

Foliations and compact leaves on 4-manifolds I. Realization and self-intersection of compact leaves

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Abstract.

We introduce an easily tractable cohomological criterion for the existence of 2-dimensional foliations with a prescribed compact leaf on a 4-manifold relying on standard methods, Milnor's inequality for the existence of a flat connection on an \mathbb{R}^2 -bundle over a surface, and Thurston's h -principle. This is used to investigate the self-intersection numbers of compact leaves of foliations on the product of two surfaces, in particular the question whether these numbers are bounded on a given 4-manifold.

§1. Introduction

This article is the first in a series of papers to study 2-dimensional foliations with compact leaves on 4-manifolds, in particular the self-intersection of these compact leaves. By [HM] the self-intersection homology class of a Ruelle-Sullivan foliation cycle without compact leaves in its support is zero. Thus only the self-intersection classes of compact leaves in the support of a foliation cycle contribute to the self-intersection class of this cycle. From this fact results our interest in self-intersection numbers of compact leaves of n -dimensional foliations on $2n$ -manifolds, with 2-dimensional foliations on 4-manifolds being the first case of interest.

This article treats two themes. First we give a general criterion in (co)homological terms for the existence of a foliation on a given closed

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4-manifold with a prescribed embedded closed surface as a compact leaf. This will be explained in §4. The basic tools for solving this problem are Thurston's h -principle for foliations of codimension greater than one ([Th]), Milnor's inequality for the existence of a flat connection on an \mathbb{R}^2 -bundle over a surface ([M]), and well known relations for characteristic classes concerning the existence of plane fields on 4-manifolds due to Hirzebruch-Hopf ([HH]).

To deal with this question was prompted by the second theme. Here we ask whether for any given closed oriented 4-manifold M there exists an upper bound for the self-intersection numbers of compact leaves of 2-dimensional foliations on M . This question originated in the work of the first author in his thesis [Mi1] under the direction of Professor Shigeyuki Morita. See also [Mi2].

In 1999 the authors showed that there exists for each $g \geq 2$ and any even number k a foliation on $M = \Sigma_g \times T^2$ with a compact leaf with self-intersection number k . The method to produce these foliations is a good illustration in the use of Thurston's h -principle, so that we feel justified to include it. This we do in §3. The arguments we use in our treatment of the first theme can be regarded as cohomological improvements of this method.

While the results concerning our first theme apply to all closed orientable 4-manifolds, we only consider products of two orientable closed surfaces for the second. We show in §6 that the self-intersection numbers of compact leaves are unbounded for 2-dimensional foliations on products of closed orientable surfaces where either one factor is a torus or no factor is a sphere. In all other cases, *i.e.*, at least one factor is a 2-sphere and no factor is a torus, the set of intersection numbers is bounded. This is proved in §7. In fact, the set of homology classes in $H_2(\Sigma_g \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ which can be realized as a compact leaf of some foliation is essentially bounded.

Apart from the sections mentioned above there are three more. We begin in Section 2 by recalling some basic facts about compact leaves of foliations. Section 5 contains the proof of a Theorem of Section 4 which says that two splittings of an \mathbb{R}^4 -bundle over a surface into a sum of two plane bundles are homotopic as splittings if their Euler classes agree. In the final section, apart from raising several questions related to this article, we supplement the information of Section 7 on the foliations with compact leaves on $\Sigma_g \times S^2$. As will become clear, there are many interesting foliations with compact leaves representing a surprising number of elements in $H_2(\Sigma_g \times S^2)$.

We only deal with smooth oriented foliations on smooth oriented manifolds in this article, but everything will work also in the C^r -category, $r \geq 2$.

The authors are grateful to Dieter Kotschick for suggesting to them to introduce cohomological considerations for these problems. As was mentioned earlier, the cohomological method gives rise to the existence of some very interesting foliations. In the second article of the present series we will describe some of them explicitly.

§2. Fundamentals on compact leaves and basic examples

In this section and in the next, we explain the constraints on a prescribed surface to be a compact leaf of some foliation. The first one is due to the integrability of the plane field.

Before we describe this we fix some notation which we will use throughout this paper. M will always denote an oriented closed 4-manifold, \mathcal{F} a 2-dimensional smooth foliation on a 4-manifold, τ [*resp.* ν] the tangent [*resp.* normal] bundle to a foliation, e the Euler class of bundles, and Σ_g will denote the closed orientable surface of genus g with some fixed orientation. We deal only with smooth objects.

The normal bundle $\nu\mathcal{F}$ of any foliation \mathcal{F} admits a so-called Bott connection, that is, a connection which is flat along the leaves of $\mathcal{F}([B])$. (This is true in all dimensions.)

On the other hand, Milnor's seminal inequality ([M]) asserts that an oriented vector bundle of rank 2 over Σ_g , $g \geq 1$, admits a flat connection if and only if the Euler number of the bundle is bounded in absolute value by one half of the absolute value of the Euler characteristic of the surface. Combining these two theorems, we have the following.

Proposition 2.1. *A compact leaf $L \cong \Sigma_g$, $g \geq 1$, of a foliation \mathcal{F} on M satisfies the following inequality between its homological self-intersection $[L]^2 = [L] \cdot [L] \in \mathbb{Z}$ and its Euler characteristic.*

$$|\langle e(\nu\mathcal{F}), [L] \rangle| = |[L]^2| \leq \frac{1}{2} |\langle e(\tau\mathcal{F}), [L] \rangle| = \frac{1}{2} |\chi(L)| = g - 1$$

Conversely, if an embedded surface $L \cong \Sigma_g \subset M$ ($g \geq 1$) satisfies $|[L]^2| \leq g - 1$, then L is a leaf of some foliation of some open neighborhood of L .

The following example is standard and fundamental. It also shows how to compactify and modify a foliated tubular neighbourhood of a closed surface into a foliated closed manifold.

Example 2.2. Take an oriented flat \mathbb{R}^2 -vector bundle E_ρ over a closed oriented surface Σ_g of genus $g \geq 1$ with holonomy $\rho : \pi(\Sigma_g) \rightarrow SL(2; \mathbb{R})$. Its Euler number $k = \langle e(E_\rho), [\Sigma_g] \rangle$ satisfies Milnor's inequality $|k| \leq g - 1$. Take the product $\bar{\rho}$ of the one-dimensional trivial representation and ρ , namely, embed $SL(2; \mathbb{R})$ into the second and the third rows and columns of $SL(3; \mathbb{R})$. By taking the associated action of $SL(3; \mathbb{R})$ on the space of oriented lines, we obtain an oriented flat S^2 -bundle \bar{E}_ρ and consequently a foliation \mathcal{F}_ρ on \bar{E}_ρ .

The points $P_\pm \in S^2$ which correspond to $(\pm 1, 0, 0) \in \mathbb{R}^3$ are fixed points of the action (in fact, the only fixed points, if $k \neq 0$), and the tangential representation at P_+ exactly coincides with ρ , while at P_- we obtain ρ with the orientation reversed. Therefore the foliation \mathcal{F}_ρ has two compact leaves L_\pm corresponding to P_\pm with $[L_\pm]^2 = \pm k$. The hemispheres $H_\pm \{(x, y, z) \in S^2; \pm x > 0\}$ and the equator $S^1 = \{x = 0\}$ are also invariant. The flat H_+ -bundle is diffeomorphic as an oriented foliated manifold to the original flat vector bundle E_ρ . The same holds for the H_- -bundle with the reversed orientation.

Also remark that if k is even, the S^2 -bundle is diffeomorphic to the product bundle once the flat structure is forgotten. The identification via this diffeomorphism does not preserve the equator nor P_\pm . In the product bundle the compact leaf L_\pm is identified with the graph of a map $f_\pm : \Sigma_g \rightarrow S^2$ with $\deg f_\pm = \pm k/2$. For k odd, the bundle is twisted. See also Proposition 5.1. \square 2.2.

§3. Unbounded self-intersection on $\Sigma_g \times T^2$. A geometric construction

In this section, we construct a family of foliations with prescribed compact leaves on $M = \Sigma_g \times T^2$ ($g \geq 2$), relying on Thurston's h -principle, and we also explain how the h -principle is used in our context.

Theorem 3.1. (Thurston [Th]) *Let ξ be a smooth 2-plane field on a smooth n -manifold N , $n \geq 4$, let $K \subset N$ be closed, and assume that ξ is completely integrable (i.e., it defines a 2-dimensional foliation) in a neighborhood of K . Then ξ is homotopic to a completely integrable plane field via a homotopy which is constant on K .*

There is an unpublished somewhat simplified proof of this theorem, which is due to A. Haefliger. It makes the resulting foliations almost visible. We will present this proof in the forthcoming paper [MV].

Example 3.2. Fix $g \geq 2$. Then for each pair of positive integers a and b satisfying $1 \leq b \leq g - 1$, there exists a foliation $\{\mathcal{F}_{a,b}\}$ on $M = \Sigma_g \times T^2$ with a compact leaf $L_{a,b}$, which has self-intersection

number $[L_{a,b}]^2 = 2ab$. Especially, on each such M , the self-intersection numbers of closed orientable surfaces which are leaves of foliations on M are not bounded.

General strategy: The following is the basic strategy for the construction, which will again be used in the next section.

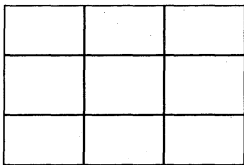
First we fix an embedded surface L satisfying Milnor's inequality, *i.e.*, $|[L]^2| \leq |\chi(L)|/2$. Then as we have seen in 2.1, we can foliate a neighbourhood of L . Next we extend the tangent plane field of this foliation to a plane field on all of M . In this section this process is done by a geometric argument, while a cohomological argument replaces this in the next section. Then Thurston's h -principle modifies the plane field into a foliation keeping L as a compact leaf.

