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New examples of elements in the kernel of the Magnus representation of the Torelli group

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Abstract.

From our previous paper, it is known that the Magnus representation of the Torelli group is not faithful. In this paper, we show a certain kind of elements in the kernel of the representation.

§1. Introduction

Let Σ_g be an oriented closed surface of genus g and $\Sigma_{g,1}$ a compact surface removed an open disk from Σ_g. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_{g,1}$ be the Torelli group of $\Sigma_{g,1}$, that is, the normal subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act trivially on the first homology group of $\Sigma_{g,1}$.

A group is said to be linear, if it admits a finite dimensional faithful representation. In [6] and [1], it is proved that the mapping class group of a closed surface of genus two is linear. However, the linearity of the mapping class group of a surface of higher genera is still not known.

It is shown in [13] that the Magnus representation of the Torelli group

$$r_1 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}[H])$$

is not faithful for $g \geq 2$, where $H = H_1(\Sigma_{g,1}; \mathbb{Z})$. Then this representation cannot determine the linearity of the Torelli group $\mathcal{I}_{g,1}$. In [16], we characterized which commutators of two BSCC maps lie in the kernel of r_1, where the Dehn twist along a bounding simple closed curve is called BSCC map.

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Let $\mathcal{K}_{g,1}$ be the normal subgroup of $\mathcal{I}_{g,1}$ generated by all BSCC maps. Morita constructed a homomorphism

$$d_1 : \mathcal{K}_{g,1} \longrightarrow \mathbb{Z},$$

which is a secondary characteristic class of surface bundles. See [8], [10], [11] and [12] for details. The following open problem is mentioned in [12]: determine whether the Magnus representation of the Torelli group detects d_1 or not. In other words, does there exist some $\varphi \in \ker r_1$ so that $d_1(\varphi) \neq 0$.

In order to investigate the above problem, we will present new examples of elements in the kernel of the Magnus representation of the Torelli group in this paper. In particular, we will describe a relation between the kernel of $\mathcal{I}_{g,1} \to \mathcal{I}_g$, which is denoted by G, and the kernel of r_1, where $\mathcal{I}_g$ is the Torelli group of Σ_g. More precisely, the following is proved in this paper:

Main Theorem . (1) *Let ψ be a BSCC map and $\tilde{\gamma} \in G$. Then*

$$[\psi, \tilde{\gamma}\psi\tilde{\gamma}^{-1}] \in \ker r_1.$$

(2) *The 3-rd derived subgroup of G is contained in the kernel of r_1.*
(3) *Let φ_1, φ_2 be elements of the commutator subgroup of G. Then*

$$[\varphi_1, \varphi_2][\varphi_1, \varphi_2^{-1}] \in \ker r_1.$$

However, the value of d_1 for each element in the kernel of r_1 which is shown in this paper is zero. Then these elements are not useful to answer the above problem of Morita. We note that all the elements mentioned in [13], [16] map to zero by d_1.

In Section 2, we review the definition and the irreducible decomposition of the Magnus representation of the Torelli group. In Section 3, we study some properties and examples of the Magnus representation, which are needed to prove the main results. In Section 4, the main results of this paper are shown, that is, we present a certain kind of elements in the kernel of r_1, which do not appear in the previous papers [13], [16].

§2. Definition and irreducible decomposition of the Magnus representation of the Torelli group

In this section, we recall the definition of the Magnus representation of the Torelli group and its irreducible decomposition from [9], [13] and [14].

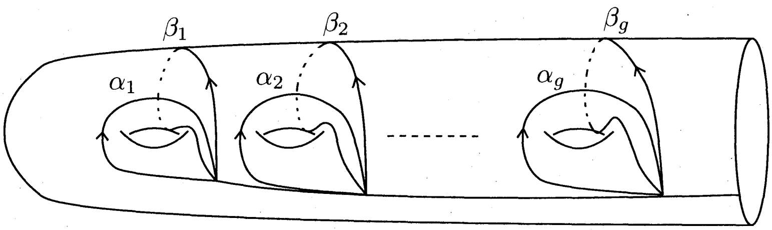

Fig. 1. Generators of Γ_0

We fix a system of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the free group $\Gamma_0 = \pi_1(\Sigma_{g,1})$ as shown in Figure 1. Let us simply write $\gamma_1, \ldots, \gamma_{2g}$ for them.

