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On the simplicity of the group of contactomorphisms

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Abstract.

We consider the group $\mathrm{Cont}^r_c(M^{2n+1},\alpha)$ of C^r contactomorphisms with compact support of a contact manifold (M^{2n+1},α) of dimension $(2n+1)$ with the C^r topology. We show that the first homology group of the classifying space $B\overline{\mathrm{Cont}}^r_c(M^{2n+1},\alpha)$ for the C^r foliated M^{2n+1} products with compact support with transverse contact structure α is trivial for $1 \leq r < n+(3/2)$. This implies that the identity component $\mathrm{Cont}^r_c(M^{2n+1},\alpha)_0$ of the group $\mathrm{Cont}^r_c(M^{2n+1},\alpha)$ of contactomorphisms with compact support of a connected contact manifold (M^{2n+1},α) is a simple group for $1 \leq r < n+(3/2)$.

§1. Introduction

The groups of diffeomorphisms play an important role in the theory of foliations. This relationship is clear in the theory developed by Mather and Thurston, which asserts the relationship between the topology of the classifying space for C^r foliations of codimension n and that of the group $\mathrm{Diff}^r_c(\boldsymbol{R}^n)$ of C^r diffeomorphisms of $\boldsymbol{R}^n$ with compact support: $H_*(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n);\boldsymbol{Z}) \cong H_*(\Omega^n B\overline{\Gamma}^r_n;\boldsymbol{Z})$. Here, $B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n)$ is the classifying space for the C^r foliated $\boldsymbol{R}^n$ products with compact support, $B\overline{\Gamma}^r_n$ is the classifying space for the Γ^r_n structures (C^r foliations of codimension n) with trivialized normal bundles, and Ω^n means the n-fold loop space.

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The homology of $H_*(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})$ is known for several cases and this gives the information on the connectivity of $B\overline{\Gamma}_n^r$: $B\overline{\Gamma}_n^r$ is n-connected; $B\overline{\Gamma}_n^\infty$ is $(n+1)$-connected (Mather [9], Thurston [12]); $B\overline{\Gamma}_n^r$ is $(n+1)$-connected $(r \neq n+1)$ (Mather [10]); $B\overline{\Gamma}_n^0$ and $B\overline{\Gamma}_n^L$ (L stands for Lipschitz) are contractible (Mather [8]); $B\overline{\Gamma}_n^1$ is contractible (Tsuboi [17]); The connectivity of $B\overline{\Gamma}_n^r$ increases as $r \searrow 1$ (Tsuboi [15]); $\pi_{2n+1}(B\overline{\Gamma}_n^r) \geq \boldsymbol{R}$ $(r > 2 - 1/(n+1))$ (by using the characteristic classes for foliations). This connectivity information is closely related to the construction of foliations (Haefliger [3], [4], Thurston [13], [14]).

It is also known that if $H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$, that is, if $r = 0$, $r = L$, $1 \leq r < n+1$, or $n+1 < r \leq \infty$, the identity component $\mathrm{Diff}_c^r(M)_0$ of the group $\mathrm{Diff}_c^r(M)$ of C^r diffeomorphisms with compact support of a connected n-dimensional manifold M is a simple group (Thurston [12], see also Banyaga [2]).

Now we are interested in the group of volume preserving diffeomorphisms, the group of symplectomorphisms of a symplectic manifold and the group of contactomorphisms of a contact manifold. The homology of the group of volume preserving diffeomorphisms is studied by McDuff ([7]). The homology of the group of symplectomorphisms is studied by Banyaga ([1]). The book [2] by Banyaga is a good reference for these groups.

In this paper, we apply the techniques of Mather [10] and Tsuboi [15] to the group of contactomorphisms to show that the first homology group of the classifying space $B\overline{\mathrm{Cont}}_c^r(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ for the C^r foliated $\boldsymbol{R}^{2n+1}$ products with compact support with transverse contact structure α_{st} is trivial for $1 \leq r < n + (3/2)$ (Theorem 5.1 for the standard contact manifold $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$). Then by the fragmentation technique (Thurston [12], Banyaga [2]), $H_1(B\overline{\mathrm{Cont}}_c^r(M^{2n+1}, \alpha); \boldsymbol{Z})$ is trivial for any contact manifold (M^{2n+1}, α) (Theorem 5.1), and the identity component $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)_0$ of the group $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$ of C^r contactomorphisms with compact support of a connected contact manifold (M^{2n+1}, α) is a simple group for $1 \leq r < n + (3/2)$ (Theorem 5.2).

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§2. Contactomorphisms of class C^r

First we need to define what is a C^r contactomorphism for $1 \leq r < \infty$.

Let (M^{2n+1}, α) be a contact manifold of dimension $(2n+1)$, where α is a C^∞ 1-form on M^{2n+1} such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form. A (positive) C^r contactomorphism φ of (M^{2n+1}, α) is a C^r diffeomorphism of M^{2n+1} such that $\varphi^*\alpha = w_\varphi \alpha$ where w_φ is a positive C^r function on M^{2n+1} depending on φ. In other words, a contactomorphism φ is a diffeomorphism whose tangent map $T\varphi$ preserves the C^∞ contact hyperplane field $\xi = \ker\alpha$ with transverse orientation.