Geometric Construction: To begin the construction, let us take a distinct points $\{P_1, \dots, P_a\}$ of T^2 and b distinct points $\{Q_1, \dots, Q_b\}$ of Σ_g . The singular oriented surface

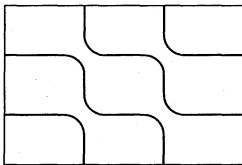
$$L'_{a,b} = \left(\bigcup_{i=1}^a \Sigma_g \times P_i\right) \cup \left(\bigcup_{j=1}^b Q_j \times T^2\right)$$

has ab double points. Resolving the double points by rounding off, *i.e.*, by removing small disk neighborhoods of the points from the horizontal and vertical branches of $L'_{a,b}$, and reconnecting each pair of the resulting boundary circles by an annulus respecting the orientation, we obtain a connected closed surface $L_{a,b}$ with the desired self-intersection number. However, we do this slightly more carefully so that we can easily extend the tangent plane field of this surface to all of M .

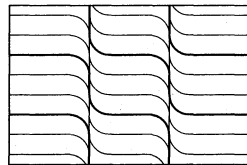
Step 1 (resolving the double points): Take a small holomorphic coordinate neighbourhood (U_j, z) around Q_j in Σ_g so that at Q_j we have $z = 0$. Also fix a complex structure on T^2 as $\mathbb{C}/\mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}$ and place P_1 at 0. Let w denote the standard holomorphic local coordinate on T^2 . Then around the double point $(Q_j, P_1) = (0, 0)$ consider the graph of $zw = \varepsilon^4$ for a small constant $0 < \varepsilon \ll 1$. Inside the polydisk $\{(z, w); |z|, |w| \leq \varepsilon^2\}$ of radius ε^2 we adopt the graph as part of the connecting smooth annulus.



singular surface $L'_{a,b}$



surface $L_{a,b}$



plane field ξ_0

Outside the polydisk, we deform the graph as follows. In $\{(z, w); |z| \leq \varepsilon^2 \leq |w|\}$ we deform the graph horizontally, *i.e.*, in z -direction, so that outside $\{(z, w); \varepsilon \leq |w|\}$ it is vertical, *i.e.*, it coincides with $\{z = 0\}$. In $\{(z, w); |w| \leq \varepsilon^2 \leq |z|\}$ it is deformed vertically so that outside $\{(z, w); \varepsilon \leq |z|\}$ it is horizontal. This can be done so that the resulting surface is smooth and is away from $\{z = 0\}$ the graph of a function. We can achieve this by a C^1 -small deformation if we take ε small enough.

The connecting annulus is then the part of the deformed graph inside $\{(z, w); |z|, |w| \leq \varepsilon\}$ and the disks $\{(z, 0); |z| \leq \varepsilon\}$ and $\{(0, w); |w| \leq \varepsilon\}$ have been removed from $F'_{a,b}$.

Step 2 (coherent resolutions and the plane field): The remaining double points get resolved in exactly the same way using exactly the same graphs and deformations as above where at (Q_j, P_i) we choose the same coordinate system (U_j, z) around Q_j as before and around P_i the translate of the coordinate system of P_1 .

The resulting smooth oriented embedded surface $L_{a,b}$ has the property that any translation of M in the T^2 -direction which maps a point x of $L_{a,b}$ to another point y of $L_{a,b}$ also maps a neighborhood of x in $L_{a,b}$ diffeomorphically onto a neighborhood of y . Therefore, by translations in the T^2 direction, we can extend the tangent plane field of $L_{a,b}$ to an oriented plane field ξ_0 on M . This plane field is smooth in the complement of $\bigcup_{j=1}^b Q_j \times T^2$, but globally only C^0 .

Let ξ_1 be a smooth approximation of ξ_0 , close enough so that the arguments in the next step are valid.

Step 3 (application of 2.1 and 3.1): The surface $L_{a,b}$ represents the homology class $a[\Sigma_g] + b[T^2]$ and thus has self-intersection number $2ab$. Also it is easy to see that $\chi(L_{a,b}) = 2a(1 - g - b)$. Therefore, our condition $1 \leq b \leq g - 1$ is nothing but Milnor's inequality in 2.1 provided that a and b are positive. It follows that there is a foliation of some open neighbourhood V of $L_{a,b}$ which has $L_{a,b}$ as a compact leaf.

If V is small enough (and ξ_1 close enough to ξ_0), the plane field of this foliation is smoothly homotopic to ξ_1 on V . Therefore, by taking a closed neighborhood $U \subset V$ of $L_{a,b}$ and a smooth partition of unity subordinate to $\{M \setminus U, V\}$ we find an oriented plane field ξ on M which coincides with the tangent plane field of the foliation on U (and with ξ_1 outside of V).

Now, Thurston's h -principle 3.1 enables us to deform the plane field ξ into the tangent plane field of a foliation on M which admits $L_{a,b}$ as a compact leaf. This completes the construction of Example 3.2. \square 3.2.

Remark 3.3. The surface $L_{a,b}$ is genus-minimizing in its homology class, *i.e.*, the genus of any connected embedded surface which is homologous to $L_{a,b}$ is not smaller than the genus of $L_{a,b}$.

This is so, because $L_{a,b}$ is close to an almost holomorphic submanifold and so is a symplectic submanifold for the product symplectic structure on $M = \Sigma_g \times T^2$. Now, the validity of the symplectic Thom conjecture ([OS]) assures that $L_{a,b}$ is genus-minimizing in its homology class. We can also realize $L_{a,b}$ as a holomorphic submanifold of M .

§4. A cohomological criterion

We introduce a (co)homological method to decide when the tangent plane field of a given surface can be extended to the ambient manifold. The following result is well known, and it is an immediate corollary of a theorem accredited in [DW] to Pontrjagin [P]. This theorem states that for $n \geq 4$ isomorphism classes of principal $SO(n)$ -bundles over a 4-complex K without 2-torsion in $H^4(K; \mathbb{Z})$ are completely classified by w_2, p_1 , and W_4 . Here, $w_2 \in H^2(K; \mathbb{Z}/2)$ is the second Stiefel-Whitney class, $p_1 \in H^4(K; \mathbb{Z})$ the first Pontrjagin class, and W_4 the fourth Stiefel-Whitney class in $H^4(K; \mathbb{Z})$, $n = 4$, or in $H^4(K; \mathbb{Z}/2)$, $n \geq 5$. (See also the corollary in Section 3 of [DW]). Note that for $SO(4)$ -bundles, W_4 is just the Euler class.

Theorem 4.1. *Let M be a closed oriented connected 4-manifold and assume that the two cohomology classes e_1 and $e_2 \in H^2(M; \mathbb{Z})$ satisfy the following three conditions.*

- (1) $e_1^2 + e_2^2 = p_1(M) \in H^2(M; \mathbb{Z})$
- (2) $\langle e_1 \cup e_2, [M] \rangle = \chi(M)$
- (3) $e_1 + e_2 \equiv w_2(M) \in H^2(M; \mathbb{Z}/2)$

Then, there exists a pair of oriented \mathbb{R}^2 -bundles \mathcal{L}_1 and \mathcal{L}_2 over M such that $\mathcal{L}_1 \oplus \mathcal{L}_2$ is isomorphic to the tangent bundle TM and $e(\mathcal{L}_i) = e_i$ for $i = 1, 2$. In other words, e_1 and e_2 are the Euler classes of a transverse pair of oriented plane fields on M .

Assume now that we have a pair of transverse plane fields of M . The next theorem will provide us with a simple check involving the Euler classes of these fields to decide when the first one of them can be homotoped into one which is tangent to a given embedded closed surface. A similar statement for splittings of \mathbb{R}^2 -bundles over the circle is not true.

Theorem 4.2. *Let E be an oriented \mathbb{R}^4 -bundle over a closed oriented surface L and $\mathcal{L}_1 \oplus \mathcal{L}_2 = E$ and $\mathcal{L}'_1 \oplus \mathcal{L}'_2 = E$ be two splittings of E*

into sums of oriented \mathbb{R}^2 -subbundles. Then the following two statements are equivalent.

- (4) $e(\mathcal{L}_1) = e(\mathcal{L}'_1)$ and $e(\mathcal{L}_2) = e(\mathcal{L}'_2)$.
- (5) $\mathcal{L}_1 \oplus \mathcal{L}_2$ and $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ are homotopic as splittings of E .

The proof will be given in the next section.

Corollary 4.3. *Let L be an embedded closed oriented connected surface in M , e_1 and $e_2 \in H^2(M; \mathbb{Z})$ be two cohomology classes satisfying the conditions (1) - (3) of Theorem 4.1 and the following condition.*

$$(4') \langle e_1, [L] \rangle = \chi(L) \text{ and } \langle e_2, [L] \rangle = [L]^2.$$

Then the tangent plane field $TL \subset TM$ of L extends to an oriented plane field \mathcal{L} on M .

Proof of Corollary 4.3.

This follows from Theorems 4.1 and 4.2 in a straight forward way.

Assuming the hypotheses of Corollary 4.3, Theorem 4.1 assures the existence of a transverse pair of plane fields \mathcal{L}_1 and \mathcal{L}_2 on M whose Euler classes are e_1 and e_2 . Theorem 4.2 asserts that $\mathcal{L}_1|_L$ and TL are homotopic as subbundles of $E := TM|_L$. Let $L'_t, 0 \leq t \leq 1$, be a continuous family of 2-dimensional oriented subbundles of E with $L'_t = TL, 0 \leq t \leq 1/2$, and $L'_1 = \mathcal{L}_1|_L$.