Definition 2.1. We call the mapping

$$\begin{array}{rccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g;\mathbb{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j} \end{array}$$

the Magnus representation of the mapping class group, where $\frac{\partial}{\partial \gamma_i}$: $\mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the Fox derivation of the integral group ring $\mathbb{Z}[\Gamma_0]$ and $^- : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the antiautomorphism induced by the mapping $\gamma \mapsto \gamma^{-1}$.

The mapping r is not a homomorphism but a crossed homomorphism. In other words, we have the following.

Proposition 2.2 (Morita [9]). *For any elements $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

where ${}^{\varphi}r(\psi)$ denotes the matrix obtained from $r(\psi)$ by applying the automorphism $\varphi : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ on each entry.

Since the Torelli group acts trivially on H, we obtain a genuine representation of the Torelli group as follows. We restrict r to the Torelli group $\mathcal{I}_{g,1}$ and reduce the coefficients to $\mathbb{Z}[H]$:

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g;\mathbb{Z}[H]).$$

That is to say, this homomorphism r_1 is the composite $r^{\mathfrak{a}}$ of the mapping r by the abelianization $\mathfrak{a} : \Gamma_0 \to H$. We call r_1 the Magnus representation of the Torelli group.

The Magnus representation of the Torelli group is reducible, but not completely reducible. More precisely, we have the following irreducible decomposition.

Theorem 2.3 ([14]). *For $g \geq 2$ there exists a non-singular matrix $P \in GL(2g; R)$ such that for any element $\varphi \in \mathcal{I}_{g,1}$*

$$P^{-1}\, r_1(\varphi)\, P = \left(\begin{array}{c|ccc|c} 1 & & \rho_{b_1}(\varphi) & & \rho_{b_3}(\varphi) \\ \hline 0 & & & & \\ \vdots & & \rho_B(\varphi) & & \rho_{b_2}(\varphi) \\ & & & & \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Moreover, ρ_B is a $(2g-2)$-dimensional irreducible representation of $\mathcal{I}_{g,1}$. Here $R = \mathbb{Z}[{x_i}^{\pm 1}, {y_i}^{\pm 1}, \frac{1}{1-y_i}]$ ($\supset \mathbb{Z}[H]$) where x_i, y_i ($i = 1, \ldots, g$) are obtained by abelianizing α_i, β_i respectively.

Though the actual value of P is slightly complicated, we express it explicitly according to [14]. That is to say,

$$P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_g \end{pmatrix} \cdot \left(I_1 \oplus \left(\begin{pmatrix} 0 & -{}^t P_{11} \\ I_{g-1} & 0 \end{pmatrix}^{-1} {Q_2}^{-1} \right) \right)$$

where I_k is the identity matrix of degree k and

$$P_{11} = -\begin{pmatrix} 1-{y_1}^{-1} & & & 0 \\ 1-{y_2}^{-1} & 1-{y_2}^{-1} & & \\ \vdots & \vdots & \ddots & \\ 1-{y_g}^{-1} & 1-{y_g}^{-1} & \cdots & 1-{y_g}^{-1} \end{pmatrix}$$

$$P_{21} = \begin{pmatrix} 1-{x_1}^{-1} & & & 0 \\ 1-{x_2}^{-1} & 1-{x_2}^{-1} & & \\ \vdots & \vdots & \ddots & \\ 1-{x_g}^{-1} & 1-{x_g}^{-1} & \cdots & 1-{x_g}^{-1} \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

In this paper, we write $r_1'(\varphi)$ for $P^{-1} r_1(\varphi) P$ for simplicity.

§3. Some properties of $\rho_B, \rho_{b_1}, \rho_{b_2}, \rho_{b_3}$

In this section, we show the product formulas of $\rho_B, \rho_{b_1}, \rho_{b_2}, \rho_{b_3}$ and calculate them for elements of the kernel of the projection $\mathcal{I}_{g,1} \to \mathcal{I}_g$, where $\mathcal{I}_g$ is the Torelli group of Σ_g.

