Advanced Studies in Pure Mathematics 52, 2008
Groups of Diffeomorphisms
pp. 443–468

Symplectic automorphism groups of nilpotent quotients of fundamental groups of surfaces

Shigeyuki Morita

Abstract.

We describe the *group version* of the trace maps given in [29]. This gives rise to abelian quotients of *symplectic* IA-automorphism groups of nilpotent quotients of the fundamental groups of compact surfaces. By making use of them, we construct a representation of the group $\mathcal{H}_{g,1}$ of homology cobordism classes of homology cylinders introduced by Garoufalidis and Levine [6]. We define various cohomology classes of $\mathcal{H}_{g,1}$ and propose a few problems concerning them. In particular, we mention a possible relation to additive invariants for the group $\Theta^3_{\mathbb{Z}}$ of homology cobordism classes of homology 3-spheres.

§1. Introduction

As is well known, the mapping class group $\mathcal{M}(\Sigma_g)$ of a closed oriented surface Σ_g of genus $g \geq 2$ can be identified with the orientation preserving outer automorphism group of $\pi_1\Sigma_g$. This is the classical theorem of Dehn and Nielsen. In the case of a compact oriented surface Σ^0_g with one boundary component, the corresponding mapping class group $\mathcal{M}(\Sigma^0_g, \mathrm{rel}\ \partial\Sigma^0_g)$ relative to the boundary is canonically isomorphic to the subgroup of the automorphism group of $\pi_1\Sigma^0_g$ consisting of elements which preserve a particular element representing the boundary curve. Henceforth, we simply denote this group $\mathcal{M}(\Sigma^0_g, \mathrm{rel}\ \partial\Sigma^0_g)$ by $\mathcal{M}_{g,1}$.

Since the action of $\mathcal{M}_{g,1}$ on $\pi_1\Sigma^0_g$ preserves its lower central series, it induces a series of representations ρ_d of $\mathcal{M}_{g,1}$ into the automorphism groups of free nilpotent quotients of $\pi_1\Sigma^0_g$ with nilpotency class $d = 1, 2, \cdots$.

Received October 15, 2007.
Revised January 9, 2008.
Partially supported by JSPS Grant 19204003.

On the other hand, the kernels of these representations induce a filtration on the Torelli subgroup $\mathcal{I}_{g,1} \subset \mathcal{M}_{g,1}$ and the associated graded Lie algebra can be mapped injectively into the graded Lie algebra consisting of all the derivations of the free graded Lie algebra generated by the first homology group $H = H_1(\Sigma_g^0; \mathbb{Z})$. This embedding is called the Johnson homomorphism, which we denote by τ. It can be said that τ gathers the "differences" between the successive representations ρ_d and ρ_{d+1} for all d.

Many works have been done concerning the Johnson homomorphism τ and through them the image of τ has been shown to capture a smaller and smaller part of the target. First it was shown to satisfy a certain symplecticity condition and then it was proved to be in the kernel of certain abelian quotients of the relevant Lie algebra which we called the *traces* (see [29]).

The purpose of the present paper is to *lift* the traces as homomorphisms with abelian targets defined on certain symplectic automorphism groups of nilpotent quotients of $\pi_1\Sigma_g^0$ rather than in the context of Lie algebra homomorphisms.

We expect that these lifts will be useful in the study of the group $\mathcal{H}_{g,1}$ of homology cobordism classes of homology cylinders over Σ_g^0 which was introduced by Garoufalidis and Levine [6]. This is because of their result that the above series of representations ρ_d can be extended to their group $\mathcal{H}_{g,1}$ and these are all *surjective* onto some symplectic automorphism groups of nilpotent quotients of $\pi_1\Sigma_g^0$.

Part of the results of this paper were sketched in section 11 of [33].

§2. Symplectic automorphism groups

Let Σ_g denote a closed oriented surface of genus g and we denote by $\Sigma_g^0 = \Sigma_g \setminus \mathrm{Int}D^2$ a compact oriented surface of genus g with one boundary component. Consider the mapping class group

$$\mathcal{M}_{g,1} = \pi_0\mathrm{Diff}(\Sigma_g^0, \mathrm{rel}\ \partial\Sigma_g^0)$$

of Σ_g^0 *relative* to the boundary. Let $\Gamma = \pi_1\Sigma_g^0$ denote the fundamental group of Σ_g^0 which is a free group of rank $2g$. We have a distinguished element

$$\zeta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \in \Gamma$$

which corresponds to the boundary curve of Σ_g^0 so that it descends to the single defining relation for the fundamental group $\pi_1\Sigma_g$ of the closed surface, where α_i, β_i denotes a standard system of generators of Γ. Then by the classical theorem of Dehn-Nielsen adapted to the case of one boundary component due to Zieschang, we can write

$$\mathcal{M}_{g,1} \cong \mathrm{Aut}_0\Gamma = \{\varphi \in \mathrm{Aut}\,\Gamma; \varphi(\zeta) = \zeta\}. \tag{1}$$

Now we consider the lower central series

$$\Gamma_0 = \Gamma,\ \Gamma_1 = [\Gamma, \Gamma],\ \Gamma_2 = [\Gamma_1, \Gamma], \cdots,\ \Gamma_d = [\Gamma_{d-1}, \Gamma], \cdots$$

of Γ and also consider the quotient groups

$$N_d = \Gamma/\Gamma_d \quad (d = 1, 2, \cdots)$$

which we call the d-th nilpotent quotient of Γ. The first one N_1 is nothing but the abelianization of Γ so that we have an identification

$$N_1 = H = H_1(\Sigma_g^0; \mathbb{Z}).$$

We also denote by ζ_d the image of ζ in N_d under the natural projection $\Gamma \to N_d$. The first one is trivial, namely $\zeta_1 = 0 \in N_1 = H$. However the second one can be identified as

$$\zeta_2 = \omega_0 = \sum_{i=1}^{g} x_i \wedge y_i \in \Lambda^2 H \cong \Gamma_1/\Gamma_2 \subset N_2$$

where ω_0 is the symplectic class and x_i, y_i denote the homology classes of α_i, β_i respectively.

Definition 2.1. We define two subgroups of $\mathrm{Aut}\,N_d$ as follows.

$$\begin{aligned} \mathrm{Aut}_0' N_d &= \{\varphi \in \mathrm{Aut}\,N_d; \varphi(\zeta_d) = \zeta_d\} \\ \mathrm{Aut}_0 N_d &= p(\mathrm{Aut}_0' N_{d+1}) \end{aligned}$$

where $p : \mathrm{Aut}\,N_{d+1} \to \mathrm{Aut}\,N_d$ denotes the natural projection.

The subgroup $\mathrm{Aut}_0 N_d$ was introduced by Garoufalidis and Levine [6] which plays a fundamental role in their theory of the group, denoted by $\mathcal{H}_{g,1}$, of homology cobordism classes of homology cylinders over Σ_g^0. The concept of homology cylinders in turn was introduced by Habiro [9] and Goussarov [7] independently. The above definition is a slight modification of their original one and the reason why we consider another subgroup $\mathrm{Aut}_0' N_d$ will be explained in Remark 2.2 below.

It is easy to see that the first group $\mathrm{Aut}_0 N_1 = \mathrm{Aut}_0 H$ can be identified with the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ (whereas notice that $\mathrm{Aut}_0' N_1 \cong$

$\mathrm{GL}(2g,\mathbb{Z})$). By definition, these subgroups $\mathrm{Aut}_0 N_d$ $(d=1,2,\cdots)$ make a projective system of groups

$$\cdots \longrightarrow \mathrm{Aut}_0 N_d \longrightarrow \mathrm{Aut}_0 N_{d-1} \longrightarrow \cdots \longrightarrow \mathrm{Aut}_0 N_2 \longrightarrow \mathrm{Aut}_0 N_1.$$

The expression (1) induces representations

$$\rho_d : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}_0 N_d \tag{2}$$

and the totality of these representations gives rise to a homomorphism

$$\rho_\infty : \mathcal{M}_{g,1} \cong \mathrm{Aut}_0 \Gamma \longrightarrow \varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d$$

which is known to be *injective.*

The induced filtration

$$\{\mathcal{M}_{g,1}(d)\}_d, \quad \mathcal{M}_{g,1}(d) = \mathrm{Ker}(\rho_d : \mathcal{M}_{g,1} \to \mathrm{Aut}_0 N_d)$$

of the mapping class group was considered by Johnson [14] and it is called the *Johnson filtration.* The first group $\mathcal{M}_{g,1}(1)$ is nothing other than the Torelli group denoted by $\mathcal{I}_{g,1}$.