Next take a tubular neighbourhood $p : U_3 \rightarrow L$ of L in M which we view as a normal disk bundle of radius 3 and let $S_1 \subset U_3$ be the corresponding unit sphere bundle. Every point of U_3 can and will be written in the form $r \cdot x, 0 \leq r \leq 3, x \in S_1$, and we identify $0 \cdot x$ with $p(x) \in L$. Also for $s < 3, U_s$ denotes $\{r \cdot x \in U_3 \mid 0 \leq r \leq s\}$.

The bundle $TM|_{U_3}$ is isomorphic to

$$p^*E = \{(r \cdot x, v) \in U_3 \times E \mid v \in E_{0 \cdot x}\},$$

where as usual E_b denotes the fibre of E over $b \in L$. We identify $TM|_{U_3}$ with p^*E .

The restriction $\mathcal{L}|_{U_3}$ of any oriented 2-dimensional subbundle \mathcal{L} of TM is described by a map L which assigns to each $r \cdot x \in U_3$ an oriented 2-dimensional subspace $L(r \cdot x)$ of $E_{0 \cdot x}$. With this notation we define a continuous family $\mathcal{L}_t, 0 \leq t \leq 1$, of oriented plane fields of M , where \mathcal{L}_1 is our original \mathcal{L}_1 from above, as follows.

- (a) $\mathcal{L}_t|_{M \setminus U_2} = \mathcal{L}_1|_{M \setminus U_2}$
- (b) $L_t(r \cdot x) = \begin{cases} L_1(\frac{2(r-1+t)}{1+t} \cdot x), & 1-t \leq r \leq 2, \\ L'_{t+r}(0 \cdot x), & 0 \leq r \leq 1-t. \end{cases}$

The plane field \mathcal{L}_0 restricted to $U_{1/2}$ is the pull back of TL by $p : U_{1/2} \rightarrow L$. Therefore it is tangent to L and smooth in the interior of $U_{1/2}$, and we may approximate it by a smooth field, still called \mathcal{L}_0 , which is tangent to L and homotopic to \mathcal{L}_1 as an oriented subbundle of TM . \square 4.3.

Applying our general strategy stated in Example 3.2, we obtain by combining Milnor's inequality 2.1, Thurston's h -principle 3.1, and Corollary 4.3 the following theorem, which is one of the main results of the present article.

Theorem 4.4. *Let L be a closed oriented connected embedded surface of genus greater than 0 in a closed oriented 4-manifold M which satisfies Milnor's inequality 2.1. Assume that two cohomology classes e_1 and $e_2 \in H^2(M; \mathbb{Z})$ satisfy the following five conditions.*

- (1) $e_1^2 + e_2^2 = p_1(M) \in H^2(M; \mathbb{Z})$,
- (2) $\langle e_1 \cup e_2, [M] \rangle = \chi(M)$,
- (3) $e_1 + e_2 \equiv w_2(M) \in H^2(M; \mathbb{Z}/2)$,
- (4) $\langle e_1, [L] \rangle = \chi(L)$,
- (5) $\langle e_2, [L] \rangle = [L]^2$.

Then there exists a foliation \mathcal{F} on M which has L as a compact leaf.

This theorem translates the existence problem of foliations with prescribed compact leaves into a purely cohomological problem, *i.e.*, looking for solutions of a system of quadratic equations in the integral cohomology ring of the ambient 4-manifold. For 4-manifolds with accessible cohomology rings these equations can be dealt with. We will see examples of this in Sections 6, 7, and 8.

Remark 4.5. The above theorem implies that the question of whether an embedded surface L is a leaf of a foliation depends only on its homology class and its genus. Obviously, given a homology class and genus satisfying the equations of the theorem, we can realize it by an embedded surface if and only if the genus is not smaller than the minimal genus of a connected embedded surface in this homology class. So it is important to know this minimal genus. Of course, quite often the solution of the symplectic Thom conjecture [OS] provides the answer.

Here is an immediate corollary to the theorem, which is already interesting.

Corollary 4.6. *Let \mathcal{F} be a 2 dimensional oriented foliation on a 4-manifold and let L be a homologically trivially embedded 2-torus in M , *i.e.*, $[L] = 0 \in H_2(M; \mathbb{Z})$. Then we can modify the foliation in a*

neighbourhood of L into new foliations \mathcal{F}_1 and \mathcal{F}_2 such that the following holds.

- (1) \mathcal{F}_1 has L as a compact leaf.
- (2) \mathcal{F}_2 is transverse to L .

Of course one is tempted to construct these foliations explicitly. Some of these constructions will be presented in the forthcoming paper [MV]. Scorpan showed the existence of foliations like \mathcal{F}_2 in the framework of singular foliations [S].

§5. Proof of Theorem 4.2

In this section we give a proof of Theorem 4.2, though it might be a folk theorem. It can be explained in a more general setting, as a general story about homotopy classes of subplane bundle of an oriented \mathbb{R}^4 -bundle. (See for example the two excellent articles of Dold and Whitney [DW] and Hirzebruch and Hopf [HH]). However, we deal with the material in a rather down-to-earth manner which works in our particular range of dimensions. In this section, Σ refers to the oriented connected closed surface L in the statement of Theorem 4.2.

Let us start with reviewing the following well known facts, one of which was already mentioned in Example 2.2. Let $\mathcal{L}(k)$ denote an oriented \mathbb{R}^2 -bundle over a closed oriented connected surface Σ with the Euler number $\langle e(\mathcal{L}(k)), [\Sigma] \rangle = k$. We also regard $\mathcal{L}(k)$ as a complex line bundle with $c_1 = k$. Let ε^r denote the product \mathbb{R}^r -bundle and $\varepsilon_{\mathbb{C}}^r$ the product \mathbb{C}^r -bundle. The isomorphism classes of complex line bundles or oriented \mathbb{R}^2 -bundles over surfaces are determined by $c_1 = e$.

The statements in the following proposition are also fairly elementary and well known. Nevertheless, we review proofs of them, because this will aid in understanding the arguments leading up to the proof of Theorem 4.2.

Proposition 5.1.

- 1) Isomorphism classes of oriented \mathbb{R}^3 -bundles over Σ are determined by $w_2 \in H^2(\Sigma; \mathbb{Z}/2)$. In other words, any oriented \mathbb{R}^3 -bundle over Σ is isomorphic to $\mathcal{L}(k) \oplus \varepsilon^1$ for some $k \in \mathbb{Z}$ and $\mathcal{L}(k_1) \oplus \varepsilon^1$ and $\mathcal{L}(k_2) \oplus \varepsilon^1$ are isomorphic if and only if $k_1 \equiv k_2 \pmod{2}$.
- 2) For any oriented \mathbb{R}^3 -bundle over Σ with $w_2 = 0$ [resp. $w_2 \neq 0$], the self-intersection number of any cross section of the associated unit S^2 -bundle is an even [resp. odd] number.
- 3) Isomorphism classes of oriented \mathbb{R}^4 -bundles over Σ are also determined by $w_2 \in H^2(\Sigma; \mathbb{Z}/2)$. Any oriented \mathbb{R}^4 -bundle over

Σ is isomorphic to some $\mathcal{L}(k) \oplus \varepsilon^2$ and its isomorphism class is determined by the parity of k .

- 4) \mathbb{C}^2 -bundles over Σ are determined by $c_1 \in H^2(\Sigma; \mathbb{Z})$. In other words, any \mathbb{C}^2 -bundle over Σ is isomorphic to $\mathcal{L}(k_1) \oplus \mathcal{L}(k_2)$ for some k_1 and $k_2 \in \mathbb{Z}$, and $\mathcal{L}(k_1) \oplus \mathcal{L}(k_2)$ and $\mathcal{L}(m_2) \oplus \mathcal{L}(m_2)$ are isomorphic as \mathbb{C}^2 -bundles if and only if $k_1 + k_2 = m_1 + m_2$.

Proof. 1) Reversing the steps in the construction for Example 2.2, where a trivial \mathbb{R} -bundle was added to an oriented \mathbb{R}^2 -bundle, gives rise to the proof. Let \bar{E} be an oriented \mathbb{R}^3 -bundle over Σ and $S^2\bar{E}$ be the associated S^2 -bundle. Because the fibre of $S^2\bar{E}$ is simply connected, it is easy to find a section of $S^2\bar{E}$. This section defines a trivial 1-dimensional subbundle of \bar{E} which gives rise to an isomorphism $\bar{E} \cong \mathcal{L}(k) \oplus \varepsilon^1$ for some $k \in \mathbb{Z}$. Also notice here that the normal bundle of this section in $S^2\bar{E}$ is canonically isomorphic to the complementary subbundle $\mathcal{L}(k)$ and thus $k = |[L]^2|$, if L denotes the image of this section.

We can change the homotopy class of the cross section locally over a disc $D \subset \Sigma$. There the bundle looks like a product and the section is a graph of a smooth map $f : D \rightarrow S^2$. Up to homotopy we may assume that the map is constant on the boundary ∂D . So $f : (D, \partial D) \rightarrow (S^2, f(\partial D))$ has a degree. Changing the degree of this map by l changes the self-intersection of the cross section by $2l$.

The statement 2) also follows from the argument above. Remark also that if $w_2 = 0$, there is no homology class whose self-intersection is an odd number.

3) It is also easy to find a cross section to the associated S^3 -bundle of a given oriented \mathbb{R}^4 -bundle, which splits off a trivial line bundle. Then what we have to show reduces to statement 1) for the complementary \mathbb{R}^3 -bundle.