Lemma 3.1. *For any elements $\varphi_1, \varphi_2 \in \mathcal{I}_{g,1}$, we have*

(1) $\rho_B(\varphi_1\varphi_2) = \rho_B(\varphi_1)\rho_B(\varphi_2)$,
(2) $\rho_{b_1}(\varphi_1\varphi_2) = \rho_{b_1}(\varphi_2) + \rho_{b_1}(\varphi_1)\rho_B(\varphi_2)$,
(3) $\rho_{b_2}(\varphi_1\varphi_2) = \rho_B(\varphi_1)\rho_{b_2}(\varphi_2) + \rho_{b_2}(\varphi_1)$,
(4) $\rho_{b_3}(\varphi_1\varphi_2) = \rho_{b_3}(\varphi_2) + \rho_{b_1}(\varphi_1)\rho_{b_2}(\varphi_2) + \rho_{b_3}(\varphi_1)$.

Proof. By $r_1'(\varphi_1\varphi_2) = r_1'(\varphi_1)r_1'(\varphi_2)$, we get

$$\left(\begin{array}{c|c|c} 1 & \rho_{b_1}(\varphi_1\varphi_2) & \rho_{b_3}(\varphi_1\varphi_2) \\ \hline 0 & & \\ \vdots & \rho_B(\varphi_1\varphi_2) & \rho_{b_2}(\varphi_1\varphi_2) \\ & & \\ \hline 0 & 0 \cdots 0 & 1 \end{array}\right) =$$

$$\left(\begin{array}{c|c|c} 1 & \rho_{b_1}(\varphi_1) & \rho_{b_3}(\varphi_1) \\ \hline 0 & & \\ \vdots & \rho_B(\varphi_1) & \rho_{b_2}(\varphi_1) \\ & & \\ \hline 0 & 0 \cdots 0 & 1 \end{array}\right) \left(\begin{array}{c|c|c} 1 & \rho_{b_1}(\varphi_2) & \rho_{b_3}(\varphi_2) \\ \hline 0 & & \\ \vdots & \rho_B(\varphi_2) & \rho_{b_2}(\varphi_2) \\ & & \\ \hline 0 & 0 \cdots 0 & 1 \end{array}\right).$$

Q.E.D.

Next, we consider the kernel of the projection $\mathcal{I}_{g,1} \to \mathcal{I}_g$. The kernel of $\mathcal{I}_{g,1} \to \mathcal{I}_g$, denoted by G, is isomorphic to the fundamental group $\pi_1(T_1\Sigma_g)$, where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g. It is known that G is generated by $\tau_\zeta, \tilde{\alpha}_i, \tilde{\beta}_i$, $(i = 1, \ldots, g)$, see [4], [7] for details. These generators $\tau_\zeta, \tilde{\alpha}_i, \tilde{\beta}_i$ are given as follows. Let $\alpha_{i+}, \alpha_{i-}, \beta_{i+}, \beta_{i-}, \zeta$ be the curves in Figure 2, 3 and 4. Then $\tilde{\alpha}_i \in \mathcal{I}_{g,1}$ (respectively $\tilde{\beta}_i \in \mathcal{I}_{g,1}$), which is a lift of α_i (respectively β_i) to $\pi_1(T_1\Sigma_g)$, is a product of the Dehn twist along α_{i+} (respectively β_{i+}) and the inverse of the Dehn twist along α_{i-} (respectively β_{i-}). Besides, τ_ζ is the Dehn twist along a curve ζ which is parallel to the boundary of $\Sigma_{g,1}$.