Note that the action of the C^r diffeomorphism on the tangent bundle TM^{2n+1} or on the cotangent bundle T^*M^{2n+1} is usually of class C^{r-1}, however, it is well defined that the function w_φ is of class C^r and such diffeomorphisms form a group. For, the equations ${\varphi_j}^*\alpha = w_{\varphi_j}\alpha$ $(j = 1, 2)$ imply

$$(\varphi_1\varphi_2)^*\alpha = {\varphi_2}^*{\varphi_1}^*\alpha = {\varphi_2}^*(w_{\varphi_1}\alpha) = ({\varphi_2}^*w_{\varphi_1}){\varphi_2}^*\alpha = ({\varphi_2}^*w_{\varphi_1})w_{\varphi_2}\alpha.$$

Hence $w_{\varphi_1\varphi_2} = ({\varphi_2}^*w_{\varphi_1})w_{\varphi_2}$, and if w_{φ_1} and w_{φ_2} are of class C^r, then so is $w_{\varphi_1\varphi_2}$. We see also that $w_{\varphi^{-1}} = \varphi^{-1*}w_\varphi$. Thus C^r contactomorphisms of (M^{2n+1}, α) form a group.

Note also that C^r contactomorphisms of (M^{2n+1}, α) are determined by the C^∞ contact hyperplane field ξ. For, if $\xi = \ker\alpha$ is defined by another C^∞ 1-form $\beta = k\alpha$ $(k > 0)$, then $\varphi(k\alpha) = (\varphi^*k)w_\varphi\alpha = \frac{\varphi^*k}{k}w_\varphi(k\alpha)$, hence if w_φ is of class C^r, then so is $\frac{\varphi^*k}{k}w_\varphi$.

§3. Local contractibility and fragmentation property

Let $\mathrm{Cont}^r_c(M^{2n+1}, \alpha)$ or $\mathrm{Cont}^r_c(M^{2n+1}, \xi)$ denote the group of contactomorphisms of (M^{2n+1}, α) or of (M^{2n+1}, ξ) with compact support.

An element φ of this group, C^1 close to the identity, corresponds to a C^{r+1} function on $M = M^{2n+1}$. Consider the contact form $w\alpha_1 - \alpha_2$ on $M \times M \times \boldsymbol{R}_{>0}$, where w is the coordinate of $\boldsymbol{R}_{>0}$, $\alpha_i = p_i^*\alpha$ and $p_i : M \times M \times \boldsymbol{R}_{>0} \longrightarrow M$ is the projection to the i-th component $(i = 1,\ 2)$. A C^r diffeomorphism φ of M belongs to $\mathrm{Cont}^r_c(M, \xi)$ (i.e., $\varphi^*\alpha = w_\varphi\alpha$ with w_φ being a positive C^r function) if and only if the graph of $(\varphi, w_\varphi) : M \longrightarrow M \times \boldsymbol{R}_{>0}$;

$$\{(u, \varphi(u), w_\varphi(u)) \mid u \in M\}$$

is a Legendrian submanifold of $(M \times M \times \boldsymbol{R}_{>0}, w\alpha_1 - \alpha_2)$ of class C^r (see Lychagin [6], Banyaga [2]). There is a C^∞ contactomorphism from a neighborhood of the graph of $(\mathrm{id}, 1)$ to the space $J^1(M, \boldsymbol{R})$ of 1-jets of functions on M. A C^r Legendrian submanifold of $J^1(M, \boldsymbol{R})$, C^1 close to 0 is the prolongation of a C^{r+1} function f on M, C^2 close to 0. In this way, a neighborhood of the identity in $\mathrm{Cont}^r_c(M^{2n+1}, \alpha)$ is diffeomorphic to a neighborhood of 0 in the space $C^{r+1}_c(M^{2n+1})$ of C^{r+1} functions on M^{2n+1} with compact support. This shows the local contractibility of $\mathrm{Cont}^r_c(M^{2n+1}, \alpha)$.

Note here that the identity of M^{2n+1} corresponds to the zero function, and we have a canonical C^∞ path to the identity for an element of $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$, C^1 close to the identity. In fact, for an element φ C^1 close to the identity, we obtain the C^{r+1} function f_φ. Then tf_φ is the canonical C^∞ path in the space $C_c^{r+1}(M^{2n+1})$ of C^{r+1} functions and this corresponds to a C^∞ path in the space of graphs, that is, to a C^∞ path in $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$ to the identity.

We also see the fragmantation property of $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$. If we have a contactomorphism φ with compact support C^1 close to the identity, we choose finitely many Darboux coordinate neighborhoods $\{U_i\}$ which cover the support of φ. We choose a covering by smaller neighborhoods $W_i \subset \overline{W}_i \subset V_i \subset \overline{V}_i \subset U_i$, and using the C^r function f_φ which is associated to φ and a bump function with support in U_i, we obtain a C^r function with support in U_i which coincides with f_φ on V_i. Thus we have a contactomorphism with support in U_i which coincides with φ on W_i. In this way, φ can be fragmented. See Banyaga [2].