Of course, an analogous statement is true for oriented vector bundles over Σ of any rank greater than 2.

4) The proof is again as in 1). Consider the $\mathbb{C}P^1$ -bundle associated to the given \mathbb{C}^2 -bundle E . Because the fibre is $\mathbb{C}P^1 \cong S^2$, we can again find a cross section, which gives rise to some complex line bundle $\mathcal{L} \subset E$. Therefore we obtain a splitting $E = \mathcal{L}_1 \oplus \mathcal{L}_2$ of E into two complex line bundles $\mathcal{L}_1 \cong \mathcal{L}(k_1)$ and $\mathcal{L}_2 \cong \mathcal{L}(k_2)$ for some $k_1, k_2 \in \mathbb{Z}$.

As $c_1(E) = k_1 + k_2$ is an invariant, any splitting satisfies this relation. On the other hand, assume $m_1 + m_2 = k_1 + k_2$. Then we can locally change the splitting so as to have $E \cong \mathcal{L}(m_1) \oplus \mathcal{L}(m_2)$ by changing the degree of the cross section to the associated $\mathbb{C}P^1$ -bundle, as the following famous example illustrates. Take $E = \varepsilon_{\mathbb{C}}^2$ over $\Sigma = \mathbb{C}P^1$. Then we have two canonical splittings $\varepsilon_{\mathbb{C}}^2 = \varepsilon_{\mathbb{C}}^1 \oplus \varepsilon_{\mathbb{C}}^1$ and $\varepsilon_{\mathbb{C}}^2 = \gamma^1 \oplus \bar{\gamma}^1$ where γ^1 denotes

the tautological line bundle and $\bar{\gamma}^1$ denotes its complement. Of course we have $\gamma^1 \cong \mathcal{L}(-1)$ and $\bar{\gamma}^1 \cong \mathcal{L}(1)$. □5.1.

We will also use the following proposition.

Proposition 5.2. *For an oriented S^2 -bundle over Σ , two cross sections are homotopic if and only if their self-intersections coincide.*

Next, we review some basic facts about the oriented Grassmannian. Let us fix the standard orientation and the inner product on \mathbb{R}^4 . We denote by $\widetilde{Gr}(4, 2)$ the set of oriented planes through the origin in \mathbb{R}^4 with its natural structure as a smooth 4-manifold. Let L be an oriented plane in \mathbb{R}^4 and L^\perp be its oriented orthogonal complement. Then we obtain a pair (J_L^+, J_L^-) of complex structures on \mathbb{R}^4 as follows.

$$\begin{aligned} J_L^+|_L & \text{ rotates } L \text{ by } \pi/2. \\ J_L^+|_{L^\perp} & \text{ rotates } L^\perp \text{ by } \pi/2. \\ J_L^-|_L & \text{ rotates } L \text{ by } \pi/2. \\ J_L^-|_{L^\perp} & \text{ rotates } L^\perp \text{ by } -\pi/2. \end{aligned}$$

Notice that J_+ defines the standard orientation on \mathbb{R}^4 while J_- defines the opposite one. We call a complex structure on \mathbb{R}^4 positive or negative depending on whether it induces the standard orientation or not. Also, we assume that any complex structure on \mathbb{R}^n is an orthogonal transformation of \mathbb{R}^n . Let \mathcal{J}_\pm denote the set of positive/negative complex structures considered as subspaces of $O(4)$. We have obtained the map $\Phi : \widetilde{Gr}(4, 2) \rightarrow \mathcal{J}_+ \times \mathcal{J}_-$.

Proposition 5.3.

- 1) Both of \mathcal{J}_\pm are diffeomorphic to S^2 .
- 2) The map $\Phi : \widetilde{Gr}(4, 2) \rightarrow \mathcal{J}_+ \times \mathcal{J}_-$ is a diffeomorphism.

Proof. 1) Let $\mathbf{e}_1, \dots, \mathbf{e}_4$ be the standard basis of \mathbb{R}^4 . A positive (or negative) almost complex structure J is uniquely determined by $J(\mathbf{e}_1)$ which lies in the unit two sphere $S^2(\langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle)$ in the span $\mathbf{e}_1^\perp = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ of $\mathbf{e}_2, \mathbf{e}_3$, and \mathbf{e}_4 , because J rotates $L = \langle \mathbf{e}_1, J(\mathbf{e}_1) \rangle$ positively and L^\perp positively (or negatively). Here \langle, \rangle denotes the linear span.

Conversely a free choice of the image of \mathbf{e}_1 from $S^2(\mathbf{e}_1^\perp)$ determines a positive (or negative) almost complex structure.

We orient $S^2(\mathbf{e}_1^\perp)$ as the unit sphere in $\langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ and orient \mathcal{J}_\pm accordingly.

2) $SO(4)$ naturally acts transitively on $\widetilde{Gr}(4, 2)$. It is not difficult to check at one point of $\widetilde{Gr}(4, 2)$ that Φ is a local diffeomorphism. Therefore Φ is a covering map over a simply connected space and hence is a diffeomorphism. □ 5.3.

For a given oriented \mathbb{R}^4 -bundle E over Σ , we always assume that a fibre-wise inner product has been fixed. Then denote by $\widetilde{Gr}(E)$ the associated Grassmannian bundle and by $\mathcal{J}_\pm(E)$ the associated bundles of positive and negative complex structures. The S^2 -bundles $\mathcal{J}_\pm(E)$ over Σ carry natural orientations.

With regard to the next lemma, the proof of Proposition 5.3 is more important than the statement.

Lemma 5.4. *If $w_2(E) \neq 0$, both bundles $\mathcal{J}_\pm E$ are twisted as S^2 -bundles, i.e., $w_2(\mathcal{J}_\pm E) \neq 0$. If $w_2(E) = 0$, of course, both bundles $\mathcal{J}_\pm(E)$ are trivial.*

Proof. By Proposition 5.1 3), we may assume that $E \cong \varepsilon^1 \oplus E'$ with $E' = \varepsilon^1 \oplus \mathcal{L}(k)$ for some $k \in \mathbb{Z}$, with $k \equiv \langle w_2(E), [\Sigma] \rangle \pmod{2}$. It follows then from the proof of Proposition 5.3 that both $\mathcal{J}_\pm(E)$ are oriented isomorphic to $S^2(E')$. Therefore we have $w_2(\mathcal{J}_\pm(E)) = w_2(E') = w_2(E)$. □ 5.4.

The next Theorem is the key to prove Theorem 4.2. Before stating it, we fix some notations. For an oriented \mathbb{R}^2 -subbundle $\mathcal{L} \subset E$, let $J_\pm(\mathcal{L})$ denote the complex structure of E determined by $E = \mathcal{L} \oplus \mathcal{L}^\perp$. $J_\pm(\mathcal{L})$ are also considered as sections of $\mathcal{J}_\pm(E)$.

Theorem 5.5. *For a splitting $E = \mathcal{L} \oplus \mathcal{L}^\perp$ with $\mathcal{L} \cong \mathcal{L}(k_1)$ and $\mathcal{L}^\perp \cong \mathcal{L}(k_2)$, the self-intersection of $J_\pm(\mathcal{L})$ in $\mathcal{J}_\pm(E)$ is given as follows.*

$$[J_+(\mathcal{L})]^2 = k_1 + k_2, \quad [J_-(\mathcal{L})]^2 = -k_1 + k_2.$$

Proof. First we prove this for the case $\mathcal{L} \cong \varepsilon^2 \cong \varepsilon_1^1 \oplus \varepsilon_2^1$ and $\mathcal{L}^\perp \cong \mathcal{L}(k)$. Let us follow the notations in the proof of 5.4. In this case we can identify $\mathcal{J}_+(E)$ with $S^2(E')$ and through this identification, $J_+(\mathcal{L})$ is identified with the cross section of $S^2(E')$ which corresponds to the canonical frame of $\varepsilon_2^1 \subset E'$, because $J_+(\mathcal{L})(\varepsilon_1^1) = \varepsilon_2^1$. As $E' = \varepsilon_2^1 \oplus \mathcal{L}(k)$, the proof of Proposition 5.1 1) tells us that the normal bundle of $J_+(\mathcal{L})$ in $S^2(E')$ is isomorphic to $\mathcal{L}(k)$. This implies

$$[J_+(\mathcal{L})]^2 = [J_-(\mathcal{L})]^2 = k.$$

Now let us prove the general case. By Proposition 5.1 4), we have an isomorphism $\mathcal{L}(k_1) \oplus \mathcal{L}(k_2) \cong \mathcal{L}(0) \oplus \mathcal{L}(k)$ for $k = k_1 + k_2$ as complex vector bundles, where $\mathcal{L}(k_1)$, $\mathcal{L}(k_2)$, $\mathcal{L}(0)$, and $\mathcal{L}(k)$ are considered as

complex line bundles. This implies that the complex structures on $E = \mathcal{L}(k_1) \oplus \mathcal{L}(k_2)$ given by the two splittings $\mathcal{L}(k_1) \oplus \mathcal{L}(k_2)$ and $\mathcal{L}(0) \oplus \mathcal{L}(k)$ coincide. On the other hand, the complex structure on $\mathcal{L}(k_1) \oplus \mathcal{L}(k_2)$ [resp. $\mathcal{L}(0) \oplus \mathcal{L}(k)$] is nothing but $J_+(\mathcal{L}(k_1))$ [resp. $J_+(\mathcal{L}(0))$]. Hence their self-intersections also coincide. Therefore, computing the self-intersection of the first complex structure reduces to that of the second one and we obtain $[J_+(\mathcal{L}(k_1))]^2 = k = k_1 + k_2$.