Lemma 3.2. *For any element $\tilde{\gamma} \in G = \ker(\mathcal{I}_{g,1} \to \mathcal{I}_g)$, we have*

$$\rho_B(\tilde{\gamma}) = [\gamma]^{-1} I_{2g-2}, \quad \rho_{b_1}(\tilde{\gamma}^{-1}) = -[\gamma]\rho_{b_1}(\tilde{\gamma}), \quad \rho_{b_2}(\tilde{\gamma}^{-1}) = -[\gamma]\rho_{b_2}(\tilde{\gamma}).$$

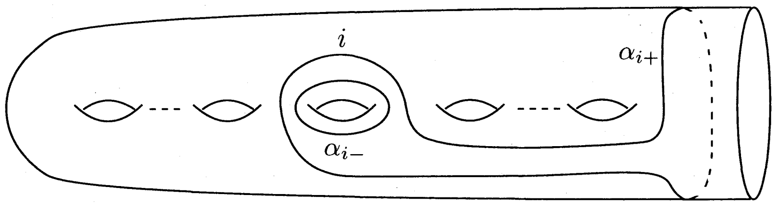

Fig. 2. Simple closed curves α_{i+} and α_{i-}

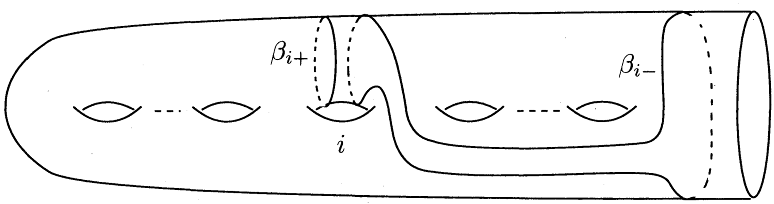

Fig. 3. Simple closed curves β_{i+} and β_{i-}

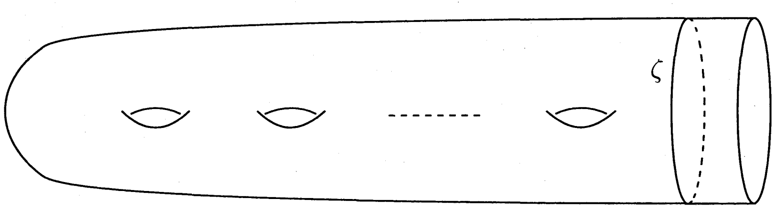

Fig. 4. Simple closed curve ζ

where $[\gamma]$ *represents the homology class of* γ *whose lift to* $\pi_1(T_1\Sigma_g)$ *is* $\tilde{\gamma}$. *In particular, if* φ *belongs to the commutator subgroup of* G, *then*

$$\rho_B(\varphi) = I_{2g-2}, \quad \rho_{b_1}(\varphi^{-1}) = -\rho_{b_1}(\varphi), \quad \rho_{b_2}(\varphi^{-1}) = -\rho_{b_2}(\varphi).$$

Proof. By direct calculations, we have

$$\tilde{\beta_1}(\alpha_j) = \begin{cases} [\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\beta_1\alpha_1\bar{\beta}_1 & j=1 \\ \beta_1[\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\alpha_j[\beta_g,\alpha_g]\cdots[\beta_1,\alpha_1]\beta_1 & 2\le j\le g \end{cases},$$

$$\tilde{\beta_1}(\beta_j) = \begin{cases} \beta_1 & j=1 \\ \beta_1[\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\beta_j[\beta_g,\alpha_g]\cdots[\beta_1,\alpha_1]\beta_1 & 2\le j\le g \end{cases}.$$

Then the Magnus matrix $r_1(\tilde{\beta_1})$ is given by the following equations:

$$\mathfrak{a}\left(\overline{\frac{\partial\tilde{\beta_1}(\alpha_j)}{\partial\alpha_i}}\right) = \begin{cases} 1-\bar{y}_i+\delta_{1,i}\bar{y}_1 & j=1 \\ \delta_{j,i}\bar{y}_1+\bar{y}_1(1-\bar{x}_j)(1-\bar{y}_i) & 2\le j\le g \end{cases}$$

$$\mathfrak{a}\left(\overline{\frac{\partial\tilde{\beta_1}(\beta_j)}{\partial\alpha_i}}\right) = \begin{cases} 0 & j=1 \\ \bar{y}_1(1-\bar{y}_i)(1-\bar{y}_j) & 2\le j\le g \end{cases}$$

$$\mathfrak{a}\left(\overline{\frac{\partial\tilde{\beta_1}(\alpha_j)}{\partial\beta_i}}\right) = \begin{cases} \delta_{1,i}(1-\bar{x}_1)-(1-\bar{x}_i) & j=1 \\ \delta_{1,i}(1-\bar{x}_j)-\bar{y}_1(1-\bar{x}_i)(1-\bar{x}_j) & 2\le j\le g \end{cases}$$

$$\mathfrak{a}\left(\overline{\frac{\partial\tilde{\beta_1}(\beta_j)}{\partial\beta_i}}\right) =$$
$$\begin{cases} \delta_{1,i} & j=1 \\ \delta_{j,i}\bar{y}_1+\delta_{1,i}(1-\bar{y}_j)-\bar{y}_1(1-\bar{x}_i)(1-\bar{y}_j) & 2\le j\le g \end{cases}.$$

