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Computing topological entropy in asymmetric Cournot duopoly games with homogeneous expectations

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Abstract.

The main aim of this paper is to analyse the dynamics of nonlinear discrete-time maps generated by duopoly games in which players have homogeneous expectations and heterogenous nonlinear cost functions. This framework leads to reaction functions that are non-monotonic and asymmetric and, in the particular case of naïve expectations, the model takes the form of an anti-triangular map, $T(x, y) = (f(y), g(x))$ characterized by a rich dynamical behavior, from stable to chaotic Nash equilibria. We also present the computation of topological entropy of this nonlinear Cournot model by using tools from symbolic dynamics and tensor products.

§1. Introduction

Oligopoly models are very simple economic structures (or games) that may lead to very complex dynamics, like multiple equilibria, indeterminacy, or chaotic dynamics, even in its most simple form which is usually denominated by duopoly (two firms producing and supplying a homogeneous good in a common market). This complexity may arise from the combination of various sources, such as the way in which each firm formulates expectations about the rival's decisions (concerning the levels of goods produced and supplied in the market or the price at which these goods are supplied by both firms), the consideration of heterogeneous cost structures, the introduction of dynamics into the firms's decision making process, or the adoption of sophisticated cooperative strategies by both players and their impact upon the equilibrium of the game, among others. The literature on these issues is extremely

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large by now and important references are, e.g., [11], [10], [5], [3], [4], [1], [9], and [8].

This paper deals with a duopoly game in a dynamic setting, assuming that firms have homogenous expectations, and production displays heterogenous cost functions which are nonlinear due to positive externalities. For simplicity, we also assume that expectations are formulated in a naive way by each firm, meaning that at every period of time each player expects his rival to offer the same quantity for sale in the current period as it did in the preceding one.[1] As we will show, a very simple economic structure may lead to extremely complex dynamics, to multiple equilibria and to chaotic dynamics. A major result obtained is that the Nash equilibrium (the noncooperative solution of the game) changes from stable and periodic to chaotic, through period-doubling bifurcations, when the system parameters are changed. The existence of chaotic motion is demonstrated by computing the topological entropy of the model through the application of tools from symbolic dynamics and tensor products [7], [6].

§2. Homogeneous naive expectations

We consider a dynamic version of a simple Cournot-type duopoly market where firms (players) produce homogeneous goods which are perfect substitutes and supply them at discrete-time periods $t = 0, 1, 2, ...$ in a common market. At each period t, every firm must formulate an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing production quantities for period $t + 1$.

If we denote by $q_i(t)$, $i = 1, 2$ the output of firm i at time t, then its production $q_i(t+1)$, $i = 1, 2$ for the next period $t + 1$ is decided by solving the following optimization problem

$$(1) \qquad \begin{cases} q_1(t+1) = \arg\max_{q_1} \Pi_1(q_1(t), q_2^e(t+1)) \\ q_2(t+1) = \arg\max_{q_2} \Pi_2(q_1^e(t+1), q_2(t)), \end{cases}$$

where the function $\Pi(\cdot,\cdot)$ denotes the profit of the i^{th} firm and $q_j^e(t+1)$ represents the expectations of firm i about the production decision of firm j $(j = 1, 2, j \neq i)$. If the optimization problem has a unique

[1] Notice that with adaptive, rational or bounded rational expectations, the dynamics would become even more complicated than in the case of naive expectations. See, eg., Mendes et al. [8].

solution, then we denote the former by

$$\begin{cases} q_1(t+1) = f(q_2^e(t+1)) \\ q_2(t+1) = g(q_1^e(t+1)), \end{cases}$$

where f and g are the reaction functions.