The computation of $[J_-(\mathcal{L})]^2$ reduces to that of $[J_+(\mathcal{L})]^2$ by reversing the orientations. Let \check{E} be the \mathbb{R}^4 -bundle which is identical to E but with the opposite orientation. We can realize this by only reversing the orientation of \mathcal{L}^\perp and leaving \mathcal{L} as it is. Therefore $\check{E} \cong \mathcal{L} \oplus \check{\mathcal{L}}^\perp$ where $\check{\mathcal{L}}^\perp$ the same as \mathcal{L}^\perp with the opposite orientation, and thus $\check{\mathcal{L}}^\perp \cong \mathcal{L}(-k_2)$. Therefore $J_-(\mathcal{L} \subset E)$ for $E = \mathcal{L} \oplus \mathcal{L}^\perp$ is nothing but $J_+(\mathcal{L} \subset \check{E})$ for $\check{E} = \mathcal{L} \oplus \check{\mathcal{L}}^\perp$. However, the orientation of $S^2(\check{E}')$ is the opposite of that of $S^2(E')$ where $\check{E}' = \varepsilon_2^1 \oplus \check{\mathcal{L}}^\perp$. Therefore we obtain

$$[J_-(\mathcal{L} \subset E)]^2 = -[J_+(\mathcal{L} \subset \check{E})]^2 = -(k_1 - k_2).$$

This completes the proof. □5.1.

Proof of Theorem 4.2.

We only have to show (4) \Rightarrow (5) because the converse is trivial. Now (4) implies

$$[J_+(\mathcal{L}_1)]^2 = [J_+(\mathcal{L}'_1)]^2 \quad \text{and} \quad [J_-(\mathcal{L}_1)]^2 = [J_-(\mathcal{L}'_1)]^2.$$

By Proposition 5.2 $J_+(\mathcal{L}_1)$ and $J_+(\mathcal{L}'_1)$ are homotopic to each other as sections of $\mathcal{J}_+(E)$ and so are $J_-(\mathcal{L}_1)$ and $J_-(\mathcal{L}'_1)$. Therefore the two sections $(J_+(\mathcal{L}_1), J_-(\mathcal{L}_2))$ and $(J_+(\mathcal{L}'_1), J_-(\mathcal{L}'_2))$ of $\widetilde{Gr}(E)$ are homotopic. This completes the proof of Theorem 4.2. □4.2.

§6. Unboundedness for $\Sigma_g \times \Sigma_h$

As an application of Theorem 4.4, we show that for most products $\Sigma_g \times \Sigma_h$ of two closed oriented surfaces there is no bound on the set of self-intersection numbers of surfaces in $\Sigma_g \times \Sigma_h$ which can be realized as leaves of a foliation on $\Sigma_g \times \Sigma_h$. Precisely, we show the following.

Theorem 6.1. *Let M be one of the following products.*

- (a) $M = \Sigma_g \times \Sigma_h$, where $g, h \geq 1$,
- (b) $M = T^2 \times S^2$.

Then, there exists a family of 2-dimensional oriented foliations on M with compact leaves such that the set of self-intersection numbers of these compact leaves is unbounded.

This result is a substantial generalization of Example 3.2. It gives an indication that there should be many more manifolds which exhibit the same phenomenon. We will see in the proof of this theorem the power and ease of use of Theorem 4.4.

Proof of (a).

First let us fix some notation, which will be used throughout this section and also in the next section. In the cohomology ring

$$H^*(M; \mathbb{Z}) \cong H^*(\Sigma_g; \mathbb{Z}) \otimes H^*(\Sigma_h; \mathbb{Z})$$

we identify elements $x \in H^*(\Sigma_g; \mathbb{Z})$ with $x \otimes 1 \in H^*(M; \mathbb{Z})$. Similarly $y \in H^*(\Sigma_h; \mathbb{Z})$ is identified with $1 \otimes y$, so that xy is identified with $x \otimes y \in H^*(M; \mathbb{Z})$. If N is one of M , Σ_g , or Σ_h , $\{N\}$ denotes the cofundamental class. With this understanding, we have

$$H^2(M; \mathbb{Z}) = H^2(\Sigma_g; \mathbb{Z}) \oplus H^1(\Sigma_g; \mathbb{Z}) \otimes H^1(\Sigma_h; \mathbb{Z}) \oplus H^2(\Sigma_h; \mathbb{Z}).$$

We use similar notations for homology as well.

Because both surfaces Σ_g and Σ_h have positive genus, there exists pairs of classes

$$c_i \in H^1(\Sigma_g; \mathbb{Z}) \quad \text{and} \quad d_j \in H^1(\Sigma_h; \mathbb{Z}) \quad \text{for} \quad i, j = 1, 2$$

which satisfy

$$c_1 c_2 = \{\Sigma_g\} \quad \text{and} \quad d_1 d_2 = \{\Sigma_h\}.$$

For two positive integers a and b , take an embedded surface $L = L_{a,b}$ in M which represents the homology class $a[\Sigma_g] + b[\Sigma_h] \in H_2(M; \mathbb{Z})$ constructed in the same way as the surface with the same name in Example 3.2. It has Euler characteristic

$$\chi(L) = 2a(1 - g) + 2b(1 - h) - 2ab$$

and is genus-minimizing in its homology class, as remarked in 3.3.

Now let us take two cohomology classes

$$e_1 = \alpha\{\Sigma_g\} + \beta\{\Sigma_h\} + \sum_{i,j=1}^2 \eta_{ij} c_i d_j$$

$$e_2 = \gamma\{\Sigma_g\} + \delta\{\Sigma_h\} + \sum_{i',j'=1}^2 \zeta_{ij} c_i d_j$$

in $H^2(M; \mathbb{Z})$ with indeterminate integers $\alpha, \beta, \gamma, \delta, \eta_{ij}$, and ζ_{ij} ($i, j = 1, 2$). These are candidates for the Euler classes of the tangent and

normal bundles of the foliations that we are looking for. We show that for an unbounded family of (a, b) 's, there exist solutions e_1 and e_2 , *i.e.*, $\alpha, \beta, \gamma, \delta, \eta_{ij}$, and ζ_{ij} ($i, j = 1, 2$), for the equations (1) - (5) in Theorem 4.4 which also satisfy Milnor's inequality. If $g \geq 2$ or $h \geq 2$ we have more room in $H^2(M; \mathbb{Z})$ and therefore more freedom to choose e_1 or e_2 , but, as we will see, even with this limited choice for the Euler classes of tangent and normal bundles of foliations there is no bound on the set of self-intersection numbers of compact leaves.

To prove the theorem, it is enough to present the solutions. However, here we demonstrate a procedure to solve the equations briefly to appreciate the power of the cohomological criterion.

Let us first assume that $h \geq 2$. Then, for the surface $L_{a,b}$, Milnor's inequality $|[L_{a,b}]^2| \leq \frac{1}{2}|\chi_{(a,b)}|$ becomes

$$a(g - 1) + b(h - 1) \geq ab.$$

Therefore, by putting $a = h - 1 \geq 1$, the inequality is satisfied for any b . The other equations (1) - (5) which have to be fulfilled are as follows in terms of the integral indeterminants.

- (1) $p_1(M) : \alpha\delta + \beta\gamma + \eta_{12}\eta_{21} - \eta_{11}\eta_{22} + \zeta_{12}\zeta_{21} - \zeta_{11}\zeta_{22} = 0,$
- (2) $\chi(M) : \alpha\beta + \gamma\delta - \eta_{11}\zeta_{22} + \eta_{12}\zeta_{21} + \eta_{21}\zeta_{12} - \eta_{22}\zeta_{11}$
 $\qquad\qquad\qquad = 4(1 - g)(1 - h),$
- (3) $w_2(M) : \alpha \equiv \gamma, \beta \equiv \delta, \eta_{ij} \equiv \zeta_{ij} \pmod{2} \ (i, j = 1, 2)$
- (4) $\chi(L_{a,b}) : a\alpha + b\beta = 2a(1 - g) + 2b(1 - h) - 2ab,$
- (5) $[L_{a,b}]^2 : a\gamma + b\delta = 2ab.$

Now (4) and (5) are also expressed as

$$\begin{aligned} \alpha &= -\frac{b}{a}\beta + 2(1 - g) + 2\frac{b}{a}(1 - h) - 2b \\ \gamma &= -\frac{b}{a}\delta + 2b. \end{aligned}$$

Therefore take b to be a multiple of a . Then any integral choice for β and δ determines integral α and γ . Also we assume here that β and δ are even so that α and γ are also even. To fulfill (3), we also assume that all η_{ij} 's and ζ_{ij} 's are even. Already (3) - (5) are fulfilled. Now put

$$\eta_{22} = \zeta_{11} = \zeta_{12} = \zeta_{21} = 0, \quad \eta_{21} = \zeta_{22} = 2.$$

Then for any choice for a, b and any even choice of β and δ , it is easy to find even integers η_{11} and η_{12} for which (1) and (2) are satisfied. Therefore the existence of a solution is proved for $a = h - 1$ and $b = k(h - 1)$ for any $k \in \mathbb{N}$.

In the case $g = h = 1$, the argument has to be slightly modified. In particular, we need extra handles on L to achieve Milnor's inequality. So add local tiny handles to $L_{a,b}$ l -times to obtain a new surface L' which belongs to the same homology class. Then we have $\chi(L') = -2ab - 2l$. Putting $l = ab$, we can achieve the inequality as an extremal case, *i.e.*, with equality. Then the five equations are unchanged except for (4), which now takes the following form;

$$\alpha = -\frac{b}{a}\beta - 4b.$$

For the rest, the same argument works as above. □ 6.1 (a).