§4. Contactmorphisms for the standard structure on $\boldsymbol{R}^{2n+1}$

For the standard contact structure $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$, we give an explicit correspondence between a neighborhood of the identity in $\mathrm{Cont}_c^r(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ and a neighborhood of 0 in $C_c^{r+1}(\boldsymbol{R}^{2n+1})$.

Let $\alpha_{\mathrm{st}} = \mathrm{d}z - \sum_{i=1}^{n} y_i \mathrm{d}x_i$ be the standard contact form on $\boldsymbol{R}^{2n+1}$. Let $\mathrm{Cont}_c^r(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ denote the group of C^r contactomorphisms of $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ with compact support.

When $\varphi \in \mathrm{Cont}_c^r(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ is C^1 close to the identity, we write the graph

$$\{(u, \varphi(u), w_\varphi(u)) \mid u \in \boldsymbol{R}^{2n+1}\} \subset \boldsymbol{R}^{2n+1} \times \boldsymbol{R}^{2n+1} \times \boldsymbol{R}_{>0}$$

in the form of 1-jets of a function of $2n+1$ variables as follows: For

$$u = (x^{(1)}, y^{(1)}, z^{(1)}) = (x_1^{(1)}, \dots, x_n^{(1)}, y_1^{(1)}, \dots, y_n^{(1)}, z^{(1)}),$$

we write $\varphi(u) = (x_1^{(2)}(u), \dots, x_n^{(2)}(u), y_1^{(2)}(u), \dots, y_n^{(2)}(u), z^{(2)}(u))$, where

$$\begin{aligned}
x_i^{(2)} &= x_i^{(2)}(x_1^{(1)}, \dots, x_n^{(1)}, y_1^{(1)}, \dots, y_n^{(1)}, z^{(1)}) \quad (i = 1, \dots, n), \\
y_i^{(2)} &= y_i^{(2)}(x_1^{(1)}, \dots, x_n^{(1)}, y_1^{(1)}, \dots, y_n^{(1)}, z^{(1)}) \quad (i = 1, \dots, n), \\
z^{(2)} &= z^{(2)}(x_1^{(1)}, \dots, x_n^{(1)}, y_1^{(1)}, \dots, y_n^{(1)}, z^{(1)}) .
\end{aligned}$$

Then the graph $\{(u, \varphi(u), w_\varphi(u)) \mid u \in \boldsymbol{R}^{2n+1}\}$ satisfies

$$\mathrm{d}z^{(2)} - \sum_{i=1}^{n} y_i^{(2)} \mathrm{d}x_i^{(2)} = w_\varphi(u)\{\mathrm{d}z^{(1)} - \sum_{i=1}^{n} y_i^{(1)} \mathrm{d}x_i^{(1)}\}.$$

That is,

$$\begin{aligned}\mathrm{d}\big(z^{(2)}(u) - z^{(1)} - \sum_{i=1}^{n} y_i^{(2)}(u)(x_i^{(2)}(u) - x_i^{(1)})\big) - (w_\varphi(u) - 1)\mathrm{d}z^{(1)} \\ + \sum_{i=1}^{n} \big(w_\varphi(u) y_i^{(1)} - y_i^{(2)}(u)\big)\mathrm{d}x_i^{(1)} + \sum_{i=1}^{n} (x_i^{(2)}(u) - x_i^{(1)})\mathrm{d}y_i^{(2)} = 0.\end{aligned}$$

Since the graph is close to $\{(u, u, 1) \mid u \in \boldsymbol{R}^{2n+1}\}$, the graph can be written as a graph with respect to the variables

$$(x^{(1)}, y^{(2)}, z^{(1)}) = (x_1^{(1)}, \ldots, x_n^{(1)}, y_1^{(2)}, \ldots, y_n^{(2)}, z^{(1)})$$

and the graph is written as

$$\begin{aligned}\{(x^{(1)}, y^{(1)}(x^{(1)}, y^{(2)}, z^{(1)}), z^{(1)}, \\ y^{(2)}, x^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}), z^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}), w(x^{(1)}, y^{(2)}, z^{(1)}))\},\end{aligned}$$

where $y^{(1)}(x^{(1)}, y^{(2)}, z^{(1)})$, $x^{(2)}(x^{(1)}, y^{(2)}, z^{(1)})$ and $z^{(2)}(x^{(1)}, y^{(2)}, z^{(1)})$ are C^r functions by the inverse function theorem, and

$$w(x^{(1)}, y^{(2)}, z^{(1)}) = w_\varphi(x^{(1)}, y^{(2)}(x^{(1)}, y^{(1)}, z^{(1)}), z^{(1)}).$$