The crucial point here is how each firm formulates expectations about the future production levels of its rival. Four different approaches have been found in the literature. Expectations can be *adaptive* if they are formulated in accordance with the rule $q_i^e(t+1) = q_i^e(t) + \xi(q_i(t) - q_i^e(t)), \xi \neq 1$ is the parameter that reflects the speed at which the expectation mistakes are corrected over time.[2] If we apply the condition $\xi = 1$, we obtain the rule $q_i^e(t+1) = q_i(t)$, which suggests *naive* behavior from both players. Fully *rational* expectations occur when $\xi \to \infty$, so that any mistake made at any period is instantaneously corrected, and we get the rule $q_i^e(t+1) = q_i(t+1)$. We may still have *bounded rational* expectations (see [1]) where firms use information based on the local estimates of the marginal profit function, such that at each time period t, each firm increases (decreases) its production q_i at the period $(t+1)$ if the marginal profit is positive (negative).

In this paper, and for the sake of brevity, we will only illustrate the case when both players choose the strategy defined by naive expectations. The model becomes

$$\begin{cases} q_1(t+1) = f(q_2(t)) \\ q_2(t+1) = g(q_1(t)), \end{cases}$$

which is known as a anti-triangular map, whose dynamics can be studied by using analytical and numerical tools.

§3. The complete model

Let us assume that the market inverse demand function is linear and decreasing:

$$P = p(Q) = a - b(q_1 + q_2), \tag{2}$$

where $Q = q_1 + q_2$ is the industry total output and $a, b > 0$. Following Kopel [5], we consider the case in which production satisfies two conditions. Firstly, there are positive production externalities, and secondly,

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[2]The fundamental point in this rule is that palyers update their expectations based on the mistakes they made in the past. Therefore, if no mistakes were done at time t, $q_i(t) = q_i^e(t)$, and so $q_i^e(t+1) = q_i^e(t)$.

for a given level of the rival's production, each firm's individual production shows constant returns to scale. The cost functions of both firms in the market may be represented by

$$C_1(q_1, q_2) = c_1(q_2)\, q_1 \text{ and } C_2(q_1, q_2) = c_2(q_1)\, q_2, \tag{3}$$

with $c_1^{'}(q_2) < 0, c_2^{'}(q_1) < 0$. Notice that each firm has a marginal cost of production $(c_i(\cdot))$ that is constant with respect to its own output but varies negatively with respect to the rival's output. We adopt here the specific asymmetric form for the functions c_i, $i = 1, 2$ as proposed by [9]:

$$\begin{cases} c_1(q_2) = a - bq_2 - 2b(\alpha q_2 - \alpha + 1)^2 \\ c_2(q_1) = a - bq_1 - 2b(\beta q_1 - 1)^2 . \end{cases}$$

Therefore, this particular structure for the marginal cost functions implies that both firms have heterogeneous costs of production and production externalities are nonlinear for both firms.

Combining (1), (2) and (3), we have the following profit functions for both firms:

$$\begin{cases} \Pi_1 = pq_1 - c_1(q_2)\, q_1 \\ \Pi_2 = pq_2 - c_2(q_1)\, q_2 \end{cases}$$

and the reaction functions are given by $f(q_2) = (\alpha q_2 - \alpha + 1)^2$ and $g(q_1) = (\beta q_1 - 1)^2$. Considering the case of naive expectations for both players the model takes the form

$$AT \begin{cases} q_1(t+1) = f(q_2(t)) = (\alpha q_2(t) - \alpha + 1)^2 \\ q_2(t+1) = g(q_1(t)) = (\beta q_1(t) - 1)^2 \end{cases},$$

and was firstly studied by Nonaka [9]. This is an anti-triangular or Cournot map and its dynamics can be rigorously studied by using analytical and numerical tools defined for example in [2], [7], [8], among others. This duopoly model admits multiple Nash equilibria, namely two, three or four, depending on the values of the two parameters α, β, and they can be obtained as the non-negative solutions of the algebraic system $(\alpha q_2 - \alpha + 1)^2 = q_1$, $(\beta q_1 - 1)^2 = q_2$.

The easiest way to study the anti-triangular map is to consider the second order composition of AT, that is,

$$T \begin{cases} (f \circ g)(q_1) = F(q_1) = (\alpha(\beta q_1 - 1)^2 - \alpha + 1)^2 \\ (g \circ f)(q_2) = G(q_2) = (\beta(\alpha q_2 - \alpha + 1)^2 - 1)^2, \end{cases} \tag{4}$$

where variables q_1 and q_2 are acting now independently from each other. This is a product map and we denote it by $T(q_1, q_2) = (F(q_1), G(q_2))$.