To describe the structure of $\mathrm{Aut}_0 N_d$ more precisely, let

$$\mathcal{L}_{g,1} = \bigoplus_{d=1}^{\infty} \mathcal{L}_{g,1}(d)$$

be the free graded Lie algebra generated by $H = H_1(\Sigma_g;\mathbb{Z})$ so that $\mathcal{L}_{g,1}(1) = H, \mathcal{L}_{g,1}(2) \cong \Lambda^2 H$, and $\mathcal{L}_{g,1}(d) \cong \Gamma_{d-1}/\Gamma_d$ in general (see [27]). Next let

$$\mathfrak{h}_{g,1} = \bigoplus_{d=0}^{\infty} \mathfrak{h}_{g,1}(d)$$

be the graded Lie algebra consisting of *symplectic* derivations of the free Lie algebra $\mathcal{L}_{g,1}$, namely those derivations which kill the symplectic class $\omega_0 \in \mathcal{L}_{g,1}(2)$. Here the submodule

$$\mathfrak{h}_{g,1}(d) = \{D \in \mathrm{Hom}(H, \mathcal{L}_{g,1}(d+1)); D(\omega_0) = 0\}$$

of $\mathfrak{h}_{g,1}$ is the one consisting of symplectic derivations with degree d. As was shown in [29], the Poincaré duality $H^* \cong H$ induces a canonical isomorphism

$$\mathfrak{h}_{g,1}(d) \cong \mathrm{Ker}([\ ,\ ] : H \otimes \mathcal{L}_{g,1}(d+1) \to \mathcal{L}_{g,1}(d+2)).$$

Now Garoufalidis and Levine proved in the above cited paper [6] that there is a short exact sequence

$$1 \longrightarrow \mathfrak{h}_{g,1}(d) \longrightarrow \mathrm{Aut}_0 N_{d+1} \longrightarrow \mathrm{Aut}_0 N_d \longrightarrow 1. \tag{3}$$

Remark 2.2. It may appear that the definition of the subgroup $\mathrm{Aut}'_0 N_d$ given in Definition 2.1 is more natural than that of $\mathrm{Aut}_0 N_d$. However this is not the case because of the following reason. Since the group N_d is defined as the quotient of Γ by its d-th commutator subgroup Γ_d and since the element ζ belongs to Γ_1, the condition of preserving the element ζ_d becomes vacuous on

$$\mathrm{Ker}(\mathrm{Aut}\, N_d \rightarrow \mathrm{Aut}\, N_{d-1}) \cong \mathrm{Hom}(H, \mathcal{L}_{g,1}(d))$$

(see [30] for the above isomorphism). The subgroup $\mathrm{Aut}_0 N_d$ is defined so as to eliminate this point. In particular, we have an isomorphism

$$\mathrm{Aut}'_0 N_d / \mathrm{Aut}_0 N_d \cong \mathrm{Hom}(H, \mathcal{L}_{g,1}(d))/\mathfrak{h}_{g,1}(d-1)$$

for any $d \geq 2$.

If we restrict the homomorphism ρ_{d+1} given in (2) to $\mathcal{M}_{g,1}(d)$, then we obtain a homomorphism

$$\tau_d = \rho_{d+1}|_{\mathcal{M}_{g,1}(d)} : \mathcal{M}_{g,1}(d) \longrightarrow \mathfrak{h}_{g,1}(d) \subset \mathrm{Hom}(H, \mathcal{L}_{g,1}(d+1))$$

which was introduced by Johnson [12][14] and is now called the d-th Johnson homomorphism (but with the target narrowed by [29]). See also Kawazumi's work [17] for his theory of Johnson *maps* which are certain extensions of Johnson homomorphism.

In our paper [29], we constructed an Sp-homomorphism

$$\mathrm{trace}(2d+1) : \mathfrak{h}_{g,1}(2d+1) \longrightarrow S^{2d+1} H$$

for any $d = 0, 1, 2, \cdots$ where $S^* H$ denotes the symmetric algebra of H. We proved that $\mathrm{trace}(2d+1)$ vanishes identically on $\mathrm{Image}\, \tau_{2d+1}$ while $\mathrm{trace}(2d+1) \otimes \mathbb{Q}$ is surjective for any $d \geq 1$. It follows that the cokernel of the homomorphism ρ_d becomes larger and larger as d tends to infinity.

On the other hand, there exists a natural homomorphism

$$\iota : \mathcal{M}_{g,1} \longrightarrow \mathcal{H}_{g,1} \tag{4}$$

from the mapping class group to the group $\mathcal{H}_{g,1}$ of Garoufalidis and Levine, already mentioned above. They proved the following remarkable theorem.

Theorem 2.3 (Garoufalidis-Levine [6], see also Habegger [8]). *For any d, there exists a homomorphism*

$$\tilde{\rho}_d : \mathcal{H}_{g,1} \longrightarrow \mathrm{Aut}_0 N_d$$

such that $\rho_d = \tilde{\rho}_d \circ \iota$. *It follows that the natural homomorphism* $\iota : \mathcal{M}_{g,1} \to \mathcal{H}_{g,1}$ *is injective because* ρ_∞ *is injective. Furthermore all the homomorphisms* $\tilde{\rho}_d$ *are surjective.*

They apply a theorem of Stallings in [38] to show the existence of $\tilde{\rho}_d$ and for the surjectivity of it, they use an argument of Kervaire-Milnor in [20]. They call their group $\mathcal{H}_{g,1}$ an *enlargement* of the mapping class group (see [25]).

In contrast to the case of the mapping class group, they point out that the induced homomorphism

$$\tilde{\rho}_\infty : \mathcal{H}_{g,1} \longrightarrow \varprojlim_{d \to \infty} \mathrm{Aut}_0 N_d$$

is *not injective* because the group $\Theta^3_{\mathbb{Z}}$ of homology cobordism classes of homology 3-spheres is contained in the center of the group $\mathcal{H}_{g,1}$ while all the homomorphisms $\tilde{\rho}_d$ vanish on it.

In relation to the problem of determining the image of $\tilde{\rho}_\infty$, Sakasai [36] considered the *acyclic closure* Γ^{acy} of Γ due to Levine (see [24]) and proved the following theorem (we refer to the above cited papers for the definition of the acyclic closure of a group because it is rather complicated and we do not use it in this paper).

Theorem 2.4 (Sakasai [36]). *There exists a natural homomorphism*

$$\rho^{acy} : \mathcal{H}_{g,1} \to \mathrm{Aut}\,\Gamma^{acy}$$

and its image can be identified as

$$\mathrm{Image}\,\rho^{acy} = \mathrm{Aut}_0\Gamma^{acy} = \{\varphi \in \mathrm{Aut}\,\Gamma^{acy}; \varphi(\zeta) = \zeta\}$$

where $\zeta \in \Gamma^{acy}$ *denotes the image of* $\zeta \in \Gamma$ *under the natural injection* $\Gamma \to \Gamma^{acy}$.

We refer to the above cited paper of Sakasai as well as [37] for further results concerning the structure of $\mathcal{H}_{g,1}$.

In view of the above results, it is a very important problem to analyze the structure of the groups $\mathrm{Aut}_0 N_d$.

§3. Rational forms of symplectic automorphism groups

In this section, we embed $\mathrm{Aut}_0 N_d$ into a linear algebraic group over $\mathbb{Q}$, which we denote by $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$, as a discrete and Zariski dense subgroup. We may call the latter group the *rational form* of the former group. We also describe the Lie algebra of $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ explicitly.

For this, we begin by recalling a few results from section 2 of [30] where we analyzed the structure of $\mathrm{Aut}\, N_d$. We showed that we can embed $\mathrm{Aut}\, N_d$ into the automorphism group $\mathrm{Aut}(N_d \otimes \mathbb{Q})$ of the Mal'cev completion $N_d \otimes \mathbb{Q}$ of N_d and also that the group $\mathrm{Aut}(N_d \otimes \mathbb{Q})$ has a natural structure of a linear algebraic group over $\mathbb{Q}$.

We shall show that almost the same argument applies to the subgroup $\mathrm{Aut}_0 N_d \subset \mathrm{Aut}\, N_d$ which consists of elements satisfying the symplectic constraint (see Definition 2.1).

