Advanced Studies in Pure Mathematics 53, 2009
Advances in Discrete Dynamical Systems
pp. 51–57

Periodic solutions of periodic difference equations

Tetsuo Furumochi [1] and Masato Muraoka

Abstract.

In this paper, we discuss the existence of periodic solutions of the periodic difference equation

$$x(n+1) = f(n, x(n)), \quad n \in \mathbf{Z}$$

and the periodic difference equation with finite delay

$$x(n+1) = f(n, x_n), \quad n \in \mathbf{Z},$$

where x and f are d-vectors, and $\mathbf{Z}$ denotes the set of integers. We show the existence of periodic solutions by using Browder's fixed point theorem, and illustrate an example by using a boundedness result due to Shunian Zhang.

§1. Introduction

The existence of periodic solutions of periodic difference equations has been discussed in some books and papers (see [2-5, 7] and their references). In particular, Dannan-Elaydi-Liu [2], Elaydi [4] and Elaydi-Zhang [5] have obtained some existence results of periodic solutions by employing various methods including homotopy techniques. In this paper, we give some new existence results of periodic solutions for periodic difference equations by using Browder's fixed point theorem, Liapunov method and Razumikhin method, and show an example by using a boundedness result due to Shunian Zhang.

Fixed point theorems are very useful tools in obtaining existence theorems for periodic solutions. First we state a fixed point theorem

---

Received October 12, 2007.

2000 *Mathematics Subject Classification.* 39A11.

Key words and phrases. Periodic solutions, difference equations, fixed point theorem.

[1]This paper is in final form and no version of it will be submitted for publication elsewhere, and partly supported in part by Grant-in-Aid for Scientific Research (C), No. 16540141, Japan Society for the Promotion of Science.

due to Browder, which we use later to show the existence of periodic solutions of periodic difference equations.

Theorem 1. *(Browder) Let S and S^1 be open convex subsets of the Banach space A, S^0 a closed convex subset of A, $S^0 \subset S^1 \subset S$, F a compact mapping of S into A. Suppose that for a positive integer m, F^m is well-defined on S^1, $\cup_{0 \le j \le m} F^j(S^0) \subset S^1$, while $F^m(S^1) \subset S^0$. Then F has a fixed point in S^0.*

Here, F is said to be a compact mapping of S, if F is a continuous mapping and $F(S)$ is contained in a compact set of A.

§2. Difference equations without delays

Consider the difference equation

$$x(n+1) = f(n, x(n)), \quad n \in \mathbf{Z}, \tag{1}$$

where x and f are d-vectors, and $\mathbf{Z}$ denotes the set of integers. Suppose that $f : \mathbf{Z} \times \mathbf{R}^d \to \mathbf{R}^d$ is continuous in x for each fixed $n \in \mathbf{Z}$, where $\mathbf{R}$ denotes the set of real numbers. For any $n_0 \in \mathbf{Z}$ and $\xi \in \mathbf{R}^d$, $x(n, n_0, \xi)$ denotes the solution of Eq.(1) with $x(n_0, n_0, \xi) = \xi$. For any $\alpha > 0$, define the set S_α by

$$S_\alpha := \{x \in \mathbf{R}^d : |x| \le \alpha\},$$

where $|\cdot|$ denotes the Euclidean norm for $\mathbf{R}^d$.

First we state the definitions of boundedness of solutions of Eq.(1).

Definition 2.1. *The solutions of* Eq.(1) *are equi-bounded, if for any $n_0 \in \mathbf{Z}$ and $\alpha > 0$, there exists a $\beta = \beta(n_0, \alpha) > 0$ such that if $\xi \in S_\alpha$, then $|x(n, n_0, \xi)| < \beta$ for $n \in \mathbf{Z} \cap [n_0, \infty)$.*

Definition 2.2. *The solutions of* Eq.(1) *are uniformly bounded, if the β in* Definition 1 *is independent of n_0.*

Definition 2.3. *The solutions of* Eq.(1) *are equiltimately bounded for bound X, if there exists an $X > 0$ and if corresponding to any $n_0 \in \mathbf{Z}$ and $\alpha > 0$, there exists a $\nu = \nu(n_0, \alpha) \in \mathbf{N}$ such that $\xi \in S_\alpha$ implies that $|x(n, n_0, \xi)| < X$ for $n \in \mathbf{Z} \cap [n_0 + \nu, \infty)$, where $\mathbf{N}$ denotes the set of positive integers.*

Definition 2.4. *The solutions of* Eq.(1) *are uniformly ultimately bounded for bound X, if the ν in* Definition 3 *is independent of n_0.*

By using Theorem 1, we can prove the following theorem.