Proof of the case (b).

Here we only present some families of the solutions. We use the previous notations. In this case, we have $H_2(M; \mathbb{Z}) \cong H_2(T^2; \mathbb{Z}) \oplus H_2(S^2; \mathbb{Z})$ so that none of c_i, d_j, η_{ij} , or ζ_{ij} appear in the equations. However we need extra handles, whose number is denoted by l .

It is easy to verify that the requirements (1)-(5) and Milnor's inequality are satisfied in the families of solutions presented below. Here again, Milnor's inequality is satisfied as an extremal case.

Claim 6.2. *For any $a, b \geq 1$, the following families satisfy all the requirements of Theorem 4.4.*

Family 1 : $\alpha = 0, \beta = -4a, \gamma = 0, \delta = 2a, l = ab + b,$

Family 2 : $\alpha = -4b, \beta = 0, \gamma = 2b, \delta = 0, l = ab + b,$

Family 3 : $\alpha = -3b, \beta = -a, \gamma = 3b, \delta = -a, l = ab + b,$

Family 4 : $\alpha = -b, \beta = -3a, \gamma = -b, \delta = 3a, l = ab + b.$

This completes the proof for case (b). □ 6.1 (b).

In case (b), one family would have been enough. There might exist many more.

§7. Boundedness for $\Sigma_g \times S^2$

Contrary to the case in the previous section, for $M = \Sigma_g \times S^2$, $H^2(M; \mathbb{Z})$ is small enough and we can conclude that the set of homology classes which can be realized as a compact leaf of some foliation is essentially bounded. More precisely the following holds.

Theorem 7.1. *For $M = \Sigma_g \times S^2$ with $g \neq 1$, there exists a number B_g so that any homology class $a[\Sigma_g] + b[S^2] \in H_2(M; \mathbb{Z}) \cong H_2(\Sigma_g; \mathbb{Z}) \oplus H_2(S^2; \mathbb{Z})$ which is represented by a compact leaf of some oriented foliation satisfies one of the following three conditions; $a = 0, b = 0$, or $a^2 + b^2 \leq B_g$.*

Corollary 7.2. *For $M = \Sigma_g \times S^2$ with $g \neq 1$, for any compact leaf $[L]$ of any foliation on M , we have $|[L]^2| \leq B_g$.*

Remark 7.3. Our simple proof below shows that we get a bound $|a|, |b| \leq 8(g - 1)^2 + 2|g - 1|$, so that $B_g \leq 2(8(g - 1)^2 + 2|g - 1|)^2$. But this bound is far from optimal, and can be improved without much effort.

Proof of Theorem 7.1.

Assume that there exists a foliation \mathcal{F} with $e(\tau\mathcal{F}) = \alpha\{\Sigma_g\} + \beta\{S^2\}$, $e(\nu\mathcal{F}) = \gamma\{\Sigma_g\} + \delta\{S^2\} \in H^2(M; \mathbb{Z}) \cong H^2(\Sigma_g; \mathbb{Z}) \oplus H^2(S^2; \mathbb{Z})$ which has a compact leaf L representing $a[\Sigma_g] + b[S^2]$. Here $\{\Sigma_g\}$ [resp. $\{S^2\}$] denotes the cofundamental class of Σ_g [resp. S^2] pulled back to M . Without loss of generality we may assume that a and b are non-negative integers. The Euler classes of the plane fields and the homology class of L must satisfy the following equations.

$$\begin{aligned} p_1(M): \quad & e(\tau\mathcal{F})^2 + e(\nu\mathcal{F})^2 = p_1(TM) \quad \text{i.e.,} \quad \alpha\beta + \gamma\delta = 0 \\ \chi(M): \quad & e(\tau\mathcal{F})^2 \cup e(\nu\mathcal{F})^2 = e(TM) \quad \text{i.e.,} \quad \alpha\delta + \beta\gamma = 4(1 - g) \\ \chi(L): \quad & \langle e(\tau\mathcal{F})^2, [L] \rangle = \chi(L) \quad \text{i.e.,} \quad a\alpha + b\beta = \chi(L) \\ [L]^2: \quad & \langle e(\nu\mathcal{F})^2, [L] \rangle = [L]^2 \quad \text{i.e.,} \quad a\gamma + b\delta = 2ab \\ \text{Milnor:} \quad & |[L]^2| \leq |\chi(L)|/2 \quad \text{i.e.,} \quad 4ab \leq |a\alpha + b\beta| \end{aligned}$$

The proof is broken down into several steps. In the first step we show that $e(\tau\mathcal{F})$ and $e(\nu\mathcal{F})$ are bounded by using the first two equations above. In the second, we prove that $[L]$ is bounded in the case $\alpha\beta\gamma\delta \neq 0$ by using equation ($[L]^2$) only. In the third step, we deal with the case $\alpha\beta\gamma\delta = 0$. Milnor's inequality is used only in this step.

Step 1 (bound for the Euler classes): From the equations above we have the following:

$$\begin{aligned} p_1(M) + \chi(M) : \quad & (\alpha + \gamma)(\beta + \delta) = -4(g - 1) \\ p_1(M) - \chi(M) : \quad & (\alpha - \gamma)(\beta - \delta) = 4(g - 1) \end{aligned}$$

This implies that

$$1 \leq |\alpha| + |\gamma|, |\beta| + |\delta| \leq 4|g - 1|,$$

and therefore

$$|\alpha|, |\beta|, |\gamma|, |\delta| < 4|g - 1|.$$

This completes step 1.

Of course this estimate is far from being optimal. The arguments used in this step do not apply to $M = T^2 \times S^2$ because then $g - 1 = 0$.

Step 2 (bound for $[L]$, when $\alpha\beta\gamma\delta \neq 0$): Assuming $\alpha\beta\gamma\delta \neq 0$, we show that a and b are bounded. The equation $([L]^2)$ reads

$$(a - \frac{\delta}{2})(b - \frac{\gamma}{2}) = \frac{\gamma\delta}{4}$$

and we have assumed $\gamma\delta \neq 0$. Therefore, in the ab -plane, (a, b) lies on a hyperbola. Put $\bar{a} = a - \delta/2$ and $\bar{b} = b - \gamma/2$. Then the hyperbola is given by $\bar{a}\bar{b} = \frac{\gamma\delta}{4}$. To our integral point (a, b) corresponds an integral or a half-integral point (\bar{a}, \bar{b}) . Therefore we have $|\bar{a}|, |\bar{b}| \geq \frac{1}{2}$. This immediately implies $|\bar{a}|, |\bar{b}| \leq 8(g - 1)^2$, because we have seen that $|\gamma|, |\delta| \leq 4|g - 1|$. Thus we obtain

$$|a|, |b| \leq 8(g - 1)^2 + 2|g - 1|.$$

Step 3 (the case $\alpha\beta\gamma\delta = 0$): The equations $(p_1(M))$ and $(\chi(M))$ imply that “ $\alpha\beta\gamma\delta = 0$ ” is equivalent to

$$“\alpha = \delta = 0, \beta\gamma = 4(1 - g)” \text{ or } “\alpha\delta = 4(1 - g), \beta = \gamma = 0”.$$

Therefore this step further splits into two cases $\alpha = \delta = 0$ and $\beta = \gamma = 0$.

If $a = 0$ or $b = 0$ there is nothing to prove. So we may assume $a, b \geq 1$.

Assume $\alpha = \delta = 0$ and $\beta\gamma = 4(1 - g)$. Then $([L]^2)$ implies $\gamma = 2b$ and (Milnor) implies $4a \leq |\beta|$. Therefore we have $a \leq |g - 1|, b \leq 2|g - 1|$. (In fact, in this case, $g > 2, a = 1, b \leq \frac{g-1}{2}, \beta b = 2(1 - g), \gamma = 2b, \beta \equiv 0 \pmod{2}$, see Remark 7.5 below).

If $\alpha\delta = 4(1 - g)$ and $\beta = \gamma = 0$, we obtain $\delta = 2a$ and $4b \leq |\alpha|$. So $b \leq |g - 1|, a \leq 2|g - 1|$. (In fact, one gets $g > 2, a = 1, \alpha = 2(1 - g), \delta = 2, b \leq \frac{g-1}{2}$, see Remark 7.5 below). □ 7.1.

The next proposition deals with the case $a \cdot b = 0$ excluded in Theorem 7.1.

Proposition 7.4. *Let $M = \Sigma_g \times S^2, g \neq 1$. Then*

- 1) *for any $b \in \mathbb{Z}$ there is a 2-dimensional oriented foliation on M which has a compact leaf representing $b[S^2] \in H_2(\Sigma_g \times S^2)$;*
- 2) *for any $a \neq 0$ there is a 2-dimensional oriented foliation on M which has a compact leaf L representing $a[\Sigma_g] \in H_2(\Sigma_g \times S^2)$, and if $g > 1$, then for any such foliation the leaf L is genus-minimizing in its homology class.*

Proof of Proposition 7.4.

We first deal with the case $a = 0, b \in \mathbb{Z}$. We may assume that $b \geq 0$ by changing the orientation of S^2 if necessary. We also know that any homologically trivial embedded torus is a leaf of a foliation in any

homotopy class of foliations. Therefore, we may assume $b > 0$. Now, pick b disjoint vertical copies of S^2 , punch out a disk in each copy and join the resulting disks by attaching $(b - 1)$ annuli in such a way that we obtain an embedded 2-sphere L' representing $b[S^2]$.