First let $\mathrm{IAut}_0 N_d$ denote the kernel of the projection

$$\mathrm{Aut}_0 N_d \longrightarrow \mathrm{Aut}_0 N_1 \cong \mathrm{Sp}(2g, \mathbb{Z}).$$

It is the subgroup of $\mathrm{Aut}_0 N_d$ consisting of elements which act on the abelianization $H_1(N_d) \cong H$ trivially and we have a short exact sequence

$$1 \longrightarrow \mathrm{IAut}_0 N_d \longrightarrow \mathrm{Aut}_0 N_d \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1.$$

The short exact sequence (3) restricts to

$$0 \longrightarrow \mathfrak{h}_{g,1}(d) \longrightarrow \mathrm{IAut}_0 N_{d+1} \longrightarrow \mathrm{IAut}_0 N_d \longrightarrow 1 \tag{5}$$

which turns out to be a *central* extension. It follows that the group $\mathrm{IAut}_0 N_d$ is a *nilpotent* group for any d. Let $(\mathrm{IAut}_0 N_d) \otimes \mathbb{Q}$ denote its Mal'cev completion.

Now it was shown in [30] (Proposition 2.5) that there exists an embedding

$$i : N_d \rightarrow N_d \otimes \mathbb{Q} \tag{6}$$

of N_d into its Mal'cev completion $N_d \otimes \mathbb{Q}$ and this induces an injective homomorphism

$$i_* : \mathrm{Aut}\, N_d \longrightarrow \mathrm{Aut}(N_d \otimes \mathbb{Q})$$

whose image is Zariski dense. Furthermore $\mathrm{Aut}(N_d \otimes \mathbb{Q})$ is a linear algebraic group over $\mathbb{Q}$ so that the extension

$$1 \longrightarrow \mathrm{IAut}(N_d \otimes \mathbb{Q}) \longrightarrow \mathrm{Aut}(N_d \otimes \mathbb{Q}) \longrightarrow \mathrm{GL}(2g, \mathbb{Q}) \longrightarrow 1 \tag{7}$$

splits, where $\mathrm{IAut}(N_d \otimes \mathbb{Q})$ denotes the kernel of the natural projection $\mathrm{Aut}(N_d \otimes \mathbb{Q}) \rightarrow \mathrm{GL}(2g, \mathbb{Q})$ and it is the maximal normal unipotent subgroup.

We have a distinguished element $i(\zeta_d) \in N_d \otimes \mathbb{Q}$ and the subgroup

$$\mathrm{Aut}'_0(N_d \otimes \mathbb{Q}) = \{\varphi \in \mathrm{Aut}(N_d \otimes \mathbb{Q}); \varphi(i(\zeta_d)) = i(\zeta_d)\}$$

of $\mathrm{Aut}(N_d \otimes \mathbb{Q})$ is clearly an algebraic subgroup because the condition for an automorphism to preserve a particular element is an algebraic one. Furthermore $i_*(\mathrm{Aut}'_0 N_d)$ is contained in $\mathrm{Aut}'_0(N_d \otimes \mathbb{Q})$ as a Zariski dense subgroup.

Finally, we consider the natural projection

$$p : \mathrm{Aut}(N_{d+1} \otimes \mathbb{Q}) \longrightarrow \mathrm{Aut}(N_d \otimes \mathbb{Q})$$

and define $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ to be the image of $\mathrm{Aut}'_0(N_{d+1} \otimes \mathbb{Q})$ under p, namely we set

$$\mathrm{Aut}_0(N_d \otimes \mathbb{Q}) = p(\mathrm{Aut}'_0(N_{d+1} \otimes \mathbb{Q})).$$

Clearly the above projection p is a rational morphism. Hence we can conclude that $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ is algebraic because it is the image of the algebraic group $\mathrm{Aut}'_0(N_{d+1} \otimes \mathbb{Q})$ by a rational morphism. It is then easy to see that $i(\mathrm{Aut}_0 N_d)$ is Zariski dense in $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$.

Summarizing the above argument, we have the following proposition.

Proposition 3.1. *The group* $\mathrm{Aut}_0 N_d$ *can be embedded into a linear algebraic group* $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ *as a Zariski dense subgroup and we have a split short exact sequence*

$$1 \longrightarrow \mathrm{IAut}_0(N_d \otimes \mathbb{Q}) \longrightarrow \mathrm{Aut}_0(N_d \otimes \mathbb{Q}) \longrightarrow \mathrm{Sp}(2g, \mathbb{Q}) \longrightarrow 1$$

where $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ *is the maximal normal unipotent subgroup. Furthermore, the subgroup* $\mathrm{IAut}_0 N_d$ *is mapped to* $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ *as a Zariski dense subgroup and this gives an explicit construction of the Mal'cev completion* $(\mathrm{IAut}_0 N_d) \otimes \mathbb{Q}$ *of the nilpotent group* $\mathrm{IAut}_0 N_d$.

Next we describe the Lie algebra of $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$. As in the previous section, let $\mathfrak{h}_{g,1}$ denote the graded Lie algebra over $\mathbb{Z}$ consisting of all the symplectic derivations of the free Lie algebra $\mathcal{L}_{g,1}$. Also let

$$\mathfrak{h}^+_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$$

be the ideal of $\mathfrak{h}_{g,1}$ consisting of all the derivations with *positive* degrees. We consider the rational forms

$$\mathfrak{h}^{\mathbb{Q}}_{g,1} = \mathfrak{h}_{g,1} \otimes \mathbb{Q} = \bigoplus_{k=0}^{\infty} \mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$$

where $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k) = \mathfrak{h}_{g,1}(k) \otimes \mathbb{Q}$ and

$$\mathfrak{h}^{\mathbb{Q}+}_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{h}^{\mathbb{Q}}_{g,1}(k).$$

In particular, we have

$$\mathfrak{h}_{g,1}^{\mathbb{Q}}(0) = \mathfrak{sp}(2g, \mathbb{Q}), \quad \mathfrak{h}_{g,1}^{\mathbb{Q}}(1) = \Lambda^3 H_{\mathbb{Q}}.$$

Definition 3.2. For each $d = 1, 2, \cdots$, we define the truncated Lie algebras, denoted by $\mathfrak{h}_{g,1}^{\mathbb{Q}}[d]$ and $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$, by setting

$$\mathfrak{h}_{g,1}^{\mathbb{Q}}[d] = \mathfrak{h}_{g,1}^{\mathbb{Q}}/I(d)$$
$$\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d] = \mathfrak{h}_{g,1}^{\mathbb{Q}+}/I(d)$$

where $I(d)$ denotes the ideal consisting of derivations with degree $\geq d$. Additively, we can write

$$\mathfrak{h}_{g,1}^{\mathbb{Q}}[d] = \bigoplus_{k=0}^{d-1} \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$$
$$\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d] = \bigoplus_{k=1}^{d-1} \mathfrak{h}_{g,1}^{\mathbb{Q}}(k).$$

Theorem 3.3. *The Lie algebras of the algebraic groups* $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ *and* $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ *are isomorphic to the truncated Lie algebras*

$$\mathfrak{h}_{g,1}^{\mathbb{Q}}[d], \quad \mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$$

respectively.

Before proving the above result, we prepare a few facts. First of all, let $\mathfrak{n}_d^{\mathbb{Q}}$ denote the Lie algebra of the Mal'cev completion $N_d \otimes \mathbb{Q}$ of the free nilpotent group N_d. Then, as is well known, the exponential map induces a natural bijection

$$\exp : \mathfrak{n}_d^{\mathbb{Q}} \cong N_d \otimes \mathbb{Q}.$$

This then gives rise to an isomorphism

$$\mathrm{Aut}\, \mathfrak{n}_d^{\mathbb{Q}} \cong \mathrm{Aut}(N_d \otimes \mathbb{Q}) \tag{8}$$

between the corresponding automorphism groups. Also the restriction of (8) gives rise to an isomorphism

$$\mathrm{IAut}\, \mathfrak{n}_d^{\mathbb{Q}} \cong \mathrm{IAut}(N_d \otimes \mathbb{Q}) \tag{9}$$

where $\mathrm{IAut}\, \mathfrak{n}_d^{\mathbb{Q}}$ is defined by the following exact sequence

$$1 \longrightarrow \mathrm{IAut}\, \mathfrak{n}_d^{\mathbb{Q}} \longrightarrow \mathrm{Aut}\, \mathfrak{n}_d^{\mathbb{Q}} \longrightarrow \mathrm{GL}(2g, \mathbb{Q}) \longrightarrow 1$$

which is essentially the same as (7). These bijections as well as isomorphisms are compatible with respect to the projection $N_d \otimes \mathbb{Q} \to N_{d-1} \otimes \mathbb{Q}$ so that the following diagram is commutative

(10)

$$\begin{array}{ccccc}
\mathrm{Hom}(H_{\mathbb{Q}}, \mathcal{L}_{g,1}^{\mathbb{Q}}(d)) & \xrightarrow{\text{inj}} & \mathrm{Aut}\,\mathfrak{n}_d^{\mathbb{Q}} & \xrightarrow{\text{surj}} & \mathrm{Aut}\,\mathfrak{n}_{d-1}^{\mathbb{Q}} \\
\Vert & & \exp\downarrow\cong & & \exp\downarrow\cong \\
\mathrm{Hom}(H_{\mathbb{Q}}, \mathcal{L}_{g,1}^{\mathbb{Q}}(d)) & \xrightarrow{\text{inj}} & \mathrm{Aut}(N_d \otimes \mathbb{Q}) & \xrightarrow{\text{surj}} & \mathrm{Aut}(N_{d-1} \otimes \mathbb{Q}).
\end{array}$$

Now let us consider the particular element

$$\tilde{\omega}_d := \exp^{-1}(i(\zeta_d)) \in \mathfrak{n}_d^{\mathbb{Q}}$$

and make the following definition.