Theorem 2. *If $f(n,x)$ in* Eq.(1) *is N-periodic in n for an $N \in \mathbf{N}$, and if the solutions of* Eq.(1) *are equiultimately bounded for bound $X > 0$, then* Eq.(1) *has an N-periodic solution $x(n)$ such that $|x(n)| < X$.*

Proof. Since equiultimate boundedness of the solutions of Eq.(1) implies equi-boundedness of the solutions of Eq.(1), and since Eq.(1) is N-periodic, the solutions of Eq.(1) are uniformly bounded. Therefore, there exists a $\beta = \beta(X) > 0$ such that if $n_0 \in \mathbf{Z}$ and $\xi \in S_X$, then $|x(n, n_0, \xi)| < \beta$ for $n \in \mathbf{Z} \cap [n_0, \infty)$. Moreover, there are numbers $\gamma > \beta$ and $\gamma^* > \gamma$ such that if $n_0 \in \mathbf{Z}$ and $\xi \in S_\beta$, then

$$|x(n, n_0, \xi)| < \gamma \quad \text{for} \quad n \in \mathbf{Z} \cap [n_0, \infty)$$

and that $n_0 \in \mathbf{Z}$ and $\xi \in S_\gamma$ imply

$$|x(n, n_0, \xi)| < \gamma^* \quad \text{for} \quad n \in \mathbf{Z} \cap [n_0, \infty).$$

From equiultimate boundedness for bound X of the solutions of Eq.(1), it follows that there exists a $\nu \in \mathbf{N}$ such that if $n \in \mathbf{Z} \cap [\nu, \infty)$ and $\xi \in S_\beta$, then $|x(n, 0, \xi)| < X$, and hence, there exists an $m \in \mathbf{N}$ for which

$$|x(mN, 0, \xi)| < X \quad \text{if} \quad \xi \in S_\beta.$$

Let $\mathbf{R}^d$ and S_X be A and S^0 in Theorem 1 respectively, and let S and S^1 be the sets defined by

$$S := \{x \in \mathbf{R}^d : |x| < \gamma\}$$

and

$$S^1 := \{x \in \mathbf{R}^d : |x| < \beta\}.$$

In addition, let $F : S \to \mathbf{R}^d$ be a mapping defined by

$$F(\xi) := x(N, 0, \xi), \quad \xi \in S.$$

Then, the mapping F is continuous, and $F(S)$ is contained in a compact set S_{γ^*}, which shows that F is a compact mapping. Moreover, the convex sets S, S^0 and S^1 satisfy the assumptions in Theorem 1. Therefore, there exists a fixed point $\xi \in S_X$, which implies the existence of an N-periodic solution $x(n) = x(n, 0, \xi)$ of Eq.(1) with $|x(n)| < X$ for $n \in \mathbf{Z} \cap [0, \infty)$. Q.E.D.

§3. Difference equations with finite delay

For a fixed $\kappa \in \mathbf{N}$, let B be the set of sequences $\phi : \mathbf{Z} \cap [-\kappa, 0] \to \mathbf{R}^d$. For any $\phi \in B$, define $\|\phi\|$ by

$$\|\phi\| := \sup\{|\phi(k)| : k \in \mathbf{Z} \cap [-\kappa, 0]\}.$$

For any $\alpha > 0$, the set B_α defined by

$$B_\alpha := \{\phi \in B : \|\phi\| \leq \alpha\}$$

is compact. For any sequence $x(k) : \mathbf{Z} \to \mathbf{R}^d$ and any fixed $n \in \mathbf{Z}$, the symbol x_n denotes the restriction of $x(k)$ on $\mathbf{Z} \cap [n - \kappa, n]$, that is, x_n is an element of B defined by