If φ is a C^r contactomorphism with compact support, then the C^r function

$$\begin{aligned}f(x^{(1)}, y^{(2)}, z^{(1)}) = z^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}) - z^{(1)} \\ - \sum_{i=1}^{n} y_i^{(2)} \left(x_i^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}) - x_i^{(1)}\right)\end{aligned}$$

has the derivatives

$$\begin{aligned}\frac{\partial f}{\partial x_i^{(1)}} &= -w(x^{(1)}, y^{(2)}, z^{(1)})\; y_i^{(1)}(x^{(1)}, y^{(2)}, z^{(1)}) + y_i^{(2)} \quad (i = 1, \ldots, n), \\ \frac{\partial f}{\partial y_i^{(2)}} &= x_i^{(1)} - x_i^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}) \quad (i = 1, \ldots, n), \\ \frac{\partial f}{\partial z^{(1)}} &= w(x^{(1)}, y^{(2)}, z^{(1)}) - 1 \quad ,\end{aligned}$$

which are C^r functions, hence $f(x^{(1)}, y^{(2)}, z^{(1)})$ is a C^{r+1} function with compact support. Note again that the identity of $\boldsymbol{R}^{2n+1}$ corresponds to the zero function.

Conversely for a C^{r+1} function $f(x^{(1)}, y^{(2)}, z^{(1)})$ with compact support, C^2 close to the zero, by putting

$$\begin{aligned}
w(x^{(1)}, y^{(2)}, z^{(1)}) &= \frac{\partial f}{\partial z^{(1)}} + 1 && , \\
x_i^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}) &= -\frac{\partial f}{\partial y_i^{(2)}} + x_i^{(1)} && (i = 1, \dots, n), \\
y_i^{(1)}(x^{(1)}, y^{(2)}, z^{(1)}) &= \frac{-\dfrac{\partial f}{\partial x_i^{(1)}} + y_i^{(2)}}{\dfrac{\partial f}{\partial z^{(1)}} + 1} && (i = 1, \dots, n), \\
z^{(2)}(x^{(1)}, y^{(2)}, z^{(1)}) &= f + z^{(1)} - \sum_{i=1}^{n} y_i^{(2)} \frac{\partial f}{\partial y_i^{(2)}} && ,
\end{aligned}$$

the graph is a Legendrian submanifold of $(\boldsymbol{R}^{2n+1} \times \boldsymbol{R}^{2n+1} \times \boldsymbol{R}_{>0}, w\alpha_1 - \alpha_2)$, C^1 close to the identity. Hence it is a graph of a C^r contactomorphism φ with compact support.

§5. Statement of result

Let $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$ be the group of C^r contactomorphisms with compact support of the contact manifold (M^{2n+1}, α) with the C^r topology, and $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)^\delta$, the same group with the discrete topology. Let $B\overline{\mathrm{Cont}}_c^r(M^{2n+1}, \alpha)$ denote the homotopy fiber of the map between their classifying spaces: $B\mathrm{Cont}_c^r(M^{2n+1}, \alpha)^\delta \longrightarrow B\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$. $B\overline{\mathrm{Cont}}_c^r(M^{2n+1}, \alpha)$ is the classifying space for the C^r foliated M^{2n+1} products with compact support with transverse contact structure defined by α. Here is our main result.

Theorem 5.1. *For* $1 \leq r < n+(3/2)$, $H_1(B\overline{\mathrm{Cont}}_c^r(M^{2n+1}, \alpha); \boldsymbol{Z}) = 0$.

For $G = \mathrm{Cont}_c^r(M^{2n+1}, \alpha)$, $B\overline{G}$ is constructed as the realization of the semi-simplicial set $S_*^\infty(G)/G$. Here $S_i^\infty(G)$ is the set of smooth singular simplices of G. C^r diffeomorphism groups have a smooth structure such that the composition

$$(g_1, g_2) \longmapsto g_1 g_2 : G \times G \longrightarrow G$$

is smooth with respect to g_1 (but it is not smooth with respect to g_2 if r is finite). An element $\sigma : \Delta^i \longrightarrow G$ $(\in S_i^\infty(G))$ defines a foliated M^{2n+1}

product over Δ^i such that the leaf passing through $(t,x) \in \Delta^i \times M^{2n+1}$ is given by

$$\{(s, \sigma(s)\sigma(t)^{-1}(x)) \in \Delta^i \times M^{2n+1} \mid s \in \Delta^i\}.$$

A C^r foliated product has a natural C^r (semi-)norm ([15]).

We note here that for a topological group G, the classifying space $B\overline{G}$ is constructed as the realization of $S_*(G)/G$, where $S_*(G)$ is the set of singular simplices of G. When G has a smooth structure, the inclusion $|S_*^\infty(G)/G| \subset |S_*(G)/G|$ is usually a homotopy equivalence. The homotopy equivalence $|S_*^\infty(G)/G| \subset |S_*(G)/G|$ is shown by approximating a singular simplex by a smooth singular simplex and by constructing a canonical homotopy between them. The construction of the canonical homotopy uses the fact explained in Section 3 that there is a canonical path to the identity for an element of $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$, C^1 close the identity.

Let $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)_0$ denote the connected component of the identity of $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)$. Theorem 5.1 implies the simplicity of this group (Thurston [12], Banyaga [2]).