If $\alpha\{\Sigma_g\} + \beta\{S^2\}, \gamma\{\Sigma_g\} + \delta\{S^2\}$ are the Euler classes of the tangent and normal bundle of a foliation having a compact leaf homologous to L' , then $([L]^2), (\chi(M)),$ and $(p_1(M))$ imply that $\alpha = \delta = 0$ and $\beta\gamma = 4(1 - g)$. Now choose any even negative β which divides $2(1 - g)$ and choose l so as to have

$$b\beta = 2(1 - l).$$

Then attach to $L' - l$ homologically trivial handles to obtain a connected orientable surface L homologous to L' with $\chi(L) = b\beta$. Finally, put

$$\gamma = \frac{4(1 - g)}{\beta}.$$

Then $(\alpha = 0, \beta, \gamma, \delta = 0, a = 0, b, g)$ satisfy all requirements of Theorem 4.4.

If $g > 1, a > 0,$ and $b = 0,$ we obtain $\gamma = \beta = 0, \alpha \equiv \delta \equiv 0 \pmod 2,$ and $\alpha\delta = 4(1 - g)$. It is easy to embed a surface L' in M such that the projection to Σ_g is a covering of degree a . Then L' is genus minimizing in its homology class. The easiest way to see this is to use the Gromov volume for surfaces, and the fact that the projection onto Σ_g is a map of degree a for any connected surface representing $[L']$. Therefore, for any surface L homologous to L' we have $\chi(L) = 2a(1 - g) - 2l$ with $l \geq 0$. Then $(\chi(L))$ reads $a\alpha = 2a(1 - g) - 2l,$ and we obtain $\delta = 2, \alpha = 2(1 - g), l = 0$. Again, all conditions of Theorem 4.4 are satisfied for these choices for $(\alpha, \beta = 0, \gamma = 0, \delta = 0, a, b = 0, g)$.

Of course we can also construct explicitly such a foliation very easily in the following way. Take a surjective homomorphism $\phi : \pi_1(\Sigma_g) \rightarrow \mathbb{Z}/a\mathbb{Z}$ and let $\mathbb{Z}/a\mathbb{Z}$ act on S^2 by mapping a generator of $\mathbb{Z}/a\mathbb{Z}$ onto a rotation of angle $2\pi/a$. Composing with ϕ we obtain an action of $\pi_1(\Sigma_g)$ on S^2 , and the corresponding suspension foliation is a foliation with all leaves compact. The leaves corresponding to the two fixed points of the rotations are projected to Σ_g bijectively, but any other leaf covers Σ_g with degree a . □ 7.4.

While Proposition 7.4 says that a or b are unbounded, if $a \cdot b = 0,$ it only deals with compact leaves with vanishing self-intersection.

In Section 2 we have seen examples of foliations on $\Sigma_g \times S^2$ with compact leaves representing $[\Sigma_g] + b[S^2],$ *i.e.*, with leaves of self-intersection number $2b,$ as long as $|2b| \leq |g - 1|.$ The corresponding foliations were

foliated S^2 -bundles \mathcal{F} with $e(\tau\mathcal{F}) = 2(1 - g)\{\Sigma_g\}$ and $e(\nu\mathcal{F}) = 2\{S^2\}$. This corresponds to $\alpha = 2(1 - g), \beta = \gamma = 0, \delta = 2$ in our notation above.

In fact, whenever $\beta\gamma = 0$ there are no other solutions to our equations. More generally, we have

Remark 7.5. Let \mathcal{F} be a 2-dimensional oriented foliation on $M = \Sigma_g \times S^2, g \neq 1$, with $e(\tau\mathcal{F}) = \alpha\{\Sigma_g\} + \beta\{S^2\}$, $e(\nu\mathcal{F}) = \gamma\{\Sigma_g\} + \delta\{S^2\} \in H^2(M; \mathbb{Z}) \cong H^2(\Sigma_g; \mathbb{Z}) \oplus H^2(S^2; \mathbb{Z})$ which has a compact leaf L representing $a[\Sigma_g] + b[S^2]$ with $a, b \geq 1$.

- 1) If $\beta\gamma = 0$, then $g \geq 3, \alpha = 2(1 - g), \beta = \gamma = 0, \delta = 2, a = 1, 2b \leq g - 1$, and L is genus minimizing in its homology class. As we have seen in Section 2 these data are realized by a foliation.
- 2) If $\alpha\delta = 0$, then $\alpha = \delta = 0, \gamma = 2b, \beta = 2(1 - g)/b, a = 1, b$ divides $g - 1, 2b \leq (g - 1)$, and L is genus minimizing in its homology class. Conversely, these data are realized by a foliation. We may choose this foliation as a pullback of a foliation on $\Sigma_{\bar{g}} \times S^2$ with $\bar{g} = (g - 1)/b + 1$ via a covering map $\Sigma_g \rightarrow \Sigma_{\bar{g}}$ where the compact leaf represents $[\Sigma_{\bar{g}}] + [S^2]$.

The proofs of these statements follow easily from the equations introduced in the proof of Theorem 7.1 and are left to the reader.

So far, with regard to a foliation on $\Sigma_g \times S^2 (g > 1)$ and its compact leaf representing $a[\Sigma_g] + b[S^2]$ with $a, b \geq 1$, if we assume $\alpha\beta\gamma\delta = 0$ we have $a = 1$. We will see in the next section that there are many foliations on the spaces $\Sigma_g \times S^2, g > 1$, having compact leaves representing $a[\Sigma_g] + b[S^2]$ where both $|a|$ and $|b|$ are large. Obviously, by Theorem 7.1, then g has to be large also.

§8. Problems and further discussions

To conclude this article, we present comments, discussions, and problems grouped together under three headings. The first is about geometric constructions of foliations guaranteed by our cohomological criteria. The second one contains comments and questions about foliations on $\Sigma_g \times S^2$ and their compact leaves, some of which resulted from computer calculations we conducted. The final one is concerned with our original motivation, the self-intersection of compact leaves of a foliation.

A. Geometric constructions

We have shown the existence of certain foliations with some specific compact leaves. We are naturally tempted to construct such foliations in more explicit ways. In the forthcoming paper, we are going to introduce such a construction of a foliation on \mathbb{R}^4 which has trivial T^2 -knot as a leaf and generalize it to those who have spun T^2 -knots as their leaves.

Problem 8.1. *Give explicit constructions for a wider class of T^2 -knots in \mathbb{R}^4 .*

Of course we first need a geometric and convenient presentation of the knot.

B. Foliations on $\Sigma_g \times S^2$

Here, we comment on the existence of foliations on $M = \Sigma_g \times S^2$ with compact leaves representing $a[\Sigma_g] + b[S^2]$ with both a and b , and therefore also g large. Also the fact that for many homotopy classes of foliations these leaves are minimal genus representatives in their homology classes, independent of the choice of foliation in its homotopy class, seems to be noteworthy. We observed some of these phenomena by running computer experiments.

As before, let $\alpha\{\Sigma_g\} + \beta\{S^2\}$ be the Euler class of the tangent bundle and $\gamma\{\Sigma_g\} + \delta\{S^2\}$ be the Euler class of the normal bundle of our foliation. By Remark 7.5 we have to turn to foliations with $\alpha\beta\gamma\delta \neq 0$, if we are looking for foliations with compact leaves representing $a[\Sigma_g] + b[S^2]$ with $a > 1, b > 0$.

Initially, we were uncertain about the existence of such foliations. But searching for solutions of the four equations, Milnor's inequality, and the congruences coming from Theorem 4.4 with the help of a computer, we saw that solutions abound. Below we present two 3-parameter families of foliations with associated homology classes of compact leaves by listing the associated values of the eight variables $g, \alpha, \beta, \gamma, \delta, a, b, l$. Without loss of generality we may assume that a and b are positive. The non-negative integer l is the number of homologically trivial handles added to the surface of minimal genus in the homology class $a[\Sigma_g] + b[S^2]$. So $l = 0$ is equivalent to the statement that the compact leaf is genus-minimizing in its homology class.

Also notice the following. If

$$\bar{g}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{a}, \bar{b}, \bar{l}$$

are the homological data for a foliation with compact leaf on $\Sigma_{\bar{g}} \times S^2$, then pulling back the bundle, foliation and compact leaf by a d -fold

covering map $\Sigma_g \rightarrow \Sigma_{\bar{g}}$ we get a foliation with compact leaf on $\Sigma_g \times S^2$ with homological data

$$g = d(\bar{g} - 1) + 1, \alpha = d\bar{\alpha}, \beta = \bar{\beta}, \gamma = d\bar{\gamma}, \delta = \bar{\delta}, a = \bar{a}, b = d\bar{b}, l = d\bar{l}.$$

Conversely, if $g - 1, \alpha, \gamma, b, l$ are divisible by d , then these data are obtained by pulling back a foliation with compact leaf via a d -fold cover.

Therefore, below we only list data not coming from coverings.

Example 8.2. (Family of solutions with trivial handles)

(F1) For integers $0 \leq x < y, 0 \leq z$ set

$$\begin{aligned} F_1(x, y, z) : \\ g &= (2z + 1)((2y + 1)^2 - (2x + 1)^2)/4 + 1 \\ \alpha &= 2x + 1, \beta = -((2y + 1)(2z + 1)), \\ \gamma &= 2y + 1, \delta = (2x + 1)(2z + 1), \\ a &= (\delta + 1)/2, b = a\gamma, \\ l &= (z + 1)(y - x)(1 + (2z + 1)(y - x)) \end{aligned}$$

It is easy to check that all equations are satisfied and that the congruence holds. Milnor's inequality is a little messy to write down, but it will hold if y is large when compared with x . For example, if $z = 0$, then $y \geq 2x + 2$, and if $z > 0$, then $y \geq 2x + 1$ will suffice. \square 8.2.