Definition 3.4. We set

$$\begin{aligned}
\mathrm{Aut}_0'\,\mathfrak{n}_d^{\mathbb{Q}} &= \{\varphi \in \mathrm{Aut}\,\mathfrak{n}_d^{\mathbb{Q}} : \varphi(\tilde{\omega}_d) = \tilde{\omega}_d\} \\
\mathrm{Aut}_0\,\mathfrak{n}_d^{\mathbb{Q}} &= p(\mathrm{Aut}_0'\,\mathfrak{n}_{d+1}^{\mathbb{Q}})
\end{aligned}$$

where

$$p : \mathrm{Aut}\,\mathfrak{n}_{d+1}^{\mathbb{Q}} \longrightarrow \mathrm{Aut}\,\mathfrak{n}_d^{\mathbb{Q}}$$

denotes the natural projection.

Then, all the above argument can be adapted to the cases of Aut_0' and Aut_0. In particular, the restrictions of (8) and (9) induce canonical isomorphisms

$$\mathrm{Aut}_0\,\mathfrak{n}_d^{\mathbb{Q}} \cong \mathrm{Aut}_0(N_d \otimes \mathbb{Q}) \tag{11}$$

$$\mathrm{IAut}_0\,\mathfrak{n}_d^{\mathbb{Q}} \cong \mathrm{IAut}_0(N_d \otimes \mathbb{Q}), \tag{12}$$

respectively.

As a final step towards the proof of Theorem 3.3, we recall a classical result (see [27]) that the Lie algebra of the Mal'cev completion $N_d \otimes \mathbb{Q}$ of the free nilpotent group N_d is isomorphic (though not canonically) to the free nilpotent Lie algebra which is defined as the truncated Lie algebra

$$\mathcal{L}_{g,1}^{\mathbb{Q}}[d] = \mathcal{L}_{g,1}^{\mathbb{Q}} / J_d$$

where J_d denotes the ideal of the free Lie algebra $\mathcal{L}_{g,1}^{\mathbb{Q}}$ consisting of elements with degree $> d$. Thus additively we can write

$$\mathcal{L}_{g,1}^{\mathbb{Q}}[d] = \bigoplus_{k=1}^{d} \mathcal{L}_{g,1}^{\mathbb{Q}}(k).$$

In this Lie algebra, we have a particular element, namely the symplectic class

$$\omega_0 \in \mathcal{L}_{g,1}^{\mathbb{Q}}(2) \cong \Lambda^2 H_{\mathbb{Q}} \subset \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \quad (d \geq 2).$$

Lemma 3.5. *For any $d \geq 2$, the natural homomorphism*

$$\operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \longrightarrow \operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d-1]$$

is surjective and we have the following exact sequence

$$1 \longrightarrow \operatorname{Hom}(H_{\mathbb{Q}}, \mathcal{L}_{g,1}^{\mathbb{Q}}(d)) \longrightarrow \operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \longrightarrow \operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d-1] \longrightarrow 1.$$

Proof. If $d = 1$, $\mathcal{L}_{g,1}^{\mathbb{Q}}[1] = H_{\mathbb{Q}}$ so that

$$\operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[1] = \mathrm{GL}(2g, \mathbb{Q}).$$

Since the abelianization of the Lie algebra $\mathcal{L}_{g,1}^{\mathbb{Q}}[d]$ can be identified with $H_{\mathbb{Q}}$, we have a representation

$$r : \operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \longrightarrow \mathrm{GL}(2g, \mathbb{Q}).$$

On the other hand, clearly the action of $\mathrm{GL}(2g, \mathbb{Q})$ on $H_{\mathbb{Q}}$ extends canonically to that on $\mathcal{L}_{g,1}^{\mathbb{Q}}[d]$ for any d so that we have a canonically *split extension*

$$1 \longrightarrow \operatorname{IAut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \longrightarrow \operatorname{Aut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \xrightarrow{r} \mathrm{GL}(2g, \mathbb{Q}) \longrightarrow 1$$

where $\operatorname{IAut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d]$ denotes the kernel of the representation r. Hence, for the surjectivity, it is enough to show that the natural homomorphism

$$\operatorname{IAut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d] \longrightarrow \operatorname{IAut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d-1]$$

is surjective. Any element $\varphi \in \operatorname{IAut} \mathcal{L}_{g,1}^{\mathbb{Q}}[d-1]$ gives rise to an *endomorphism* of $\mathcal{L}_{g,1}^{\mathbb{Q}}[d]$ which we denote by $\tilde{\varphi}$. Therefore it suffices to prove that this extended endomorphism $\tilde{\varphi}$ is in fact an isomorphism. For this, it is enough to show that for any element $u \in \mathcal{L}_{g,1}^{\mathbb{Q}}(1) = H_{\mathbb{Q}}$, there exists an element $\xi \in \mathcal{L}_{g,1}^{\mathbb{Q}}[d]$ such that

$$\tilde{\varphi}(\xi) = u.$$

Since φ is an isomorphism by the assumption, there exists an element $\xi' \in \mathcal{L}_{g,1}^{\mathbb{Q}}[d-1]$ such that

$$\varphi(\xi') = u.$$

Then, for some element $\eta \in \mathcal{L}^{\mathbb{Q}}_{g,1}(d)$, we have

$$\tilde{\varphi}(\xi') = u + \eta.$$

It is easy to see that the action of any element in $\mathrm{IAut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ on $\mathcal{L}^{\mathbb{Q}}_{g,1}(d)$ is trivial. If we set $\xi = \xi' - \eta$, then

$$\tilde{\varphi}(\xi) = u + \eta - \eta = u.$$

Therefore $\tilde{\omega}$ is an isomorphism as required.

Finally, the fact that the kernel of the homomorphism

$$\mathrm{Aut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d] \to \mathrm{Aut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d-1]$$

is isomorphic to $\mathrm{Hom}(H_{\mathbb{Q}}, \mathcal{L}^{\mathbb{Q}}_{g,1}(d))$ can be seen easily, where any element f in this module acts on $\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ by

$$H_{\mathbb{Q}} \ni u \longmapsto u + f(u) \in \mathcal{L}^{\mathbb{Q}}_{g,1}[d].$$

This completes the proof. Q.E.D.

The following is the main technical fact which will be used in the proof of Theorem 3.3.

Lemma 3.6. *Let*

$$\xi = \omega_0 + \xi_3 + \cdots + \xi_d \in \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$$

be any element with $\xi_k \in \mathcal{L}^{\mathbb{Q}}_{g,1}(k)$ $(k = 3, \cdots, d)$. Then there exists an element

$$\varphi \in \mathrm{IAut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d-1]$$

such that

$$\tilde{\varphi}(\omega_0) = \xi$$

where $\tilde{\varphi} \in \mathrm{IAut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is the extension of φ as in the proof of Lemma 3.5.