Theorem 5.2. *Let (M^{2n+1}, α) be a connected contact manifold. For $1 \le r < n + (3/2)$, $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)_0$ is a simple group.*

Proof. By the argument in Thurston [12] or Banyaga [2], $H_1(B\overline{\mathrm{Cont}_c^r}(M^{2n+1}, \alpha); \boldsymbol{Z}) = 0$ implies that the universal covering group $\widetilde{\mathrm{Cont}_c^r}(M^{2n+1}, \alpha)_0$ is a perfect group and then $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)_0$ is a perfect group. As we explained in Section 3, $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)_0$ has the fragmentation property. Again by the argument in Thurston [12] or Banyaga [2] using the fact that $\mathrm{Cont}_c^r(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})_0$ is a perfect group, $\mathrm{Cont}_c^r(M^{2n+1}, \alpha)_0$ is a simple group. Q.E.D.

To prove Theorem 5.1, by the fragmentation technique (Thurston [12], Banyaga [2]), it suffices to prove Theorem 5.1 for the standard contact manifold $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$. In other words, by taking a Darboux coordinate neighborhood $U \cong \boldsymbol{R}^{2n+1}$, by the fragmentation technique of Section 3, and the fact that for any Darboux coordinate neighborhood U_i in M, there is a contact isotopy sending U_i into U, the induced map $H_1(B\overline{\mathrm{Cont}_c^r}(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}}); \boldsymbol{Z}) \longrightarrow H_1(B\overline{\mathrm{Cont}_c^r}(M^{2n+1}, \alpha); \boldsymbol{Z})$ is surjective. Hence $H_1(B\overline{\mathrm{Cont}_c^r}(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}}); \boldsymbol{Z}) = 0$ implies $H_1(B\overline{\mathrm{Cont}_c^r}(M^{2n+1}, \alpha); \boldsymbol{Z}) = 0$.

In order to show Theorem 5.1 for $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$, we need to construct several elements of $\mathrm{Cont}_c^\infty(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ which coincide with contractions or translations on $[-1,1]^{2n+1}$. We do this in the next section. In Section

7, we give the main construction and prove Theorem 5.1 for $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ in a way similar to that in Tsuboi [15].

§6. Lie algebra for the group of smooth contactomorphisms

For $r = \infty$, the Lie algebra of $\mathrm{Cont}_c^\infty(M^{2n+1}, \alpha)$ or $\mathrm{Cont}_c^\infty(M^{2n+1}, \xi)$ is described as follows: Let $\mathcal{X}^\infty_{\mathrm{cont}}(M^{2n+1}, \alpha)$ be the space of C^∞ vector fields X such that $L_X\alpha = k_X\alpha$ for a C^∞ function k_X depending on X.

If $L_{X_j}\alpha = k_{X_j}\alpha$ ($j = 1, 2$), then

$$\begin{aligned} L_{[X_1,X_2]}\alpha &= (L_{X_1}L_{X_2} - L_{X_2}L_{X_1})\alpha = L_{X_1}(k_{X_2}\alpha) - L_{X_2}(k_{X_1}\alpha) \\ &= X_1(k_{X_2})\alpha + k_{X_2}k_{X_1}\alpha - X_2(k_{X_1})\alpha - k_{X_1}k_{X_2}\alpha \\ &= (X_1(k_{X_2}) - X_2(k_{X_1}))\alpha \end{aligned}$$

Hence $k_{[X_1,X_2]} = X_1(k_{X_2}) - X_2(k_{X_1})$ and such vector fields form a Lie algebra. For the C^r case, however, the C^r vector fields X such that $L_X\alpha = k_X\alpha$ for a C^r function k would not form a Lie algebra, because $[X_1, X_2]$ and $k_{[X_1,X_2]}$ are usually of class C^{r-1}. This reflects the fact that the composition $(\varphi_1, \varphi_2) \longmapsto \varphi_1\varphi_2$ is smooth with respect to φ_1 but not smooth with respect to φ_2 in the group of C^r diffeomorphisms.

We need later several contactomorphisms of $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ for our construction. We write them as the time 1-maps of C^∞ contact vector fields.

For the standard contact form $\alpha_{\mathrm{st}} = \mathrm{d}z - \sum_{i=1}^n y_i\mathrm{d}x_i$, an element $X = \zeta\frac{\partial}{\partial z} + \sum_{i=1}^n \xi_i\frac{\partial}{\partial x_i} + \sum_{i=1}^n \eta_i\frac{\partial}{\partial y_i}$ of $\mathcal{X}^\infty_{\mathrm{cont}}(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$ is written by the function $a = \iota(X)\alpha$ on $\boldsymbol{R}^{2n+1}$ as follows:

$$X = (a - \sum_{j=1}^n y_j\frac{\partial a}{\partial y_j})\frac{\partial}{\partial z} - \sum_{i=1}^n \frac{\partial a}{\partial y_i}\frac{\partial}{\partial x_i} + \sum_{i=1}^n(\frac{\partial a}{\partial x_i} + y_i\frac{\partial a}{\partial z})\frac{\partial}{\partial y_i}.$$

For this contact vector field X, $k_X = \dfrac{\partial a}{\partial z}$.