$$x_n(k) := x(n + k), \quad k \in \mathbf{Z} \cap [-\kappa, 0].$$

Consider the difference equation with finite delay

$$x(n+1) = f(n, x_n), \quad n \in \mathbf{Z}, \tag{2}$$

where $f : \mathbf{Z} \times B \to \mathbf{R}^d$ is continuous in ϕ for each fixed $n \in \mathbf{Z}$. For any $n_0 \in \mathbf{Z}$ and any initial sequence $\phi \in B$, there is a unique solution of Eq.(2), denoted by $x(n, n_0, \phi)$, such that it satisfies Eq.(2) for $n \in \mathbf{Z} \cap [n_0, \infty)$ and

$$x(n_0 + k, n_0, \phi) = \phi(k) \quad \text{for} \quad k \in \mathbf{Z} \cap [-\kappa, 0].$$

The definitions of boundedness of solutions of Eq.(2) are similarly obtained by replacing ξ and S_α in Definitions 1-4 by ϕ and B_α, respectively.

By using Theorem 1, we can prove the following theorem, which is similar to Theorem 2.

Theorem 3. *If $f(n, \phi)$ in* Eq.(2) *is N-periodic in n for an $N \in \mathbf{N}$, and if the solutions of* Eq.(2) *are equiultimately bounded for bound $X > 0$, then* Eq.(2) *has an N-periodic solution $x(n)$ such that $|x(n)| < X$.*

This theorem can be easily proved by replacing ξ, $\mathbf{R}^d$, S_X, S_β, S_γ and $\{x \in \mathbf{R}^d : |x| < \gamma\}$ in the proof of Theorem 2 by ϕ, B, B_X, B_β, B_γ and $S := \{\phi \in B : \|\phi\| < \gamma\}$, respectively, and considering a mapping $F : S \to B$ defined by

$$F(\phi) := x_N(0, \phi), \quad \phi \in S.$$

So, we omit the proof.

§4. Boundedness in difference equations with finite delay

In Section 3, equiultimate boundedness of solutions of Eq.(2) is an important assumption. There are many results concerning boundedness of the solutions of difference equations (see for example [6, 8] and their references). Here we state a boundedness theorem due to Shunian Zhang without a proof. By using this theorem, we show an example later.

Theorem 4. *(Zhang) Suppose that there exists a Liapunov function* $V : \mathbf{Z} \times \mathbf{R}^d \to \mathbf{R}^+ := [0, \infty)$, *which satifies the following conditions;*

(i) $a(|x|) \le V(n,x) \le b(|x|)$, *where* $a,\ b : \mathbf{R}^+ \to \mathbf{R}^+$, $a(r)$ *and* $b(r)$ *are continuous, increasing and* $a(r) \to \infty$ *as* $r \to \infty$,

(ii) $\Delta V_{(2)}(n, x(n)) := V(n+1, x(n+1)) - V(n, x(n)) \le M - c(|x(n)|)$ *whenever*

$$P\big(V(n+1, x(n+1))\big) > V(k, x(k)) \text{ for } k \in \mathbf{Z} \cap [n - \kappa, n],$$

where $x(n)$ *is a solution of* Eq.(2), M *is a positive constant,* $c : \mathbf{R}^+ \to \mathbf{R}^+$ *is continuous, increasing and* $c(r) \to \infty$ *as* $r \to \infty$, *and* $P : \mathbf{R}^+ \to \mathbf{R}^+$ *is continuous,* $P(u) > u$ *for* $u > 0$, *and* $\kappa \in \mathbf{N}$.

Then the solutions of Eq.(2) *are uniformly ultimately bounded for a bound* $X > 0$.

By using Theorem 4, we construct an example for Theorem 3.