Proof. We use the induction on d. The claim clearly holds for $d = 2$, because $\xi = \omega_0$ in this case so that we can set $\varphi = \mathrm{id}$. Suppose that it holds for the cases $2, \cdots, d-1$, and we prove the case d. By the induction assumption, there exists an element $\psi \in \mathrm{IAut}\,\mathcal{L}_{g,1}[d-2]$ such that

$$\tilde{\psi}(\omega_0) = \bar{\xi} \in \mathcal{L}^{\mathbb{Q}}_{g,1}[d-1]$$

where $\tilde{\psi} \in \mathrm{IAut}\,\mathcal{L}_{g,1}[d-1]$ denotes the extension of ψ as in the proof of Lemma 3.5 and

$$\bar{\xi} = \omega_0 + \xi_3 + \cdots + \xi_{d-1}.$$

Now consider the element

$$\tilde{\tilde{\psi}} \in \mathrm{IAut}\,\mathcal{L}_{g,1}[d]$$

which denotes further extension of $\tilde{\psi}$. Then we see that

$$\eta := \tilde{\tilde{\psi}}(\omega_0) - \xi \in \mathcal{L}^{\mathbb{Q}}_{g,1}(d)$$

because

$$\tilde{\tilde{\psi}}(\omega_0) \equiv \bar{\xi} \bmod \mathcal{L}^{\mathbb{Q}}_{g,1}(d).$$

Since the bracket operation

$$H_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{g,1}(d-1) \longrightarrow \mathcal{L}^{\mathbb{Q}}_{g,1}(d)$$

is surjective, we can write

$$\eta = \sum_{i=1}^{g} ([x_i, \gamma_i] + [y_i, \delta_i]) \quad (\gamma_i, \delta_i \in \mathcal{L}^{\mathbb{Q}}_{g,1}(d-1))$$

where $x_1, \cdots, x_g, y_1, \cdots, y_g$ is a symplectic basis of H. Now define an element $\varphi \in \mathrm{IAut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d-1]$ by setting

$$\varphi(x_i) = \tilde{\psi}(x_i) + \delta_i, \quad \varphi(y_i) = \tilde{\psi}(y_i) - \gamma_i$$

and let $\tilde{\varphi} \in \mathrm{IAut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ denotes its extension. Then we have

$$\begin{aligned}
\tilde{\varphi}(\omega_0) &= \sum_{i=1}^{g} \left\{ [\tilde{\tilde{\psi}}(x_i) + \delta_i, \tilde{\tilde{\psi}}(y_i) - \gamma_i] \right\} \\
&= \tilde{\tilde{\psi}}(\omega_0) - \sum_{i=1}^{g} \{[x_i, \gamma_i] - [\delta_i, y_i]\} \\
&= \xi + \eta - \eta = \xi
\end{aligned}$$

because

$$\tilde{\tilde{\psi}}(x_i) \equiv x_i \bmod J_1, \quad \tilde{\tilde{\psi}}(y_i) \equiv y_i \bmod J_1.$$

Thus the claim holds for the case of d and this completes the proof.
Q.E.D.

Definition 3.7. We set

$$\operatorname{Aut}'_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d] = \{\varphi \in \operatorname{Aut} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]; \varphi(\omega_0) = \omega_0\}$$
$$\operatorname{Aut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d] = p(\operatorname{Aut}'_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d+1])$$

where

$$p : \operatorname{Aut} \mathcal{L}^{\mathbb{Q}}_{g,1}[d+1] \longrightarrow \operatorname{Aut} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$$

denotes the natural projection.

Proposition 3.8. *The group* $\operatorname{Aut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ *is a linear algebraic group over* $\mathbb{Q}$ *which can be expressed as the following canonically split extension*

$$1 \longrightarrow \operatorname{IAut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d] \longrightarrow \operatorname{Aut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d] \stackrel{r}{\longrightarrow} \operatorname{Sp}(2g, \mathbb{Q}) \longrightarrow 1.$$

Furthermore the Lie algebras of $\operatorname{Aut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ *and* $\operatorname{IAut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ *are canonically isomorphic to* $\mathfrak{h}^{\mathbb{Q}}_{g,1}[d]$ *and* $\mathfrak{h}^{\mathbb{Q}+}_{g,1}[d]$, *respectively.*

Proof. For the former half of the claim, almost the same arguments as in the case of $\operatorname{Aut}_0(N_d \otimes \mathbb{Q})$ (see those arguments given before the statement of Proposition 3.1) apply to the present case as well. In fact, this Lie algebra case is even simpler than the group case. In fact, it is easy to see first that $\operatorname{Aut} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is a linear algebraic group over $\mathbb{Q}$. Then $\operatorname{Aut}'_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is shown to be an algebraic subgroup because it consists of elements which fix a particular element ω_0. Then, since the natural projection p is rational, its image $\operatorname{Aut}_0 \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is also algebraic. The splitting is canonical because $\operatorname{Sp}(2g, \mathbb{Q})$ acts naturally on $\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ as an automorphism group preserving the symplectic class ω_0.

Now we prove the latter half of the claim. By a well known fact, the Lie algebra of the Lie group $\operatorname{Aut} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is isomorphic to that of derivations of the Lie algebra $\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ which we denote by $\operatorname{Der} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$. Since $\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is a graded Lie algebra, the above derivations are also graded and we can write

$$\operatorname{Der} \mathcal{L}^{\mathbb{Q}}_{g,1}[d] = \bigoplus_{k=0}^{d} \operatorname{Hom}(H_{\mathbb{Q}}, \mathcal{L}_{g,1}(k)).$$

Furthermore we have the following canonically split exact sequence of Lie algebras

$$0 \longrightarrow \operatorname{IDer} \mathcal{L}^{\mathbb{Q}}_{g,1}[d] \longrightarrow \operatorname{Der} \mathcal{L}^{\mathbb{Q}}_{g,1}[d] \longrightarrow \mathfrak{gl}(2g, \mathbb{Q}) \longrightarrow 0$$

where $\operatorname{IDer} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ denotes the kernel of the natural homomorphism

$$\operatorname{Der} \mathcal{L}^{\mathbb{Q}}_{g,1}[d] \longrightarrow \mathfrak{gl}(2g, \mathbb{Q})$$

and it is the Lie algebra of $\mathrm{IAut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$.

Keeping in mind Definition 3.7, we define

$$\begin{aligned}\mathrm{Der}'_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d] &= \{D \in \mathrm{IDer}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d] : D(\omega_0) = 0\}\\ \mathrm{Der}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d] &= p(\mathrm{IDer}'_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d+1])\end{aligned}$$

where

$$p : \mathrm{IDer}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d+1] \longrightarrow \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$$

denotes the natural projection. Then it is easy to see that $\mathrm{Der}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is the Lie algebra of $\mathrm{Aut}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ and similarly $\mathrm{IDer}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ is the Lie algebra of $\mathrm{IAut}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$. Since

$$\mathrm{Der}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d] = \mathfrak{h}^{\mathbb{Q}}_{g,1}[d], \quad \mathrm{IDer}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d] = \mathfrak{h}^{\mathbb{Q}+}_{g,1}[d]$$

by definition, the proof is finished.

Q.E.D.

Proof of Theorem 3.3. First of all, $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ and $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ are isomorphic to $\mathrm{Aut}_0\,\mathfrak{n}^{\mathbb{Q}}_d$ and $\mathrm{IAut}_0\,\mathfrak{n}^{\mathbb{Q}}_d$ by (11) and (12), respectively.

On the other hand, it is a classical result that the Lie algebra $\mathfrak{n}^{\mathbb{Q}}_d$ is isomorphic to the truncated Lie algebra $\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ (see [27]). Although there is no canonical isomorphism between the two Lie algebras, it is easy to see that any isomorphism

$$\iota : \mathfrak{n}^{\mathbb{Q}}_d \xrightarrow{\cong} \mathcal{L}^{\mathbb{Q}}_{g,1}[d]$$

satisfies the condition

$$\iota(\tilde{\omega}_d) = \omega_0 + \xi_3 + \cdots + \xi_d$$

for some $\xi_k \in \mathcal{L}_{g,1}(k)$ $(k = 3, \cdots, d)$. By Lemma 3.6, there exists an isomorphism $\varphi \in \mathrm{Aut}\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ such that

$$\varphi(\omega_0) = \omega_0 + \xi_3 + \cdots + \xi_d.$$

Hence, replacing ι by $\varphi^{-1} \circ \iota$, we may assume

$$\iota(\tilde{\omega}_d) = \omega_0.$$

It follows that

$$\mathrm{Aut}_0\,\mathfrak{n}^{\mathbb{Q}}_d \cong \mathrm{Aut}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d].$$

Finally, we know by Proposition 3.8 that the Lie algebras of $\mathrm{Aut}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ and $\mathrm{IAut}_0\,\mathcal{L}^{\mathbb{Q}}_{g,1}[d]$ are canonically isomorphic to $\mathfrak{h}^{\mathbb{Q}}_{g,1}[d]$ and

$\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$, respectively. Summarizing the above, we can now conclude that the Lie algebras of $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ and $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ are isomorphic to $\mathfrak{h}_{g,1}^{\mathbb{Q}}[d]$ and $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$, respectively, as required.

This completes the proof.

Q.E.D.

Remark 3.9. As mentioned above, there is no canonical isomorphism between the two Lie algebras $\mathfrak{n}_d^{\mathbb{Q}}$ and $\mathcal{L}_{g,1}^{\mathbb{Q}}[d]$. The latter Lie algebra is canonically isomorphic to the graded Lie algebra associated to the lower central series of the Lie group $N_d \otimes \mathbb{Q}$. However if we consider the analogues of the above Lie algebras corresponding to a *closed surface*, then according to Hain [10] each complex structure on the surface induces a canonical isomorphism between the two Lie algebras tensored with $\mathbb{C}$, namely $\bar{\mathfrak{n}}_d^{\mathbb{C}}$ and $\mathcal{L}_g^{\mathbb{C}}[d]$. In fact, Hain showed this at the level of the Torelli group by making use of Hodge theory.

