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Discrete potential theory for iterated maps of the interval

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Abstract.

Using Markov partitions and algebraic graph theory we introduce, in the context of discrete dynamical systems, some laws which characterize the nonlinear dynamics of iterated maps of the interval. In the Markov digraphs we assume that each directed edge has a weight associated to it, given by the Markov invariant measure. This system of weights produces a diffusion process determined by a transition matrix. In this setting, we define a current and a potential which are dynamical invariants.

§1. Introduction and preliminaries

Nonlinear dynamical systems can be effectively studied and modelled by difference equations. An important problem is to describe the behavior of the discrete dynamical system determined by the iteration of a map f,

$$x_{k+1} = f(x_k).$$

Our approach uses symbolic dynamics to obtain a digraph associated with the original discrete dynamical system, as in [3]. Several invariants for the discrete dynamical system can be obtained and interpreted in the digraph setting, see for example [7] and also [4], [5]. The notions of current, potential, conductance, have its origins in the electric circuits and have been generalized to graph theory. We extend some of this notions to discrete dynamical systems.

Given an interval map f, and using the orbits of the critical points, we obtain a graph, the *Markov digraph*, directly from the transition matrix A_f and Markov partition for f. Using the Parry measure, we

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obtain a weighted digraph, and by analogy with the electric circuits we introduce a current and a potential which satisfies the general Kirchoff laws, and are important dynamical invariants. These are very general laws that can be applied in many different contexts, see [2] (and also [1]).

We define the (ergodic) potential as the Parry invariant measure p. The reason is that if we think the time evolution as an iteration, this is a stationary potential. It can be seen as the density of points in the vertices (which corresponds to the partition intervals). We define also the (ergodic) current q in a way that do not change under iteration. In a certain sense, it corresponds to the conditional probability of a vertex to have a transition defined by an edge.

1.1. Interval maps

Let $f : I \to I$ be a piecewise monotone map of the interval into itself. The *singular points* of f are the points of I in which f' is zero, not defined or discontinuous and constitutes a finite set $\{c_1, ..., c_m\}$. Therefore, there is a finite partition

$$I = I_1 \cup C_1 \cup ... \cup C_m \cup I_{m+1}$$

so that I_j is a maximal interval of monotonicity for f (*a lap*) and $C_j = \{c_j\}$. Every point in I has a unique symbolic expansion in the alphabet

$$\mathfrak{A} = \{I_1, ..., I_{m+1}, C_1, ..., C_m\},$$

denoted by $it(x)$ (*itinerary of* x), given as follows: the *address of* x, $ad(x) \in \mathfrak{A}$, is determined by $x \in ad(x)$. The itinerary is then defined by

$$it(x) = (ad(x), ad(f(x)), ad(f^2(x), ...) \in \mathfrak{A}^{\mathbb{N}}.$$

Consider the orbits of the singular points. The itineraries of the images of the singular points are called the *kneading sequences* of f, $K_{c_j} = it(f(c_j))$, (see [6]), and define the *kneading invariant* for f,

$$\mathcal{K}_f = (K_{c_1}, ..., K_{c_m}).$$

This is a very important invariant for the discrete dynamical system defined by f.

We assume that f is such that the orbits of the singular points are finite, i.e., such that all the kneading sequences are eventually periodic. Let $\{x_1, x_2, ..., x_{n'}\}$ be the union of the singular points orbits, ordered according to

$$x_1 < x_2 < ... < x_{n'}.$$

Define the intervals $J_i = [x_i, x_{i+1}]$ with $i = 1, ..., n' - 1$. The set $\{J_1, ..., J_n\}$, with $n = n' - 1$, is a Markov partition for f. Then we define the transition matrix $A_f = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 \text{ if } f(\mathrm{Int}(J_i)) \supseteq f(\mathrm{Int}(J_j)) \\ 0 \text{ otherwise.} \end{cases}$$

1.2. Digraphs

Consider a graph $\mathcal{G} = (G_0, G_1, s, r)$ where G_0 is the set of vertices and G_1 the set of edges. The maps $s, r : G_1 \to G_0$, (source and range) assign to every edge its head and its tail, respectively.

Let $C^0 = C^0(\mathcal{G}; \mathbb{R})$ be the linear space of real functions on the set G_0, and $C^1 = C^1(\mathcal{G}; \mathbb{R})$ be the linear space of real functions on the set G_1. The canonical basis for C^0 is given by $\{\delta_v : v \in G_0\}$. The same for C^1, with $\{\delta_e : e \in G_1\}$.