Here $\bar{x}_i = x^{-1}, \bar{y}_i = y^{-1}$. Moreover, the conjugation this matrix by P is described as

$$\rho_B(\tilde{\beta_1}) = \bar{y}_1 I_{2g-2}, \quad \rho_{b_1}(\tilde{\beta_1}) = (-\bar{y}_1, \underbrace{0,\dots,0}_{2g-3}),$$

$$\rho_{b_2}(\tilde{\beta_1}) = {}^t(\underbrace{0,\dots,0}_{2g-2}), \quad \rho_{b_3}(\tilde{\beta_1}) = 0.$$

Similarly, we can compute the other matrices $\rho_B(\tilde{\alpha_i})$ and $\rho_B(\tilde{\beta_i})$:

$$\rho_B(\tilde{\alpha_i}) = \bar{x}_i I_{2g-2}, \quad \rho_B(\tilde{\beta_i}) = \bar{y}_i I_{2g-2}.$$

In addition, more simple computations show

$$\rho_B(\tau_\zeta) = I_{2g-2}.$$

These proved the first equation of the statement. Furthermore, the first equation and Lemma 3.1 follow that

$$\begin{aligned} 0 &= \rho_{b_1}(\tilde{\gamma}^{-1}\tilde{\gamma}) = \rho_{b_1}(\tilde{\gamma}) + [\gamma]^{-1}\rho_{b_1}(\tilde{\gamma}^{-1}) \\ 0 &= \rho_{b_2}(\tilde{\gamma}\tilde{\gamma}^{-1}) = [\gamma]^{-1}\rho_{b_2}(\tilde{\gamma}^{-1}) + \rho_{b_2}(\tilde{\gamma}) \quad . \end{aligned}$$

This completes the proof. Q.E.D.

Lemma 3.3. *Let ψ be a BSCC map, that is, the Dehn twist along a* 0*-homologous simple closed curve. Then we obtain*

(1) $\rho_{b_1}(\psi) = \rho_{b_1}(\psi)\rho_B(\psi)$,
(2) $\rho_{b_2}(\psi) = \rho_B(\psi)\rho_{b_2}(\psi)$,
(3) $\rho_{b_1}(\psi)\rho_{b_2}(\psi) = 0$,
(4) $(\rho_B(\psi) - I_{2g-2})^2 = 0$.

Proof. It is shown in [15] that the Magnus matrix $r_1(\psi)$ of a BSCC map ψ can be written as

$$r_1(\psi) = I_{2g} + uv$$

where $u \in \mathbb{Z}[H]^{2g}, v \in {}^t\mathbb{Z}[H]^{2g}$ and $vu = 0$. Then we get

$$(1) \quad (r_1'(\psi) - I_{2g})^2 = (P^{-1}r_1(\psi)P - I_{2g})^2 = P^{-1}uvPP^{-1}uvP = O.$$

On the other hand, by Theorem 2.3 the following equation holds:

$$\begin{aligned} &(r_1'(\psi) - I_{2g})^2 \\ &= \left(\begin{array}{c|c|c} 0 & \rho_{b_1}(\psi) & \rho_{b_3}(\psi) \\ \hline 0 & & \\ \vdots & \rho_B(\psi) - I & \rho_{b_2}(\psi) \\ & & \\ \hline 0 & 0 \cdots 0 & 0 \end{array}\right) \left(\begin{array}{c|c|c} 0 & \rho_{b_1}(\psi) & \rho_{b_3}(\psi) \\ \hline 0 & & \\ \vdots & \rho_B(\psi) - I & \rho_{b_2}(\psi) \\ & & \\ \hline 0 & 0 \cdots 0 & 0 \end{array}\right) \\ &= \left(\begin{array}{c|c|c} 0 & \rho_{b_1}(\psi)\rho_B(\psi) - \rho_{b_1}(\psi) & \rho_{b_1}(\psi)\rho_{b_2}(\psi) \\ \hline 0 & & \\ \vdots & (\rho_B(\psi) - I)^2 & \rho_B(\psi)\rho_{b_2}(\psi) - \rho_{b_2}(\psi) \\ & & \\ \hline 0 & 0 \cdots 0 & 0 \end{array}\right). \end{aligned}$$

By comparing (1) with the right hand side of the above equation, we arrive at the statements of Lemma 3.3. Q.E.D.

§4. Main Theorem

In this section, we present several types of elements in the kernel of the Magnus representation of the Torelli group.

Theorem 4.1. *Let ψ be a BSCC map and $\tilde{\gamma} \in G = \ker(\mathcal{I}_{g,1} \to \mathcal{I}_g)$. Then*

$$[\psi, \tilde{\gamma}\psi\tilde{\gamma}^{-1}] \in \ker r_1.$$

Proof. If ψ' denotes $\tilde{\gamma}\psi\tilde{\gamma}^{-1}$, it is sufficient to prove $r_1'(\psi\psi') = r_1'(\psi'\psi)$. Thus we will show that

(1) $\rho_B(\psi\psi') - \rho_B(\psi'\psi) = 0$,
(2) $\rho_{b_1}(\psi\psi') - \rho_{b_1}(\psi'\psi) = 0$,
(3) $\rho_{b_2}(\psi\psi') - \rho_{b_2}(\psi'\psi) = 0$,
(4) $\rho_{b_3}(\psi\psi') - \rho_{b_3}(\psi'\psi) = 0$.

First, Lemma 3.2 says that $\rho_B(\tilde{\gamma})$ is central, so we have

$$\rho_B(\tilde{\gamma})\rho_B(\psi)\rho_B(\tilde{\gamma})^{-1} = \rho_B(\psi). \tag{2}$$

Then we obtain the following by the above equation.

$$\begin{aligned}
&\rho_B(\psi\psi') - \rho_B(\psi'\psi) \\
&\quad = \rho_B(\psi)\rho_B(\tilde{\gamma})\rho_B(\psi)\rho_B(\tilde{\gamma})^{-1} - \rho_B(\tilde{\gamma})\rho_B(\psi)\rho_B(\tilde{\gamma})^{-1}\rho_B(\psi) \\
&\quad = 0 \quad \text{by (2)}.
\end{aligned}$$

Next, we calculate ρ_{b_1}:

$$\begin{aligned}
&\rho_{b_1}(\psi\psi') - \rho_{b_1}(\psi'\psi) \\
&\quad = \rho_{b_1}(\psi') + \rho_{b_1}(\psi)\rho_B(\psi') - \rho_{b_1}(\psi) - \rho_{b_1}(\psi')\rho_B(\psi) \\
&\qquad\qquad \text{by Lemma 3.1 (2)} \\
&\quad = -[\gamma]\rho_{b_1}(\tilde{\gamma}) + [\gamma]\rho_{b_1}(\psi) + [\gamma]\rho_{b_1}(\tilde{\gamma})\rho_B(\psi) + \rho_{b_1}(\psi)\rho_B(\psi) \\
&\qquad -\rho_{b_1}(\psi) + \{[\gamma]\rho_{b_1}(\tilde{\gamma}) - [\gamma]\rho_{b_1}(\psi) - [\gamma]\rho_{b_1}(\tilde{\gamma})\rho_B(\psi)\}\rho_B(\psi) \\
&\qquad\qquad \text{by Lemma 3.2} \\
&\quad = -[\gamma]\rho_{b_1}(\tilde{\gamma})\{I - 2\rho_B(\psi) + \rho_B(\psi)^2\} \quad \text{by Lemma 3.3} \\
&\quad = -[\gamma]\rho_{b_1}(\tilde{\gamma})(I - \rho_B(\psi))^2 \\
&\quad = 0 \quad \text{by Lemma 3.3}
\end{aligned}$$