Now, the dynamic behavior of the Cournot map can be deduced from the one-dimensional maps $F(q_1)$ and $G(q_2)$, where the basis map $F(q_1) = (\alpha(\beta q_1 - 1)^2 - \alpha + 1)^2$ and the fiber map $G(q_2) = (\beta(\alpha q_2 - \alpha + 1)^2 - 1)^2$ are trimodal with the following critical points

$$q_1^{c_1} = \frac{1}{\beta},\ q_1^{c_2} = \frac{\alpha + \sqrt{\alpha^2 - \alpha}}{\alpha\beta},\ q_1^{c_3} = \frac{\alpha - \sqrt{\alpha^2 - \alpha}}{\alpha\beta}$$
$$q_2^{c_1} = \frac{\alpha - 1}{\alpha},\ q_2^{c_2} = \frac{\beta\alpha - \beta + \sqrt{\beta}}{\beta\alpha},\ q_2^{c_3} = \frac{\beta\alpha - \beta - \sqrt{\beta}}{\beta\alpha}.$$

Since we consider that $1 < \alpha < 2,\ 0 < \beta < 2$, the critical points are always well defined and the dynamics of the map T is fully characterized by the trajectories of the critical points of $F(q_1)$ and $G(q_2)$.

For example, if q_1^* is a fixed point of F, then $(q_1^*, g(q_1^*))$ is a fixed point of the Cournot map with eigenvalues $\lambda_1 = F'(q_1^*)$ and $\lambda_2 = -F'(q_1^*)$ and if the product map T has a period four orbit, then the anti-triangular map has a period eight orbit where the periodic points of AT are obtained by pairs of combinations of the periodic points of T. Moreover by considering tensor products between the one dimensional invariants associated with the critical orbits of F and G, we can compute the topological entropy of the Cournot map (see [7], [6]).

In what follows we will define the tensor product between two matrices and present the main theorem which allows the computation of the topological entropy of triangular and anti-triangular maps.

Definition 1. *Let A, B be two matrices of type $(m \times n)$ and $(p \times q)$. The tensor product of A and B is a matrix C of type $(mp \times nq)$, represented by $C = A \otimes B$ and defined by*

$$C = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

It is well known that the class of product maps studied in this paper admits a Markov partition (by rectangles) which is determined by the itineraries of the critical points associated with the basis map and the fiber map. Given a Markov partition $\mathcal{R} = \{R_j\}_{j=1}^m$, the transition matrix $A = (a_{ij})$ of type $(m \times m)$ is defined by $a_{ij} = 1$ if $int(f(R_i)) \cap int(R_j) \neq \emptyset$ and $a_{ij} = 0$ in the other case. The subshift space for A is defined as $\Sigma_A = \left\{s : \mathbb{N} \to \{1, 2, \ldots, m\} : a_{s_i s_{i+1}} = 1\right\}$. Letting σ be the shift map on the full $m-$shift, $\Sigma_m = \{1, 2, \ldots, m\}^{\mathbb{N}}$, define $\sigma_A = \sigma_{|\Sigma_A} : \Sigma_A \to \Sigma_A$, that is the subshift of finite type that characterizes the dynamics of the system.

Let us denote by $\mathcal{R}_{q_1}, \mathcal{R}_{q_2}, A_{q_1}, A_{q_2}$ the Markov partitions and the transition matrices associated with the critical orbits of the one-dimensional basis map F and the fiber map G. Then the following theorem holds [7], [6], [2]

Theorem 2. *Let I be a compact interval of the real line and let $T(q_1, q_2) = (F(q_1), G(q_2))$ be a continuous triangular map. Suppose that the basis map F admits n critical orbits of finite period $p_{1,\dots,}p_n$ and the fiber map G admits m critical orbits of finite periods $q_1, \dots, q_m$. Then the Markov partition of the map T is given by the Cartesian product $\mathcal{R}_{q_1} \times \mathcal{R}_{q_2}$ and the transition matrix A of T is given by the following tensor product: $A = A_{q_2} \otimes A_{q_1}$.*