Obviously, $F_1(x, y, z)$ comes from a covering, if and only if the the greatest common divisor of $g - 1, \alpha, \gamma, l$ is greater than 1.

Note that $l > 0$, so that L is not genus-minimizing in all these examples. Also the genus g of the base surface is always odd. In fact, we have observed in our computer calculations the following phenomenon: if $a, b > 0$ and L is not genus-minimizing, then g is odd, $\alpha > 0$, and α is small when compared with g . Furthermore, if the data do not come from a covering, then α is odd and δ is an odd multiple of α .

Therefore, in this case, α, β, γ , and δ must have the above form. Up to $g = 90$ all examples with $l > 0$ are in the family (F1) or cover an element of this family.

Problem 8.3. Do all foliations with a compact leaf which is not genus-minimizing and representing a class with $a, b \geq 1$ belong to (F1) or cover a foliation from (F1)?

The description of the next family is slightly less direct than that of (F1).

Example 8.4. (Family of solutions without trivial handles)

(F2) This family is parametrized by a rational number x with $0 < x < 1/3$, and for each such x by an arithmetic progression

for the genus g of the base surface Σ_g . The denominator and numerator of x make it into a 3-parameter family. Specifically, for given x , set

$$a = xg + 1 - 2x, b = xg/(1 - 2x) + 1, \text{ and } l = 0.$$

Choose $g > 1$, so that a and b are positive integers, and then set

$$\alpha = -g, \beta = -g + 2, \gamma = -g + 2, \text{ and } \delta = g.$$

Obviously, all equations and the congruence hold. In order that Milnor's inequality holds, g has to be sufficiently large.

The arithmetic progression for g is obtained as follows:
Let $x = p/q, (p, q) = 1, q > 3p$. Then for any $k \geq 0$ set

$$g = q'q + 2 + q(q - 2p)k,$$

where $0 < q' < q - 2p$ solves $q'q + 2 \equiv 0 \pmod{q - 2p}$. Then a and b are positive integers. In order that Milnor's inequality holds we have to choose k large enough.

Notice that in family (F2) α is always negative, and the compact leaves are genus-minimizing. Furthermore, since α and $g - 1$ are coprime, no member is the result of a pull-back via a covering map of the base surfaces. □ 8.4.

For (F2), there are no restrictions on the parity of g . In fact, in our computer calculations, solutions with $l = 0$ occur for every $g > 2$, and for odd g their number exceeds the number of solutions with $l > 0$ by a factor of at least 2. This is probably due to the fact that the foliations with genus-minimizing compact leaves occur more often as coverings. In fact, we do not know whether for large odd g the ratio of the number of solutions with $l = 0$ to the number with $l > 0$ has a limsup greater than 0, once we discard foliations which are coverings.

Problem 8.5. *What can be said about this ratio?*

With regard to the number of classes $\alpha\{\Sigma_g\} + \beta\{S^2\}$, which occur as the Euler class of a foliation with a compact leaf representing $a[\Sigma_g] + b[S^2]$ with $a, b > 0$, there are many more classes with $\alpha < 0$ than classes with $\alpha > 0$.

Problem 8.6. *What is the reason for this?*

With the usual meanings of a, b, α , we have mentioned above that for $a, b > 0$, as far as we know, the compact leaves of all foliations with $\alpha > 0$ are not genus-minimizing, while the one's with $\alpha < 0$ are genus-minimizing.

Problem 8.7. *Do the Euler classes of a foliation determine whether a compact leaf of this foliation is genus-minimizing in its homology class?*

Looking at Remark 7.5 we see that a given compact surface which is genus-minimizing in its homology class can be a leaf of foliations with different Euler classes. For example, if $g > 2$ and the surface represents $[\Sigma_g] + [S^2]$, there exist foliations with Euler class equal to any of $2(1-g)\{\Sigma_g\}$, $2(1-g)\{S^2\}$, $-g\{\Sigma_g\} + (2-g)\{S^2\}$, $(2-g)\{\Sigma_g\} - g\{S^2\}$ having this surface as a compact leaf.

There is also an occurrence where leaves in the same homology class, one genus-minimizing, the other not, are leaves of foliations. These necessarily have distinct Euler classes. But up to coverings we have only one example for this: $g = 19, a = 2, b = 10$. One foliation is given by $\alpha = -21, \beta = -5, \gamma = -15, \delta = 7$, and $l = 0$, i.e., the leaf is genus-minimizing. The other is given by $\alpha = 1, \beta = -15, \gamma = 5, \delta = 3$, and $l = 28$.

So one might pose the following

Problem 8.8. *In general, does each homology class know its foliated genus, i.e., the genus of a representing surface which is a leaf of a foliation on the manifold?*

In all our examples on $M = \Sigma_g \times \Sigma_h$ we have seen that genus-minimizing leaves representing $a[\Sigma_g] + b[\Sigma_h]$ can be chosen to be symplectic submanifolds of M with its standard symplectic structure.

Problem 8.9. *If a surface L in $M = \Sigma_g \times S^2$ is genus-minimizing in its homology class and a leaf of a foliation, does there exist a symplectic foliation, i.e., a foliation such that all leaves are symplectic submanifolds of M with respect to some symplectic structure on M , with L as a leaf? If it is not true, then, look for a condition which guarantees that L is a leaf of a symplectic foliation.*

This question might make sense for more general closed symplectic 4-manifolds, but so far, we have not looked into this.

C. Bounds of self-intersection numbers of compact leaves

In Sections 6 and 7 we settled the question for which products of two surfaces there is a bound on the self-intersection numbers of compact surfaces which occur as leaves of a foliation.

Problem 8.10. *For which 4-manifolds M is the set of self-intersection numbers of compact leaves of foliations on M bounded?*

For which 4-manifolds with positive second Betti number is the set of homology classes which are represented by some compact leaf of some foliation essentially bounded?

Here, “essentially bounded” should be interpreted in some reasonable way. For example, like in our context: bounded when restricted to classes of non-zero self-intersection.

It is premature to venture a guess whether among closed 4-manifolds admitting a 2-dimensional foliation with a compact leaf those with a bound on the self-intersection numbers of these leaves are more prevalent or not. In the small set of 4-manifolds that we considered they did occur less often.

However, in the case of foliated bundles, the situation should be different.

Problem 8.11. *Prove that for a given Σ_h -bundle over Σ_g , there is a bound for the set of self-intersection numbers of compact leaves of foliations transverse to the fibres.*

More strongly, for any given g and h , prove that there is a bound for the set of self-intersection numbers of compact leaves of any foliated Σ_h -bundle over Σ_g .

This problem is strengthened further by dropping the flatness condition of the bundle. Then of course the bound should be larger.

Problem 8.12. *For given h and g , does there exist an upper bound for the self-intersection number of any multi-section of any Σ_h -bundle over Σ_g ?*

This is no longer a problem of foliations.

Example 8.13. There exists a multi-section of $\Sigma_2 \times \Sigma_2$, which covers both the base and the fibre twice. Since its normal and tangent bundles are isomorphic, its self-intersection number is -4 .

To prove the existence of such a multi-section, it is enough to find a pair of orientation preserving free involutions σ and τ on Σ_3 such that $\sigma \circ \tau$ has no fixed point. This is done as follows.

In $\mathbb{R}^3 = \{(x, y, z)\}$, take a spatial graph $\Gamma = S^2 \cap \{(x+y)(x-y) = 0\}$ consisting of four longitudes connecting the north and south poles. We realize Σ_3 as the smooth boundary of a thin regular neighbourhood of Γ in \mathbb{R}^3 . Then the involutions σ and τ on Σ_3 are defined as follows. σ is the rotation $(x, y, z) \mapsto (x, -y, -z)$ around the x -axis by π restricted to Σ_3 . τ is the rotation around the great circle $S^2 \cap \{(x+y) = 0\}$ by π . The two 1-handles around this great circle are invariant and rotated while the other two 1-handles connecting the regions close to the poles

are exchanged by τ . In other words τ is the composition of the inversion of \mathbb{R}^3 with respect to S^2 and the reflection at the plane $x + y = 0$.

It is easy to verify that this pair of free involutions fulfills our requirements.

Then, the involutions σ and τ define two double coverings

$$\pi_\sigma, \pi_\tau : \Sigma_3 \longrightarrow \Sigma_2$$

so that each involution is the non-trivial covering transformation.

Now we define the multi-section as the image of the following map.

$$\varphi : \Sigma_3 \ni p \mapsto (\pi_\sigma(p), \pi_\tau(p)) \in \Sigma_2 \times \Sigma_2.$$

Since $\sigma \circ \tau$ has no fixed point φ is an embedding. Therefore it defines a multi-section with the required properties. \square 8.13.

We believe that this multi-section is the one with the largest self-intersection number among all multi-sections of this product bundle. Also notice that Milnor's inequality prohibits this surface to be a leaf of a foliation on $\Sigma_2 \times \Sigma_2$.

Are there methods to prove the statement about the maximality of the self-intersection number?

Of course, questions about the existence of multi-sections of surface bundles with certain properties can be interpreted as questions about the pointed mapping class groups of these surfaces.

Many of the problems above have obvious generalizations to foliations of higher dimension or codimension. But our methods are very specific for 2-dimensional foliations on 4-manifolds and so one needs some new ideas to proceed. Independent of this, we think it is important and worthwhile to pursue the study of the subjects dealt with in this paper in other dimensions. On the other hand, it is true that there is still an abundance of problems that remain to be settled in dimension 4.

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