Similarly, we get $\rho_{b_2}(\psi\psi') - \rho_{b_2}(\psi'\psi) = 0$. Finally, we obtain

$$\begin{aligned}
&\rho_{b_3}(\psi\psi') - \rho_{b_3}(\psi'\psi) \\
&= \rho_{b_3}(\psi') + \rho_{b_1}(\psi)\rho_{b_2}(\psi') + \rho_{b_3}(\psi) \\
&\quad -\rho_{b_3}(\psi) - \rho_{b_1}(\psi')\rho_{b_2}(\psi) - \rho_{b_3}(\psi') \quad \text{by Lemma 3.1 (4)} \\
&= \rho_{b_1}(\psi)\{-\rho_B(\psi)\rho_{b_2}(\tilde{\gamma}) + [\gamma]^{-1}\rho_{b_2}(\psi) + \rho_{b_2}(\tilde{\gamma})\} \\
&\quad -\{-[\gamma]\rho_{b_1}(\tilde{\gamma}) + [\gamma]\rho_{b_1}(\psi) + [\gamma]\rho_{b_1}(\tilde{\gamma})\rho_B(\psi)\}\rho_{b_2}(\psi) \\
&\qquad\qquad \text{by Lemma 3.1 and 3.2} \\
&= 0 \quad \text{by Lemma 3.3.}
\end{aligned}$$

This completes the proof. Q.E.D.

Suppose φ be an element of $\mathcal{I}_{g,1}$ such that $\rho_B(\varphi) = I_{2g-2}$, $\rho_{b_1}(\varphi) = 0$ and $\rho_{b_2}(\varphi) = 0$. Then the Magnus matrix $r_1'(\varphi)$ commutes with that of any element of $\mathcal{I}_{g,1}$. This means that $[\varphi, \mathcal{I}_{g,1}] \subset \ker r_1$. One of such elements is τ_ζ. We show other elements satisfying these properties.

We denote by $G^{(k)}$ the k-th term in the derived series of G so that $G^{(0)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$.

Theorem 4.2. *If φ belongs to $G^{(2)}$, then $\rho_B(\varphi) = I_{2g-2}$, $\rho_{b_1}(\varphi) = 0$ and $\rho_{b_2}(\varphi) = 0$.*

Proof. Let g_1, g_2, g_3, g_4 be elements of G. It is sufficient to prove the statement for the case $\varphi = [[g_1, g_2], [g_3, g_4]]$. By Lemma 3.2, for $g, g' \in G$, $\rho_B([g, g'])$ is the identity matrix. Then we get $\rho_B(\varphi) = I_{2g-2}$. Moreover, $\rho_{b_1}(\varphi)$ is computed as

$$\begin{aligned}
\rho_{b_1}(\varphi) &= \rho_{b_1}([g_1, g_2][g_3, g_4][g_2, g_1][g_4, g_3]) \\
&= \rho_{b_1}([g_1, g_2]) + \rho_{b_1}([g_3, g_4]) + \rho_{b_1}([g_2, g_1]) + \rho_{b_1}([g_4, g_3]) \\
&\qquad\qquad \text{by Lemma 3.1 and 3.2} \\
&= \rho_{b_1}([g_1, g_2]) + \rho_{b_1}([g_3, g_4]) - \rho_{b_1}([g_1, g_2]) - \rho_{b_1}([g_3, g_4]) \\
&\qquad\qquad \text{by Lemma 3.2} \\
&= 0.
\end{aligned}$$

A similar calculation follows that $\rho_{b_2}(\varphi) = 0$. This completes the proof. Q.E.D.

Corollary 4.3. *The commutator $[G^{(2)}, \mathcal{I}_{g,1}]$ is contained in $\ker r_1$. In particular,*

$$G^{(3)} \subset \ker r_1.$$

Proof. This follows directly from Theorem 4.2 and the argument before Theorem 4.2. Q.E.D.