§4. Group version of the trace maps

In this section, we give a definition of the *group version* of the trace maps given in [29].

As was already mentioned in section 2, the trace maps consist of a series of certain homomorphisms

$$\mathrm{trace}(2k+1) : \mathfrak{h}_{g,1}(2k+1) \longrightarrow S^{2k+1}H \quad (k = 0, 1, 2, \cdots)$$

where $S^{2k+1}H$ denotes the $(2k+1)$-st symmetric power of H. As before, if we denote by $H_{\mathbb{Q}}$ the rational homology group $H \otimes \mathbb{Q} = H_1(\Sigma_g^0; \mathbb{Q})$ and also by $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ the corresponding rational form of $\mathfrak{h}_{g,1}$, then we have the rational trace maps

$$\mathrm{trace}(2k+1) : \mathfrak{h}_{g,1}^{\mathbb{Q}}(2k+1) \longrightarrow S^{2k+1}H_{\mathbb{Q}} \quad (k = 0, 1, 2, \cdots)$$

which were proved to be surjective for all k. The first trace, namely trace(1) is nothing but the contraction

$$\mathfrak{h}_{g,1}^{\mathbb{Q}}(1) \cong \Lambda^3 H_{\mathbb{Q}} \rightarrow H_{\mathbb{Q}}$$

which is a unique (up to scalars) Sp-equivariant non-trivial morphism. The totality of these trace maps gives rise to an abelian quotient

$$\mathrm{trace} : \mathfrak{h}_{g,1}^{\mathbb{Q}+} \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}$$

of the ideal $\mathfrak{h}_{g,1}^{\mathbb{Q}+}$.

If we consider the truncated Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$, then we obtain an abelian quotient

$$\text{trace} : \mathfrak{h}_{g,1}^{\mathbb{Q}+}[d] \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}} \tag{13}$$

where ℓ denotes the largest integer such that $2\ell + 1 \leq d - 1$.

On the other hand, we know by Theorem 3.3 that the Lie algebra of $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ is isomorphic to $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$. Since the commutator subgroup of any Lie group corresponds to the commutator ideal of the associated Lie algebra, and since the real form $N_d \otimes \mathbb{R}$ of $N_d \otimes \mathbb{Q}$ is simply connected, we have a canonical isomorphism

$$H_1(\mathrm{IAut}_0(N_d \otimes \mathbb{Q})) \cong H_1(\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]). \tag{14}$$

By combining (13) with (14), we obtain a homomorphism

$$\widetilde{\text{trace}} : \mathrm{IAut}_0 N_d \overset{i}{\subset} \mathrm{IAut}_0(N_d \otimes \mathbb{Q}) \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}}.$$

In particular, we call the homomorphism

$$\widetilde{\text{trace}}(2k+1) : \mathrm{IAut}_0 N_d \longrightarrow S^{2k+1} H_{\mathbb{Q}} \quad (2k+1 \leq d-1)$$

the group version of the trace map. If we further use Proposition 3.1, then we can conclude that the above homomorphism can be extended to a crossed homomorphism

$$\widetilde{\text{trace}}(2k+1) : \mathrm{Aut}_0 N_d \longrightarrow S^{2k+1} H_{\mathbb{Q}} \quad (2k+1 \leq d-1)$$

on the whole group $\mathrm{Aut}_0 N_d$. Thus we obtain the following theorem.

Theorem 4.1. *For any $d \geq 2$, let ℓ denote the largest integer such that $2\ell + 1 \leq d - 1$. Then the group version of traces $\widetilde{\text{trace}}(2k+1)$ gives rise to a homomorphism*

$$\mathrm{IAut}_0 N_d \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}}$$

whose induced homomorphism on the Mal'cev completion $(\mathrm{IAut}_0 N_d) \otimes \mathbb{Q} \cong \mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ is surjective. Furthermore the above homomorphism can be extended to the whole group $\mathrm{Aut}_0 N_d$ as a crossed homomorphism so that we obtain a homomorphism

$$\mathrm{Aut}_0 N_d \longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q}).$$

Conjecture 4.2. *The above theorem gives the abelianization of the nilpotent group* $\mathrm{IAut}_0 N_d$ *modulo torsion. More precisely*

$$H_1(\mathrm{IAut}_0 N_d; \mathbb{Q}) \cong \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}}.$$

As an evidence for the above conjecture, we would like to mention Kassabov's work in [15] which treated the case $\mathrm{IAut}\, N_d$ where there is no symplectic constraint. Here we also remark that suitably modified version of the results of this section are valid for $\mathrm{Aut}\, N_d$ and $\mathrm{IAut}\, N_d$ without the symplectic constraint.

If the above conjecture were true, then we could conclude that

$$H_1(\mathrm{Aut}_0 N_d; \mathbb{Q}) = 0.$$

This follows from the following exact sequence

$$\cdots \longrightarrow H_1(\mathrm{IAut}_0 N_d; \mathbb{Q})_{\mathrm{Sp}} \longrightarrow H_1(\mathrm{Aut}_0 N_d; \mathbb{Q}) \longrightarrow H_1(\mathrm{Sp}(2g; \mathbb{Q}); \mathbb{Q}) = 0$$

because clearly we have

$$H_1 \left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}} \right)_{\mathrm{Sp}} = 0.$$

Remark 4.3. The results of this section are valid for the closed surface case as well. More precisely, let $\bar{\Gamma} = \pi_1 \Sigma_g$ be the fundamental group of Σ_g and let $\bar{N}_d$ denote its d-th nilpotent quotient. Then according to Labute [23], the graded Lie algebra associated to the lower central series of $\bar{\Gamma}$ is given by

$$\mathcal{L}_g := \mathcal{L}_{g,1}/(\omega_0)$$

where (ω_0) denotes the ideal generated by the symplectic class ω_0. Let

$$\mathrm{Out}^+ \bar{N}_d$$

denote the orientation preserving outer automorphism group of $\bar{N}_d$ and we define a subgroup of it by setting

$$\mathrm{Out}_0^+ \bar{N}_d = \mathrm{Image}\ (p : \mathrm{Out}^+ \bar{N}_{d+1} \to \mathrm{Out}^+ \bar{N}_d)$$

where p denotes the natural homomorphism. Then we have a short exact sequence

$$1 \longrightarrow \mathrm{IOut}_0^+ \bar{N}_d \longrightarrow \mathrm{Out}_0^+ \bar{N}_d \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1.$$

On the other hand, there is a surjective homomorphism

$$\mathfrak{h}_{g,1} \longrightarrow \mathfrak{h}_g := \operatorname{Der} \mathcal{L}_g / \mathrm{Inn} \tag{15}$$

of Lie algebras, where Inn denotes the Lie subalgebra consisting of all the inner derivations. We can also consider truncated versions of these Lie algebras. Now it can be shown that the traces vanish on the kernel of the above homomorphism (15). Then suitably modified versions of the arguments in this paper imply the following result.

Namely, the group version of traces $\widetilde{\mathrm{trace}}(2k+1)$ give rise to a homomorphism

$$\mathrm{Out}_0^+ \bar{N}_d \longrightarrow \left((\Lambda^3 H_{\mathbb{Q}} / H_{\mathbb{Q}}) \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

whose induced homomorphism on $\mathrm{Out}_0^+(\bar{N}_d \otimes \mathbb{Q})$ is surjective.

In this way, we can avoid rather technical arguments related to the condition for automorphisms of free nilpotent groups or Lie algebras to preserve certain particular elements related to the symplectic class ω_0. However we treated the case of compact surfaces with boundary because of the following merits. One is that it is more suited to the group $\mathcal{H}_{g,1}$ and the other is that it is much more convenient for explicit computations.

§5. A representation of $\mathcal{H}_{g,1}$

As was already mentioned in section 2, Garoufalidis and Levine [6] (see also Habegger [8]) proved that the homomorphism

$$\tilde{\rho}_d : \mathcal{H}_{g,1} \longrightarrow \mathrm{Aut}_0 N_d$$

is *surjective* for all d. On the other hand, if we combine Proposition 3.1 with Theorem 4.1, we can conclude that there exists a homomorphism

$$\mathrm{Aut}_0 N_d \longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

whose image is Zariski dense. Since these homomorphisms are compatible with respect to d, by letting d go to the infinity, we obtain the following result.