Let us assume $G_0 = \{1, ..., n\}$. The adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix, $A = (a_{ij})$, with

$$a_{ij} = \begin{cases} 1 \text{ if there is } e \in G_1 : s(e) = i, r(e) = j \\ 0 \text{ otherwise.} \end{cases}$$

The incidence operator $D : C^1 \to C^0$ is defined by $D\delta_e = \delta_{r(e)} - \delta_{s(e)}$. It is represented by the incidence matrix D. The adjoint $D^t : C^0 \to C^1$ is defined as follows: Given $\alpha \in C^0$ we have $(D^t\alpha)(e) = \alpha(r(e)) - \alpha(s(e))$. The linear space $\ker D^t$ is the space of constant functions on the connected components of $\mathcal{G}$, see [2]. Therefore, if G is connected $\ker D^t$ is the one-dimensional space generated by the function $\sum_{i=1}^{n} \delta_i$. The operator D also has an explicit formula

$$(Dg)(v) = \sum_{e:r(e)=v} g(e) - \sum_{e:s(e)=v} g(e).$$

The kernel of D is the set of functions $g \in C^1$ so that

$$\sum_{e:r(e)=v} g(e) = \sum_{e:s(e)=v} g(e)$$

for every $v \in G_0$. An element in $\ker D$ is usually called a *flow*, or a *current*. Let $Z = \ker D$ and let $Z^\perp$ be the orthogonal complement of Z, that is,

$$C^1 = Z \oplus Z^\perp = \ker D \oplus (\ker D)^\perp .$$

A *path* is a sequence of vertices (or edges) in the graph $\mathcal{G}$. A path $(v_0, ..., v_k)$ is called a closed path if $v_0 = v_k$ and is called a *cycle* if all other vertices are distinct. To each cycle γ we associate the function $z_\gamma \in C^1$

$$z_\gamma(e) = \begin{cases} 1, & \text{if the sequence } s(e), r(e) \text{ occurs in } \gamma \\ -1, & \text{if the sequence } r(e), s(e) \text{ occurs in } \gamma \\ 0, & \text{otherwise.} \end{cases}$$

Let S be a non-empty subset of G_0. Define the function $b_S \in C^1$ by

$$b_S(e) = \begin{cases} 1, & \text{if } \{s(e), r(e)\} \cap S = \{r(e)\} \\ -1, & \text{if } \{s(e), r(e)\} \cap S = \{s(e)\} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, given $S \subset G_0$, the function b_S determines which edges "go outside" or "go inside" the region determined by the vertices in S. The set of edges which have exactly one vertex in S is called a *cut* or a *cocycle* of G.

We have that $z_\gamma \in Z$, for every cycle γ and $b_S \in Z^\perp$, for every proper subset $S \subset G_0$, see [2]. Therefore, Z is usually called the *cycle space* and $Z^\perp$ the *cocycle space.*

Let $\mathcal{T}$ be a spanning tree for $\mathcal{G}$, that is, a connected subgraph containing every vertex of $\mathcal{G}$ and with no cycle. If we remove an edge from $\mathcal{T}$ we obtain two components, $\mathcal{T}^+$ and $\mathcal{T}^-$, one containing $r(e)$ and the other containing $s(e)$. The cocycle associated with the set of vertices in $\mathcal{T}^+$, denoted by $S(e) = S(\mathcal{T}, e)$, is called a *fundamental cocycle* determined by $\mathcal{T}$ and e. The importance of the fundamental cocycles is that for any spanning tree $\mathcal{T}$ the functions $b_{S(e)}$, with e being an edge in $\mathcal{T}$, constitute a basis for the cocycle space $Z^\perp$, see [2]. Now, given a spanning tree $\mathcal{T}$ for each edge e not in $\mathcal{T}$ there is a unique path in $\mathcal{T}$ with initial vertex $s(e)$ and final vertex $r(e)$. This path together with the edge e determines a cycle $\gamma(e) = \gamma(\mathcal{T}, e)$ called a *fundamental cycle* determined by $\mathcal{T}$ and e. Given a spanning tree $\mathcal{T}$, the functions $z_{\gamma(e)}$, with e an edge not in $\mathcal{T}$, determines a base for the cycle space Z. Then, we denote $C = C(\mathcal{T})$ as the matrix of the fundamental cycles, formed by the vectors in the base of Z determined by $\mathcal{T}$. In the same manner, we define $B = B(\mathcal{T})$ as the matrix of the fundamental cocycles, formed by the vectors in the base of $Z^\perp$ determined by $\mathcal{T}$.

§2. Current and potential for interval maps

Let f be a Markov map of the interval I into itself, with Markov partition $\{I_1, ..., I_n\}$, and transition matrix A_f. There is a special measure associated with f, the *Parry measure*, which is an invariant measure and the maximal entropy measure, see [8]. It can be given as follows: Let λ_f be the Perron eigenvalue of A_f. Let u be the Perron left eigenvalue of A_f and v the Perron left eigenvalue of A_f. Let $P_f = (p_{ij})$, where

$$p_{ij} = \frac{a_{ij} v_j}{\lambda v_i},$$

and $p = (p_i)$ where $p_i = \frac{u_i v_i}{\sum u_j v_j},$. We have $p = pP_f$, that is, p is the left Perron eigenvalue of P_f, and $\sum_{i=1}^{n} p_i = 1$. Therefore, p represents an invariant probability measure, which is precisely the Parry measure.