Corollary 3. *The topological entropy of a continuous triangular map $T(q_1, q_2) = (F(q_1), G(q_2))$ is given by the sum of the topological entropies of F and G, that is $h(T) = h(F) + h(G)$.*

Now, the topological entropy of a generic Cournot map is formally characterized by:

Corollary 4. *Let I be a compact interval of the real line and let $T : I^2 \to I^2$, $AT(q_1, q_2) = (f(q_2), g(q_1))$ be a continuous Cournot map. We denote by*

$$T(q_1, q_2) = ((f \circ g)(q_1), (g \circ f)(q_2)) = (F(q_1), G(q_2))$$

the second iterate (compose) of the map AT which is by construction a triangular (product) map. Then

$$h(AT) = \frac{h(T)}{2} = \frac{h(F) + h(G)}{2}.$$

The Cournot map shows a very rich dynamic behavior which can be easily observed in Figure 1, where several bifurcation diagrams of the variable q_1 are presented when β is varied between 1 and 2, and α takes the values $\alpha = 1, 1.2, 1.4, 1.8$. We can observe period-doubling routes to chaos with several stability windows and chaotic regions, and some inverse bifurcations when $\alpha = 1.2$. In all of these cases the dynamics of variable q_2 is very similar. We found stable Nash equilibria for values of $\beta < 1$.

Let us consider the map T defined in (4) and fix $\alpha = 1.8$. In order to compute the topological entropy of the Cournot map, we have to find some periodic critical orbits for the basis map F and for the fiber map G. By observing the bifurcation diagram (Figure 1) we decided to consider the period four stability windows located immediately after $\beta = 1$, since $1 < \beta < 2$ presents more complex dynamics. This is a periodic orbit with

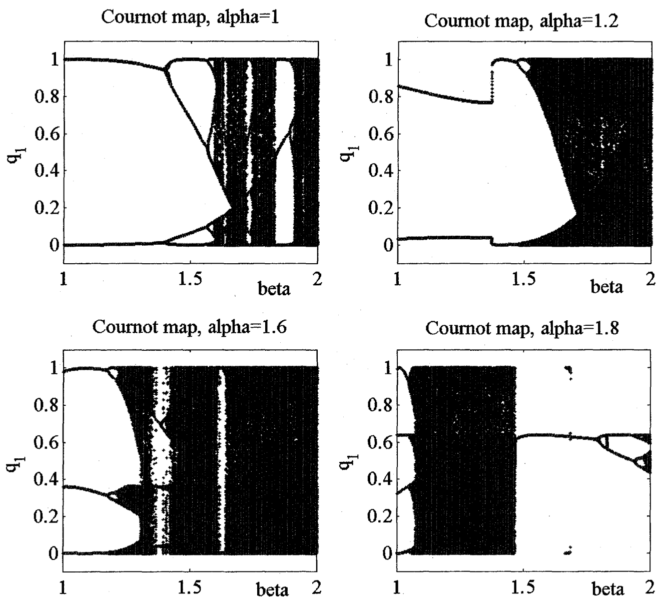

Fig. 1. Several bifurcation diagram for the q_1 variable when the parameter β is varied between 1 and 2 and the parameter α assumes the following values: $1, 1.2, 1.4, 1.8$.

chaos and by solving $(f \circ g)^4 (q_1, \beta) = q_1$ for $q_1^{c_3}$ and $\alpha = 1.8$ we find the exact value of β where the bifurcation takes place. We obtain then, that for $\beta = 1.00991$ the critical orbit of the trimodal map $(f \circ g)$ initiated in $q_1^{c_3}$ is of period four. The critical point $q_1^{c_3}$ is the only significative one, since the diagonal $q_1 = (f \circ g)(q_1)$ intersects only once the trimodal map $(f \circ g)(q_1)$ in the vicinity of this critical point. By the same reason we obtain that $G(q_2) = (g \circ f)(q_2)$ has also a period four critical orbit, which is given by the trajectory of the significant critical point, $q_2^{c_2}$. We may completely ignore the other orbits (critical or non-critical) since the kneading theory assures that the critical orbit is predominant and fully characterizes the dynamics, giving the maximal entropy of the system.

