_1}{dp}\right|_{p=0} = -\frac{N}{2} < 0.$$

By the continuity of the roots with respect to p, this implies that if $p > 0$ is sufficiently small then all the roots of (2.1) lie inside the unit disk. Next, Proposition 2.3 shows that p_0 is a positive minimum value of p such that a root of (2.1) intersects C as p increases from 0. Then by Proposition 2.2, if $p \geq p_0$, then there exists a root of (2.1) which lies outside the unit disk. From these arguments, we conclude that all the roots of (2.1) lie inside the unit disk if and only if $0 < p < p_0$. Therefore, the zero zolution of (1.3) is asymptotically stable if and only the condition (1.4) holds. Q.E.D.

Remark 2.1. For the case k and l are odd positive integers, N must also be odd (otherwise, $F(z)$ will have a root at -1 so that the zero solution of (1.3) is not asymptotically stable). Note that M is still an even integer. When $N = 1$ the same argument as in *Case I* of the proof of Proposition 2.3 shows that p_0 is the positive minimum of $|p_m|$ for $m = 0, 1, ..., \frac{M}{2} - 1$. When $N = 3$, the same argument as in *Case III* of the proof of Proposition 2.3 shows that p_0 is the positive minimum of $|p_m|$ for $m = 0, 1, ..., \left[\frac{M}{4} - \frac{1}{2}\right]$. However, we can not conclude from the proof in *Case III* of Proposition 2.3 that p_0 is the positive minimum of $|p_m|$ for $m = 0, 1, ..., \frac{M}{2} - 1$. We then have the following conclusion:

Theorem 2.4. Let k, l, and N be positive integers with k and l odd and $k > (N-1)l$. Then the zero solution of (1.3) is asymptotically stable if and only if

$$0 < p < p_0^*$$

where $M = 2k - (N-1)l$, $p_0^* = \min\{p_0, p^*\}$, $p_0 = \frac{2\sin(\frac{\pi}{M})\sin(\frac{l\pi}{2M})}{\sin(\frac{Nl\pi}{2M})}$ and

$$p^* = \min\left\{p_m : m = \left[\frac{M}{4} - \frac{1}{2}\right] + 1, \left[\frac{M}{4} - \frac{1}{2}\right] + 2, ..., \frac{M}{2} - 1\right\}.$$

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