Theorem 5.1. *There exists a homomorphism*

$$\tilde{\rho} : \mathcal{H}_{g,1} \longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q}) \tag{16}$$

whose image is Zariski dense.

Let $\mathcal{IH}_{g,1}$ denote the kernel of the homomorphism

$$\mathcal{H}_{g,1} \longrightarrow \mathrm{Sp}(2g, \mathbb{Q})$$

which is the analogue of the Torelli group in the context of the group of homology cylinders. Then the restriction of the representation $\tilde{\rho}$ to the subgroup $\mathcal{IH}_{g,1}$ gives rise to a homomorphism

$$\tilde{\rho} : \mathcal{IH}_{g,1} \longrightarrow \Lambda^3 H_{\mathbb{Q}} \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \tag{17}$$

whose image is Zariski dense. Hence we obtain the following corollary.

Corollary 5.2. *The subgroup $\mathcal{IH}_{g,1}$ of $\mathcal{H}_{g,1}$ is not finitely generated because the rank of its abelianization is already infinite.*

Here we mention that Sakasai [37] proved that the abelianization of the IA automorphism group $\mathrm{IAut}\, F_n^{\mathrm{acy}}$ of the acyclic closure F_n^{acy} of a free group F_n of rank $n \geq 2$ has infinite rank.

The above result shows a sharp contrast with the case of the Torelli group $\mathcal{I}_{g,1}$ which is known to be *finitely generated* by Johnson [13] for any $g \geq 3$.

Question 5.3. *Does the homomorphism $\tilde{\rho}$ in* (17) *give the abelianization of the group $\mathcal{IH}_{g,1}$ modulo torsion?*

This question should be very difficult to answer. If it would be affirmatively answered, then it would follow that

$$H_1(\mathcal{H}_{g,1}; \mathbb{Q}) = 0$$

because of the following exact sequence

$$\cdots \longrightarrow H_1(\mathcal{IH}_{g,1}; \mathbb{Q})_{\mathrm{Sp}} \longrightarrow H_1(\mathcal{H}_{g,1}; \mathbb{Q}) \longrightarrow H_1(\mathrm{Sp}(2g; \mathbb{Z}); \mathbb{Q}) = 0.$$

§6. Cohomology classes of $\mathcal{H}_{g,1}$

The homomorphism

$$\tilde{\rho} : \mathcal{H}_{g,1} \longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

given in Theorem 5.1 induces the following homomorphism in cohomology

$$(18)\quad \mathbb{Q}[c_1, c_3, \cdots] \otimes H^*\left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}\right)^{\mathrm{Sp}} \longrightarrow H^*(\mathcal{H}_{g,1}; \mathbb{Q})$$

where

$$\mathbb{Q}[c_1, c_3, \cdots] = \lim_{g\to\infty} H^*(\mathrm{Sp}(2g, \mathbb{Q}); \mathbb{Q})$$

corresponds to the stable cohomology of $\mathrm{Sp}(2g, \mathbb{Q})$ determined by Borel. Thus we obtain many cohomology classes of the group $\mathcal{H}_{g,1}$ and it is an important problem to determine how non-trivial classes they are.

The mapping class group $\mathcal{M}_{g,1}$ is a subgroup of $\mathcal{H}_{g,1}$ by [6] so that we can consider the restriction of the above cohomology classes to $H^*(\mathcal{M}_{g,1}; \mathbb{Q})$. Since we know that the traces vanish identically on $\mathcal{M}_{g,1}$, it is enough to consider the following composition of homomorphisms.

$$(19)\quad \mathbb{Q}[c_1, c_3, \cdots] \otimes H^*(\Lambda^3 H_{\mathbb{Q}})^{\mathrm{Sp}} \longrightarrow H^*(\mathcal{H}_{g,1}; \mathbb{Q}) \longrightarrow H^*(\mathcal{M}_{g,1}; \mathbb{Q}).$$

It was proved in [31] that the factor $H^*(\Lambda^3 H_{\mathbb{Q}})^{\mathrm{Sp}}$ gives rise to all the Mumford-Morita-Miller classes in $H^*(\mathcal{M}_{g,1}; \mathbb{Q})$. Then by making use of Kawazumi's result in [16], an explicit formula for the above homomorphism was given in [18][19] thereby it was proved that its image is precisely the subalgebra generated by the above classes. Thus we can conclude that the Mumford-Morita-Miller classes are already defined as cohomology classes in $H^*(\mathcal{H}_{g,1}; \mathbb{Q})$.

On the other hand, the Grothendieck-Riemann-Roch theorem (or the Atiyah-Singer index theorem for families), applied to the Hodge bundle over the moduli space of curves, implies that there are close relations between the Chern classes of the Hodge bundle, which come from $\mathbb{Q}[c_1, c_3, \cdots]$ above, and the Mumford-Morita-Miller classes of *odd* indices (see [34][28]). Thus we have the following question.

Question 6.1. *Do the aforementioned relations between the Chern classes of the Hodge bundle and the odd Mumford-Morita-Miller classes, which hold at the level of $H^*(\mathcal{M}_{g,1}; \mathbb{Q})$, continue to hold in $H^*(\mathcal{H}_{g,1}; \mathbb{Q})$?*

There is a natural homomorphism

$$\mathcal{H}_{g,1} \longrightarrow \mathcal{H}_{g+1,1}$$

and all the cohomology classes defined above are *stable* with respect to g. Thus, keeping in mind Harer's stability theorem [11] as well as recent remarkable result of Madsen and Weiss [26] for the cohomology of the mapping class groups, we can ask the following.

Question 6.2. *Does the (rational) cohomology groups of $\mathcal{H}_{g,1}$ stabilize with respect to g? If so, what is the stable cohomology of $\mathcal{H}_{g,1}$?*

§7. Group of homology cobordism classes of homology 3-spheres

In this section, we consider a series of certain elements in the second cohomology of $\mathrm{Aut}_0 N_d$ and $\mathcal{H}_{g,1}$. These elements for the latter group are the simplest special cases of the construction of elements of $H^*(\mathcal{H}_{g,1};\mathbb{Q})$ given in the previous section. They are also the group version of previously defined classes given in the context of Lie algebras which we now recall.

We defined in [32][33] a series of cohomology classes

$$t_{2k+1} \in H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})$$

by setting

$$t_{2k+1} = \mathrm{trace}(2k+1)^*(\iota_{2k+1})$$

where $\mathrm{trace}(2k+1) : \mathfrak{h}_{g,1}^{\mathbb{Q}} \to S^{2k+1}H_{\mathbb{Q}}$ is the $(2k+1)$-st trace map and $\iota_{2k+1} \in H^2(S^{2k+1}H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q}$ is the generator.

These cohomology classes t_{2k+1} correspond to certain elements

$$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2};\mathbb{Q})$$

via a theorem of Kontsevich [21][22] (see also Conant and Vogtmann [1]) which relates $H^*(\mathfrak{h}_{g,1}^{\mathbb{Q}})$ with the rational homology groups of the outer automorphism groups $\mathrm{Out}\, F_n$ of free groups F_n with $n \geq 2$, or the quotient space of the Outer Space of Culler and Vogtmann [3] by $\mathrm{Out}\, F_n$. In particular, the non-triviality of t_{2k+1} is equivalent to that of μ_k. The non-triviality of t_3 was proved in [32] and that of both elements μ_1 and μ_2 was proved by Conant and Vogtmann [1] where they interpreted and extended these classes in the context of the geometry of the Outer Space.

However the non-triviality of higher classes t_{2k+1}, μ_k for $k \geq 3$ is still unknown at present.

Now we define the group version of the above classes as follows. By Theorem 4.1, we have a crossed homomorphism

$$\widetilde{\mathrm{trace}}(2k+1) : \mathrm{Aut}_0\, N_d \longrightarrow S^{2k+1}H_{\mathbb{Q}} \quad (2k+1 \leq d-1).$$

Definition 7.1. We define

$$\tilde{t}_{2k+1} \in H^2(\mathrm{Aut}_0\, N_d; \mathbb{Q}) \quad (2k+1 \leq d-1)$$

by setting

$$\tilde{t}_{2k+1} = \widetilde{\mathrm{trace}}(2k+1)^*(\iota_{2k+1}).$$

Since these classes are compatible with respect to d, we may consider

$$\tilde{t}_{2k+1} \in H^2(\varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d; \mathbb{Q})$$

As for the non-triviality of $\tilde{t}_{2k+1}$, we have the following result.