For this parameters setting ($\alpha = 1.8$ and $\beta = 1.00991$) the Cournot map admits 2 unstable nonnegative Nash equilibrium points given by $(0.1753, 0.6771)$ and $(2.0967, 1.2489)$ where $q_1^1 = 0.1753$, $q_1^2 = 2.0967$ and $q_2^1 = 0.6771$, $q_2^2 = 1.2489$ are also the fixed points of the one-dimensional

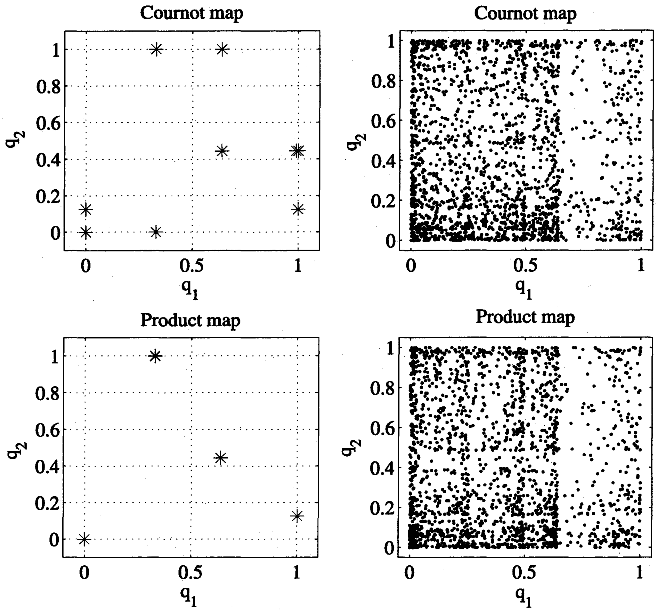

Fig. 2. Period 8 orbit and strange attractor for the antriangular map and period 4 orbit and strange attractor for the product map when $\alpha = 1.8$ and $\beta = 1.00991$, $\beta = 1.2$

trimodal maps $F(q_1)$ and $G(q_2)$. The study of the local stability of these fixed points is based on the localization, on the complex circle, of the eigenvalues of the Jacobian matrix of the two-dimensional map, which is given by

$$J(q_1, q_2) = \begin{bmatrix} 0 & 2\alpha(\alpha q_2 - \alpha + 1) \\ 2\beta(\beta q_1 - 1) & 0 \end{bmatrix}.$$

For the first fixed point we have complex conjugate eigenvalues given by $\lambda = \pm 1.5830i$, with modulus greater than one, while for the second fixed point we have $\lambda = \pm 3.4303$, both displaying real eigenvalues with modulus greater than one, which shows the instability of the Nash equilibria.

The main dynamics of the Cournot map is fully characterized by the dynamics of the critical orbits. We have that the kneading sequences of the two period four critical orbits associated with the maps $(f \circ g)$ and $(g \circ f)$ are given by $(CRLL, crll)$, where $\mathcal{A}_{q_1} = \{L, C, R\}$ and $\mathcal{A}_{q_2} = \{l, c, r\}$ are the alphabets associated with the generated partitions of the phase space of these maps. The transition matrices of the period four critical orbits are given by

$$A_{q_1} = A_{q_2} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the transition matrix A of the map T is then

$$A = A_{q_1} \otimes A_{q_2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the logarithm of the spectral radius of a Markov transition matrix gives the topological entropy of a multimodal map, we have that $h\,(fog) = h\,(gof) = \log{(\lambda_{\max})} = 0.6094$ which means that the topological entropy of the triangular map and of the Cournot map is $h\,(T) = 1.2188$ and $h\,(AT) = 0.6094$.

Figure 2 represents the period 8 orbit of the Cournot map and the period 4 orbit of the product map, for the considered parameters setting. It shows also, as a mere example, some strange attractors simulated for $\beta = 1.2$ and $\alpha = 1.8$.

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