Example. Consider the scalar N-periodic difference equation

$$x(n+1) = \sum_{k=0}^{\kappa} f_k(n, x(n-k)) + G \cos \frac{2\pi}{N} n, \quad n \in \mathbf{Z}, \tag{3}$$

where G is a nonzero constant, $\kappa,\ N \in \mathbf{N}$ with $N > 1$, $f_k(n,x) : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ is continuous in x for each fixed n, $f_k(n,x)$ is N-periodic in n,

$$|f_k(n,x)| \le c_k |x|$$

and

$$\sum_{k=0}^{\kappa} c_k < 1. \tag{4}$$

From (4), we can choose a constant $\rho > 1$ with

$$c_0 + \rho \sum_{k=1}^{\kappa} c_k < 1.$$

Let $P(u) = \rho r$ and $V(n, x) = |x|$, and let $x(n) = x(n, n_0, \phi)$ be any solution of Eq.(3) with $n_0 \in \mathbf{Z}$ and $\phi \in B$. If we have

$$P\big(V(n+1, x(n+1))\big) = \rho|x(n+1)|$$
$$> |x(n-k)| = V(n-k, x(n-k)), \ \ k \in \mathbf{Z} \cap [0, \kappa],$$

then we obtain

$$|x(n+1)| \le |f_0(n, x(n))| + \sum_{k=1}^{\kappa} |f_k(n, x(n-k))| + |G \cos \frac{2\pi}{N} n|$$

$$\le c_0|x(n)| + \sum_{k=1}^{\kappa} c_k|x(n-k)| + |G| \le c_0|x(n)| + \rho \sum_{k=1}^{\kappa} c_k|x(n+1)| + |G|,$$

which implies

$$|x(n+1)| \le \frac{c_0|x(n)|}{1 - \rho \sum_{k=1}^{\kappa} c_k} + \frac{|G|}{1 - \rho \sum_{k=1}^{\kappa} c_k}.$$

Thus, for the number M defined by

$$M := \frac{|G|}{1 - \rho \sum_{k=1}^{\kappa} c_k},$$

we have

$$\Delta V_{(3)}(n, x(n)) = |x(n+1)| - |x(n)| \le M - \frac{1 - c_0 - \rho \sum_{k=1}^{\kappa} c_k}{1 - \rho \sum_{k=1}^{\kappa} c_k} |x(n)|,$$

which shows that all assumptions of Theorem 4 are satisfied with $a(r) = b(r) = r$ and

$$c(r) = \frac{1 - c_0 - \rho \sum_{k=1}^{\kappa} c_k}{1 - \rho \sum_{k=1}^{\kappa} c_k} r.$$

Thus, by Theorems 3 and 4, Eq.(3) has an N-periodic solution.

References

[1] F. E. Browder, On a generalization of the Schauder fixed point theorem, Duke Math. J., **26** (1959), 291–303.

[2] F. Dannan, S. Elaydi and P. Liu, Periodic solutions of difference equations, J. Differ. Equations Appl., **6** (2000), 203–232.

[3] S. Elaydi, An Introduction to Difference Equations 3rd ed., Springer, 2005.

[4] S. Elaydi, Periodicity and stability of linear Volterra difference systems, J. Math. Anal. Appl., **181** (1994), 483–492.

[5] S. Elaydi and S. Zhang, Stability and periodicity of difference equations with finite delay, Funkcial. Ekvac., **37** (1994), 401–413.
[6] T. Furumochi, K. Hishitani and K. Tsuchihashi, Liapunov functions and boundedness in difference equations, Dynamics of Continuous, Discrete and Impulsive Systems, to appear.
[7] W. G. Kelley and A. C. Peterson, Difference Equations, Academic Press, 2001.
[8] S. Zhang, Boundedness of finite delay difference systems, Ann. Differential Equations, **9** (1993), 107–115.

Tetsuo Furumochi
Department of Mathematics
Shimane University
1060 Nishikawatsucho
Matsue 690-8504
Japan

Masato Muraoka
Department of Mathematics
Shimane University
1060 Nishikawatsucho
Matsue 690-8504
Japan

E-mail address: `furumochi@riko.shimane-u.ac.jp`
`muuuuuu@ezweb.ne.jp`