Corollary 4.4. *Let φ_1, φ_2 be elements of $G^{(1)}$. Then*

$$[\varphi_1, \varphi_2][\varphi_1, \varphi_2^{-1}] \in \ker r_1.$$

Proof. Since $[\varphi_1, \varphi_2]$ and $[\varphi_1, \varphi_2^{-1}]$ belong to $G^{(2)}$, it is sufficient to show

$$\rho_{b_3}([\varphi_1, \varphi_2][\varphi_1, \varphi_2^{-1}]) = 0.$$

We recall that $\rho_{b_1}([\varphi_1, \varphi_2]) = 0$, $\rho_{b_2}([\varphi_1, \varphi_2^{-1}]) = 0$, $\rho_B(\varphi_i) = I_{2g-2}$, $\rho_{b_1}(\varphi_i^{-1}) = -\rho_{b_1}(\varphi_i)$, $\rho_{b_2}(\varphi_i^{-1}) = -\rho_{b_2}(\varphi_i)$ by Lemma 3.2.

$$\begin{aligned}
&\rho_{b_3}([\varphi_1, \varphi_2][\varphi_1, \varphi_2^{-1}]) \\
&= \rho_{b_3}([\varphi_1, \varphi_2]) + \rho_{b_3}([\varphi_1, \varphi_2^{-1}]) + \rho_{b_1}([\varphi_1, \varphi_2])\rho_{b_2}([\varphi_1, \varphi_2^{-1}]) \\
&= \rho_{b_3}(\varphi_1\varphi_2) + \rho_{b_3}(\varphi_1^{-1}\varphi_2^{-1}) + \rho_{b_1}(\varphi_1\varphi_2)\rho_{b_2}(\varphi_1^{-1}\varphi_2^{-1}) \\
&\quad + \rho_{b_3}(\varphi_1\varphi_2^{-1}) + \rho_{b_3}(\varphi_1^{-1}\varphi_2) + \rho_{b_1}(\varphi_1\varphi_2^{-1})\rho_{b_2}(\varphi_1^{-1}\varphi_2) \\
&= \rho_{b_3}(\varphi_1) + \rho_{b_3}(\varphi_2) + \rho_{b_1}(\varphi_1)\rho_{b_2}(\varphi_2) \\
&\quad + \rho_{b_3}(\varphi_1^{-1}) + \rho_{b_3}(\varphi_2^{-1}) + \rho_{b_1}(\varphi_1^{-1})\rho_{b_2}(\varphi_2^{-1}) \\
&\quad + \{\rho_{b_1}(\varphi_1) + \rho_{b_1}(\varphi_2)\}\{-\rho_{b_2}(\varphi_1) - \rho_{b_2}(\varphi_2)\} \\
&\quad + \rho_{b_3}(\varphi_1) + \rho_{b_3}(\varphi_2^{-1}) + \rho_{b_1}(\varphi_1)\rho_{b_2}(\varphi_2^{-1}) \\
&\quad + \rho_{b_3}(\varphi_1^{-1}) + \rho_{b_3}(\varphi_2) + \rho_{b_1}(\varphi_1^{-1})\rho_{b_2}(\varphi_2) \\
&\quad + \{\rho_{b_1}(\varphi_1) - \rho_{b_1}(\varphi_2)\}\{-\rho_{b_2}(\varphi_1) + \rho_{b_2}(\varphi_2)\} \\
&= 2\{\rho_{b_3}(\varphi_1) + \rho_{b_3}(\varphi_1^{-1}) - \rho_{b_1}(\varphi_1)\rho_{b_2}(\varphi_1)\} \\
&\quad + 2\{\rho_{b_3}(\varphi_2) + \rho_{b_3}(\varphi_2^{-1}) - \rho_{b_1}(\varphi_2)\rho_{b_2}(\varphi_2)\} \\
&= 2\{\rho_{b_3}(\varphi_1) + \rho_{b_3}(\varphi_1^{-1}) + \rho_{b_1}(\varphi_1)\rho_{b_2}(\varphi_1^{-1})\} \\
&\quad + 2\{\rho_{b_3}(\varphi_2) + \rho_{b_3}(\varphi_2^{-1}) + \rho_{b_1}(\varphi_2)\rho_{b_2}(\varphi_2^{-1})\} \\
&= 0
\end{aligned}$$

This completes the proof. Q.E.D.

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