The reason is as follows: Since

$$\begin{aligned} L_X\alpha = \iota(X)\mathrm{d}\alpha + \mathrm{d}\iota(X)\alpha &= \sum_{i=1}^n(\xi_i\mathrm{d}y_i - \eta_i\mathrm{d}x_i) + \mathrm{d}(\zeta - \sum_{i=1}^n y_i\xi_i) \\ &= -\sum_{i=1}^n \eta_i\mathrm{d}x_i + \sum_{i=1}^n \frac{\partial\zeta}{\partial x_i}\mathrm{d}x_i + \sum_{i=1}^n \frac{\partial\zeta}{\partial y_i}\mathrm{d}y_i + \frac{\partial\zeta}{\partial z}\mathrm{d}z \\ &\quad - \sum_{i,j=1}^n y_i\frac{\partial\xi_i}{\partial x_j}\mathrm{d}x_j - \sum_{i,j=1}^n y_i\frac{\partial\xi_i}{\partial y_j}\mathrm{d}y_j - \sum_{i=1}^n y_i\frac{\partial\xi_i}{\partial z}\mathrm{d}z, \end{aligned}$$

it satisfies

$$\begin{cases} \dfrac{\partial \zeta}{\partial x_i} - \eta_i - \displaystyle\sum_{j=1}^{n} y_j \frac{\partial \xi_j}{\partial x_i} = -y_i k_X \\ \dfrac{\partial \zeta}{\partial y_i} - \displaystyle\sum_{j=1}^{n} y_j \frac{\partial \xi_j}{\partial y_i} = 0 \\ \dfrac{\partial \zeta}{\partial z} - \displaystyle\sum_{j=1}^{n} y_j \frac{\partial \xi_j}{\partial z} = k_X \end{cases}$$

By differentiating $\dfrac{\partial \zeta}{\partial y_i} = \displaystyle\sum_{j=1}^{n} y_j \frac{\partial \xi_j}{\partial y_i}$ by y_k, we obtain

$$\frac{\partial^2 \zeta}{\partial y_i \partial y_k} = \frac{\partial \xi_k}{\partial y_i} + \sum_{j=1}^{n} y_j \frac{\partial^2 \xi_j}{\partial y_i \partial y_k}$$

which is symmetric in i and k. Hence we see that $\dfrac{\partial \xi_k}{\partial y_i} = \dfrac{\partial \xi_i}{\partial y_k}$. This implies that there is a function $a(x, y, z)$ such that $\xi_i = -\dfrac{\partial a}{\partial y_i}$. With this a,

$$\begin{aligned} \zeta(x, y, z) &= \int_{-\infty}^{y_i} \frac{\partial \zeta}{\partial y_i} \mathrm{d}y_i = \int_{-\infty}^{y_i} \sum_{j=1}^{n} y_j \frac{\partial \xi_j}{\partial y_i} \mathrm{d}y_i \\ &= \left[\sum_{j=1}^{n} y_j \xi_j \right]_{y_i=-\infty}^{y_i} - \int_{-\infty}^{y_i} \xi_i \mathrm{d}y_i \\ &= \sum_{j=1}^{n} y_j \xi_j + \int_{-\infty}^{y_i} \frac{\partial a}{\partial y_i} \mathrm{d}y_i \\ &= \sum_{j=1}^{n} y_j \xi_j + a = a - \sum_{j=1}^{n} y_j \frac{\partial a}{\partial y_j} \end{aligned}$$

Then

$$\begin{aligned} k_X(x, y, z) &= \frac{\partial \zeta}{\partial z} - \sum_{j=1}^{n} y_j \frac{\partial \xi_j}{\partial z} \\ &= \frac{\partial a}{\partial z} - \sum_{j=1}^{n} y_j \frac{\partial^2 a}{\partial y_j \partial z} + \sum_{j=1}^{n} y_j \frac{\partial^2 a}{\partial y_j \partial z} = \frac{\partial a}{\partial z} \end{aligned}$$

and

$$\begin{aligned}\eta_i(x,y,z) &= \frac{\partial\zeta}{\partial x_i} - \sum y_j \frac{\partial \xi_j}{\partial x_i} + y_i k_X \\ &= \frac{\partial a}{\partial x_i} - \sum_{j=1}^{n} y_j \frac{\partial^2 a}{\partial y_j \partial x_i} + \sum_{j=1}^{n} y_j \frac{\partial^2 a}{\partial y_j \partial x_i} + y_i \frac{\partial a}{\partial z} \\ &= \frac{\partial a}{\partial x_i} + y_i \frac{\partial a}{\partial z}\end{aligned}$$

Here are the contact vector fields we are interested in.

For the function $a(x,y,z) = 1$, the vector field $X = \frac{\partial}{\partial z}$, and its time t map is θ_t, where $\theta_t(x^{(1)}, y^{(1)}, z^{(1)}) = (x^{(1)}, y^{(1)}, z^{(1)} + t)$.