Proposition 7.2. *Suppose that $t_{2k+1} \in H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})$ is non-trivial. Then $\tilde{t}_{2k+1} \in H^2(\mathrm{Aut}_0\, N_d; \mathbb{Q})$ is also non-trivial. Conversely, the non-triviality of the latter class implies that of the former class.*

Proof. The class $t_{2k+1} \in H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})$ is already defined as a second cohomology class of the truncated Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}[d]$ for any $d \geq 2k+2$. Hence if t_{2k+1} is non-trivial, it is so at the level of $H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}}[d])$. Since $\mathfrak{h}_{g,1}^{\mathbb{Q}}[d]$ is the semi-direct product of $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$ with $\mathfrak{sp}(2g, \mathbb{Q})$, the above non-triviality condition is equivalent to that of the restricted element

$$t_{2k+1} \in H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d])^{\mathrm{Sp}}$$

to the ideal $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$. On the other hand, by Theorem 3.3, $\mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$ is isomorphic to the Lie algebra of $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ which is a nilpotent Lie group. Therefore, by a theorem of Nomizu [35], the non-triviality of the above restricted element is equivalent to that of the restricted element

$$\tilde{t}_{2k+1} \in H^2(\mathrm{IAut}_0(N_d \otimes \mathbb{Q}))^{\mathrm{Sp}}$$

which is the same as the non-triviality of the class $\tilde{t}_{2k+1} \in H^2(\mathrm{Aut}_0(N_d \otimes \mathbb{Q}))$ because $\mathrm{Aut}_0(N_d \otimes \mathbb{Q}))$ is the semi-direct product of $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ with $\mathrm{Sp}(2g, \mathbb{Q})$. The result follows from this. Q.E.D.

Thus we know only the non-triviality of $\tilde{t}_3$ and $\tilde{t}_5$. However it seems to be reasonable to expect that all the classes would be non-trivial.

As was already mentioned in section 2, we know by Garoufalidis and Levine that the group $\Theta_{\mathbb{Z}}^3$ of homology cobordism classes of oriented homology 3-spheres is contained in their group $\mathcal{H}_{g,1}$ as a *central* subgroup. Hence we can consider the quotient group

$$\overline{\mathcal{H}}_{g,1} = \mathcal{H}_{g,1}/\Theta_{\mathbb{Z}}^3.$$

Since the representations $\tilde{\rho}_d$ are all trivial on $\Theta^3_{\mathbb{Z}}$, they induce a homomorphism

$$\bar{\rho}_\infty : \overline{\mathcal{H}}_{g,1} \longrightarrow \varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d.$$

Conjecture 7.3. *All the classes*

$$\bar{\rho}^*_\infty(\tilde{t}_{2k+1}) \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

are non-trivial while all the classes

$$\tilde{\rho}^*_\infty(\tilde{t}_{2k+1}) \in H^2(\mathcal{H}_{g,1};\mathbb{Q})$$

are trivial.

If this conjecture would have an affirmative solution, then the 2-cocycles representing $\tilde{\rho}^*_\infty(\tilde{t}_{2k+1})$ would be coboundaries of certain 1-cochains which are functions $\mathcal{H}_{g,1}\to\mathbb{Q}$. The restriction of these functions would yield certain homomorphisms $\Theta^3_{\mathbb{Z}}\to\mathbb{Q}$. We expect that the above argument would be useful to analyze the structure of $\Theta^3_{\mathbb{Z}}$.

Recall here that Furuta [5] proved that the rank of $\Theta^3_{\mathbb{Z}}$ is infinite. See also Fintushel and Stern [4] for another proof.

Acknowledgment The author would like to express his hearty thanks to R. Hain, N. Kawazumi and T. Sakasai for useful information.

References

[1] J. Conant and K. Vogtmann, On a theorem of Kontsevich, Algebr. Geom. Topol., **3** (2003), 1167–1224.

[2] J. Conant and K. Vogtmann, Morita classes in the homology of automorphism group of free groups, Geom. Topol., **8** (2004), 1471–1499.

[3] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups, Invent. Math., **84** (1986), 91–119.

[4] R. Fintushel and R. Stern, Instanton homology of Seifert fibred homology three spheres, Proc. London Math. Soc., **61** (1990), 109–137.

[5] M. Furuta, Homology cobordism group of homology 3-spheres, Invent. Math., **100** (1990), 339–355.

[6] S. Garoufalidis and J. Levine, Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism, In: Graphs and Patterns in Mathematics and Theoretical Physics, Proc. Sympos. Pure Math., **73**, Amer. Math. Soc., 2005, pp. 173–205.

[7] M. Goussarov, Finite type invariants and n-equivalence of 3-manifolds, C. R. Math. Acad. Sci. Paris, **329** (1999), 517–522.

[8] N. Habegger, Milnor, Johnson, and tree level perturbative invariants, preprint.

[9] K. Habiro, Claspers and finite type invariants of links, Geom. Topol., **4** (2000), 34–43.

[10] R. Hain, Infinitesimal presentations of the Torelli groups, J. Amer. Math. Soc., **10** (1997), 597–651.

[11] J. Harer, Stability of the homology of the mapping class group of orientable surfaces, Ann. of Math. (2), **121** (1985), 215–249.

[12] D. Johnson, An abelian quotient of the mapping class group $\mathcal{I}_g$, Math. Ann., **249** (1980), 225–242.

[13] D. Johnson, The structure of the Torelli group I: A finite set of generators for $\mathcal{I}_g$, Ann. of Math. (2), **118** (1983), 423–442.

[14] D. Johnson, A survey of the Torelli group, Contemp. Math., **20** (1983), 165–179.

[15] M. Kassabov, On the automorphism tower of free nilpotent groups, thesis, Yale Univ., 2003.

[16] N. Kawazumi, A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces, Invent. Math., **131** (1998), 137–149.

[17] N. Kawazumi, Cohomological aspects of Magnus expansions, preprint, math.GT/0505497.

[18] N. Kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, Math. Res. Letters, **3** (1996), 629–641.

[19] N. Kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the Mumford-Morita-Miller classes, preprint.

[20] M. Kervaire and J. Milnor, Groups of homotopy spheres, I, Ann. of Math. (2), **77** (1963), 504–537.

[21] M. Kontsevich, Formal (non-)commutative symplectic geometry, In: The Gelfand Mathematical Seminars 1990-1992, Birkhäuser Verlag, 1993, pp. 173–188.

[22] M. Kontsevich, Feynman diagrams and low-dimensional topology, In: Proceedings of the First European Congress of Mathematicians, Vol II, Paris, 1992, Progr. Math., **120**, Birkhäuser Verlag, 1994, pp. 97–121.

[23] J. Labute, On the descending central series of groups with a single defining relation, J. Algebra, **14** (1970), 16–23.

[24] J. Levine, Algebraic closure of groups, Contemp. Math., **109** (1990), 99–105.

[25] J. Levine, Homology cylinders: an enlargement of the mapping class group, Algebr. Geom. Topol., **1** (2001), 243–270.

[26] I. Madsen and M. Weiss, The stable moduli space of Riemann surfaces: Mumford's conjecture, Ann. of Math. (2), **165** (2007), 843–941.

[27] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, Wiley, New York, 1966.

[28] S. Morita, Characteristic classes of surface bundles, Bull. Amer. Math. Soc., **11** (1984), 386–388.

[29] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J., **70** (1993), 699–726.

[30] S. Morita, The extension of Johnson's homomorphism from the Torelli group to the mapping class group, Invent. Math., **111** (1993), 197–224.

[31] S. Morita, A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles, In: Proceedings of the Taniguchi Symposium on Topology and Teichmülller Spaces, Finland, July 1995, World Scientific, 1996, pp. 159–186.

[32] S. Morita, Structure of the mapping class groups of surfaces: a survey and a prospect, In: Proceedings of the Kirbyfest, Geom. Topol. Monog., **2**, Geom. Topol. Publ., 1999, pp. 349–406.

[33] S. Morita, Cohomological structure of the mapping class group and beyond, In: Problems on Mapping Class Groups and Related Topics, (ed. Benson Farb), Proc. Sympos. Pure Math., **74**, Amer. Math. Soc., 2006, pp. 329–354.

[34] D. Mumford, Towards an enumerative geometry of the moduli space of curves, In: Arithmetic and Geometry, Progr. Math., **36**, Birkhäuser, 1983, pp. 271–328.

[35] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. of Math. (2), **59** (1954), 167–189.

[36] T. Sakasai, Homology cylinders and the acyclic closure of a free group, Algebr. Geom. Topol., **6** (2006), 603–631.

[37] T. Sakasai, The Magnus representation and higher-order Alexander invariants for homology cobordisms of surfaces, preprint, math.GT/0507266.

[38] J. Stallings, Homology and central series of groups, J. Algebra, **2** (1965), 170–181.

Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Tokyo 153-8914
Japan