Let $\mathcal{G}_f = (G_0, G_1, r, s)$ be the digraph associated with the Markov partition of f so that the transition matrix A_f is the adjacency matrix of $\mathcal{G}_f$. Following the notation introduced in Section 1, let D_f be the incidence matrix of $\mathcal{G}_f$. Note that G_1 coincides with the set of admissible words of size 2. More precisely, there is a bijection

$$ij \text{ admissible word} \longleftrightarrow (ij) \in G_1.$$

Definition 1. *The (ergodic) current is the function $q_f \in C_f^1$ defined by*

$$q_f(e) = p_{s(e)} P_{s(e)r(e)}.$$

The (ergodic) potential is the function $p_f \in C_f^0$.

We will see that q is in fact a function of Z_f, therefore, is completely justified the name of current or flow. Next, we present our main result:

Theorem 2. *Let f be a Markov interval map. Then we have the following:*

$$D_f \; q_f = 0.$$

For every $b \in Z_f^{\perp}$, $\langle b, q_f \rangle = 0$ (current Kirchoff law).

For every $z \in Z_f$, $\left\langle z, D_f^t \; p_f \right\rangle = 0$ (voltage Kirchoff law).

Proof. We have

$$\begin{aligned}
(D_f q_f)(j) &= \sum_{e:r(e)=j} q_f(e) - \sum_{e:s(e)=j} q_f(e) \\
&= \sum_{e:r(e)=j} p_{s(e)} P_{s(e)r(e)} - \sum_{e:s(e)=j} p_{s(e)} P_{s(e)r(e)} \\
&= \sum_i p_i P_{ij} - \sum_i p_j P_{ji} = \sum_i p_i P_{ij} - p_j \sum_i P_{ji}.
\end{aligned}$$

Since $\sum_i P_{ji} = 1$ (the vector $(1, ..., 1)$ is a right eigenvector of P_f) and $\sum_i p_i P_{ij} = p_j$ (p_f is a left eigenvector of P_f) we have $D_f q_f = 0$. Since

$(D_f q_f)(j) = 0$ corresponds to the equation $\langle b, q_f \rangle = 0$ for the cocycle $b = (0, ..., 1, .., 0)$ (1 in j position) we have by linearity that $\langle b, q_f \rangle = 0$, for every $b \in Z_f^{\perp}$. Now, since $\mathrm{Im}(D_f^t) \cong \ker(D_f)^{\perp}$ is orthogonal to Z, therefore the result follows. Q.E.D.

Example 3. *Let us consider the unimodal map $f_b(x) = 4bx(1-x)$, with $b = 0.9764266...$, i.e., such that the orbit of the critical point is periodic of period 5 with kneading sequence RLLRC. Its topological entropy is equal to*

$$h_{top}(f_b) = \log(1.72208...).$$

The transition matrix associated with the Markov partition (using the partition generated by the critical point orbit) is

$$A_f = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The spectral radius is $\lambda_{\max}(A_f) = 1.72208...$ and the normalized Perron right eigenvector is (approximately)

$$v = (0.277916, 0.141392, 0.213326, 0.123877, 0.243489).$$

The normalized Perron left eigenvector is (approximately)

$$u = (0.141392, 0.223497, 0.195811, 0.195811, 0.243489).$$

The matrix P_f is (approximately) given by

$$P_f = \begin{pmatrix} 0 & 0.295431 & 0.445734 & 0.258834 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.337203 & 0.662797 \\ 0 & 0 & 1 & 0 & 0 \\ 0.662797 & 0.337203 & 0 & 0 & 0 \end{pmatrix},$$

and the vector p_f, which corresponds to the Parry measure, is (approximately)

$$p_f = (0.20027, 0.161055, 0.212892, 0.123624, 0.302159).$$

The incidence matrix of the digraph is

$$D_f^t = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

The matrix of a basis of the fundamental cycles is given by

$$C_f = \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix of a basis of the fundamental cocycles (cuts) is given by

$$B_f = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

The current vector, $q_f = p_f P_f$, is (approximately)

$$\begin{aligned} q_f \quad = \quad & (0.059166, 0.0892672, 0.0717877, 0.141104, \\ & \quad 0.101889, 0.161055, 0.123624, 0.0518367, 0.20027). \end{aligned}$$

The divergence $D_f\ q = 0$ and $B_f\ q_f = 0$, the total current that crosses each cocycle is zero. Moreover the difference of potential along each cycle is zero, $C_f D_f^t\, p_f = 0$.

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