For the function $a(x,y,z) = x_i$, the vector field $X = x_i \frac{\partial}{\partial z} + \frac{\partial}{\partial y_i}$, and its time t map is ψ_i^t, where $\psi_i^t(x^{(1)}, y^{(1)}, z^{(1)}) = (x^{(1)}, y^{(1)} + t1_i, z^{(1)} + tx_i)$.

For the function $a(x,y,z) = y_i$, the vector field $X = -\frac{\partial}{\partial x_i}$, and its time t map is φ_i^{-t}, where $\varphi_i^t(x^{(1)}, y^{(1)}, z^{(1)}) = (x^{(1)} + t1_i, y^{(1)}, z^{(1)})$.

For the function $a(x,y,z) = 2z - \sum_{i=1}^{n} x_i y_i$, the vector field $X = 2z\frac{\partial}{\partial z} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} y_i \frac{\partial}{\partial y_i}$, and its time t map is ε^t, where $\varepsilon^t(x^{(1)}, y^{(1)}, z^{(1)}) = (e^t x^{(1)}, e^t y^{(1)}, e^{2t} z^{(1)})$.

The supports of these vector fields are not compact. We take the product of the function a and a bump function and make the time t maps θ_t, ε_t, φ_i^t, ψ_i^t be with compact support and coincide with the original ones for $|t| \leq 1$ on a given compact subset of $\boldsymbol{R}^{2n+1}$.

§7. Main construction and the proof of Theorem 5.1

Using the above contractions and translations, we perform a construction similar to that in [15].

For a positive real number A, take the rectangle $R = [-1,1]^n \times [-A/n, A/n]^n \times [-1,1]$. Then

$$\begin{aligned}&\varepsilon^{-\log(2+A)}(R) \\ &= [-\frac{1}{2+A}, \frac{1}{2+A}]^n \times [-\frac{A/n}{2+A}, \frac{A/n}{2+A}]^n \times [-\frac{1}{(2+A)^2}, \frac{1}{(2+A)^2}].\end{aligned}$$

For $\gamma = (\gamma_1, \dots, \gamma_n)$, $(\gamma_i = \pm 1)$,

$$R_\gamma = \psi_1^{\gamma_1 \frac{(1+A)A/n}{2+A}} \dots \psi_n^{\gamma_n \frac{(1+A)A/n}{2+A}} (\varepsilon^{-\log(2+A)}(R))$$

is a parallelepiped contained in

$$[-\frac{1}{2+A}, \frac{1}{2+A}]^n \times J_{\gamma_1} \times \cdots \times J_{\gamma_n}$$
$$\times [- \frac{1}{(2+A)^2} - \Big|\sum_{i=1}^{n} \gamma_i\Big| \cdot \frac{(1+A)A/n}{2+A},$$
$$\frac{1}{(2+A)^2} + \Big|\sum_{i=1}^{n} \gamma_i\Big| \cdot \frac{(1+A)A/n}{2+A}],$$

which intersects a line parallel to z axis in a segment of length $\frac{2}{(2+A)^2}$ or not at all. Here, $J_{-1} = [-A/n, -\frac{A^2/n}{2+A}]$ and $J_{+1} = [\frac{A^2/n}{2+A}, A/n]$.

Now for $k = \pm 1$, ± 3, the parallelepipeds $\theta_{\frac{k}{(2+A)^2}}(R_\gamma)$ are disjoint. Since

$$\begin{aligned} &\frac{1}{(2+A)^2} + \Big|\sum_{i=1}^{n} \gamma_i\Big| \cdot \frac{(1+A)A/n}{2+A} + \frac{3}{(2+A)^2} \\ \leq\ & \frac{1}{(2+A)^2} + \frac{(1+A)A}{2+A} + \frac{3}{(2+A)^2} \\ =\ & \frac{4+2A+3A^2+A^3}{(2+A)^2} = 1 - \frac{A(2-2A-A^2)}{(2+A)^2} \leqq 1 \end{aligned}$$

for small A $(< \frac{1}{2})$, they are contained in R.

For $\delta = (\delta_1, \ldots, \delta_n)$, $(\delta_i = \pm 1)$, the images

$$\varphi_1^{\delta_1 \frac{1+A}{2+A}} \cdots \varphi_n^{\delta_n \frac{1+A}{2+A}} (\theta_{\frac{k}{(2+A)^2}}(R_\gamma))$$

are contained in $I_{\delta_1} \times \cdots \times I_{\delta_n} \times J_{\gamma_1} \times \cdots \times J_{\gamma_n} \times [-1, 1]$, where $I_{-1} = [-1, -\frac{A}{2+A}]$ and $I_{+1} = [\frac{A}{2+A}, 1]$.

Thus we obtain $2^n \times 2^2 \times 2^n = 2^{2n+2}$ contractions

$$\varphi_1^{\delta_1 \frac{1+A}{2+A}} \cdots \varphi_n^{\delta_n \frac{1+A}{2+A}} \theta_{\frac{k}{(2+A)^2}} \psi_1^{\gamma_1 \frac{(1+A)A/n}{2+A}} \cdots \psi_n^{\gamma_n \frac{(1+A)A/n}{2+A}} \varepsilon^{-\log(2+A)}$$

with images of R being disjointly contained in R. See Figure 1.

For $G = \mathrm{Cont}_c^r(\boldsymbol{R}^{2n+1}, \alpha)$, let $\sigma \in S_1^\infty(G)$ represent a foliated product over the interval Δ^1 with support in R.

We divide the foliated product into 2^{2n+2} segments of length $\frac{1}{2^{2n+2}}$ in the direction of Δ^1. Then $\sigma = \sum_{j=1}^{2^{2n+2}} \sigma_j$ in $H_1(B\overline{G}; \boldsymbol{Z})$. Because of the

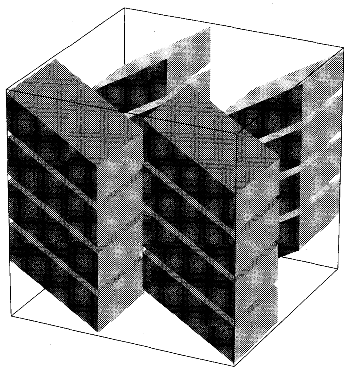

Fig. 1. The images of 2^4 contractions for $n=1$

reparametrization in the direction of the interval Δ^1, the C^r norm of σ_j is $\dfrac{1}{2^{2n+2}}$ of that of σ.

Then we map each foliated product σ_j by the action of one of the above 2^{2n+2} contractions.

By the action of $\varepsilon^{-\log(2+A)}$, the C^r norm of the foliated product is multiplied by $\dfrac{1}{2+A}(2+A)^{2r}$. By the action of $\psi_1^{\gamma_1 \frac{(1+A)A/n}{2+A}} \dots \psi_n^{\gamma_n \frac{(1+A)A/n}{2+A}}$, the C^r norm is multiplied by $(1+\dfrac{(1+A)A}{2+A})^{r+1}$. By the action of the translations $\theta_{\frac{k}{(2+A)^2}}$ and $\varphi_1^{\delta_1 \frac{1+A}{2+A}} \dots \varphi_n^{\delta_n \frac{1+A}{2+A}}$, the norm is unchanged.

Thus the foliated products obtained have the C^r norm

$$\frac{1}{2^{2n+2}}\frac{1}{2+A}(2+A)^{2r}(1+\frac{(1+A)A}{2+A})^{r+1}$$

times that of σ.

If $A=0$, the factor is $2^{-2n-2}2^{2r-1} = 2^{2(r-n-\frac{3}{2})}$. Hence if $r < n+\dfrac{3}{2}$, by taking A small, the factor is smaller than 1.

Now we can show Theorem 5.1 for $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$.

Proof of Theorem 5.1 *for* $(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$. Take $\sigma \in S_1^\infty(G) = S_1^\infty(\mathrm{Cont}_c^r(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}}))$ representing a foliated product over the interval Δ^1 with support in $[-\dfrac{A}{2+A}, \dfrac{A}{2+A}]^n \times [-A/n, A/n]^n \times [-1,1]$. We show that it is written as a boundary in $B\overline{G} = B\overline{\mathrm{Cont}_c^r}(\boldsymbol{R}^{2n+1}, \alpha_{\mathrm{st}})$.

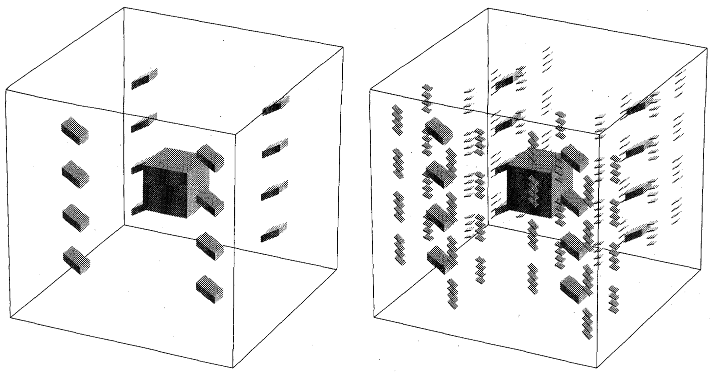

Fig. 2. The support of first and second iterates for σ with support in $[-\frac{A}{2+A}, \frac{A}{2+A}]^n \times [-A/n, A/n]^n \times [-1, 1]$

We perform the construction described above for the foliated product σ and repeat it infinitely many times. Since the supports of the resultant foliated products are disjoint, we take the union of all of them. See Figure 2. Let $I'\sigma$ be the (infinite) union without σ, and $I\sigma$, the (infinite) union with σ.

Then if we do the construction again, we see that $I\sigma = I'\sigma$ in $H_1(B\overline{G}; \boldsymbol{Z})$. Since $I\sigma = \sigma + I'\sigma$ in $H_1(B\overline{G})$, $\sigma = 0$ in $H_1(B\overline{G}; \boldsymbol{Z})$. See [15]. Q.E.D.

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