Advanced Studies in Pure Mathematics 52, 2008
Groups of Diffeomorphisms
pp. 401–413

Stable length in stable groups

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Abstract.

We show that the stable commutator length vanishes for certain groups defined as infinite unions of smaller groups. The argument uses a group-theoretic analogue of the Mazur swindle, and goes back to the works of Anderson, Fisher, and Mather on homeomorphism groups.

§1. Introduction

In this paper we show that the stable commutator length vanishes for certain groups defined as unions of subgroups that have many conjugate embeddings that commute element-wise. The argument used is a group-theoretic analogue of the Mazur swindle. Informally, it can be paraphrased by saying that as the size of a table grows, it becomes easier to sort objects on its surface, so that commutation can be done very efficiently on a table of infinite size[1]. In the words of Poénaru [18]: "The infinite comes with magic and power."

Let Γ be a group. The commutator length $c(g)$ of an element g in the commutator subgroup $[\Gamma,\Gamma] \subset \Gamma$ is the minimal number of factors needed to write g as a product of elements that can be expressed as single commutators. The stable commutator length of g is defined to be

$$||g|| = \lim_{n\to\infty} \frac{c(g^n)}{n} .$$

Direct proofs of the vanishing of the stable commutator length sometimes proceed by showing a lot more, namely that the commutator length itself is bounded. A prototypical argument for this in the context of homeomorphism groups goes back to Anderson [2] and Fisher [11], and was later used and refined by Mather [16] and by Matsumoto–Morita [17].

Received August 24, 2007.
Revised June 16, 2008.

[1]I owe this description to N. A'Campo.

This argument uses an infinite iteration that leads to complicated behaviour near one point, and is therefore not suitable for the study of diffeomorphism groups. The argument we employ here is a variation on this classical one. It works for diffeomorphism groups because instead of an infinite iteration we do only a finite iteration, however it is important that the finite number of iterations can be taken to be arbitrarily large. This argument does not prove that the commutator length is bounded, it only proves the weaker conclusion that the stable commutator length vanishes.

Bavard [5], using a Hahn–Banach argument influenced by that of Matsumoto and Morita [17], showed that the vanishing of the stable commutator length on the commutator subgroup is equivalent to the injectivity of the comparison map $H_b^2(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$, where $H_b^*(\Gamma)$ denotes bounded group cohomology in the sense of Gromov. Another way to express this condition is to say that every homogeneous quasi-homomorphism $\Gamma \to \mathbb{R}$ is in fact a homomorphism. If Γ is a perfect group, then the stable commutator length is defined on the whole of Γ, and vanishes identically if and only if there are no non-trivial homogeneous quasi-homomorphisms from Γ to $\mathbb{R}$.

Bavard's result suggests that the commutator calculus arguments discussed above should have a dual formulation in terms of quasi-homomorphisms. It is this dual formulation that we shall discuss, although we could equally well argue directly on the commutator side. Thus, what we prove for certain groups is that every homogeneous quasi-homomorphism is a homomorphism. Using Bavard's result this is equivalent to the vanishing of the stable commutator length.

In Section 2 we give the algebraic mechanism behind the vanishing results we shall prove. Section 3 applies this mechanism to the stable mapping class group, the braid group on infinitely many strands, and the stable automorphism groups of free groups. In Section 4 we give applications to diffeomorphism groups, and in Section 5 we compare our methods and results to those of Burago, Ivanov and Polterovich [8].

We refer the reader to [5, 15] for background on quasi-homomorphisms. In fact, the proofs of the vanishing results below are somewhat reminiscent of the discussion of weak bounded generation in [15].

§2. The algebraic vanishing result

A map $f\colon \Gamma \to \mathbb{R}$ is called a quasi-homomorphism if its deviation from being a homomorphism is bounded; in other words, there exists a

constant $D(f)$, called the defect of f, such that

$$|f(xy) - f(x) - f(y)| \leq D(f)$$

for all $x, y \in \Gamma$. We will always take $D(f)$ to be the smallest number with this property, i.e. it is the supremum of the left hand sides over all x and $y \in \Gamma$.

Every quasi-homomorphism can be homogenized by defining

$$\varphi(g) = \lim_{n\to\infty} \frac{f(g^n)}{n} .$$

Then φ is again a quasi-homomorphism, is homogeneous in the sense that $\varphi(g^n) = n\varphi(g)$, and is constant on conjugacy classes. (Compare [5], Proposition 3.3.1.) Throughout this paper we shall only consider homogeneous quasi-homomorphisms.

We begin with the following preliminary result.

Lemma 2.1. *Let $\varphi\colon \Gamma \to \mathbb{R}$ be a homogeneous quasi-homomorphism. Then the following two properties hold:*

(1) *If $x, y \in \Gamma$ commute, then $\varphi(xy) = \varphi(x) + \varphi(y)$.*
(2) *If φ vanishes on every single commutator, then φ is a homomorphism.*

Proof. By homogeneity we have the following:

$$\begin{aligned} |\varphi(xy) - \varphi(x) - \varphi(y)| &= \lim_{n\to\infty} \frac{1}{n} |\varphi((xy)^n) + \varphi(x^{-n}) + \varphi(y^{-n})| \\ &\leq \lim_{n\to\infty} \frac{1}{n} \left(|\varphi((xy)^n x^{-n} y^{-n})| + 2D(\varphi)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} |\varphi((xy)^n x^{-n} y^{-n})| . \end{aligned}$$

If x and y commute, then $\varphi((xy)^n x^{-n} y^{-n}) = 0$ for every n, giving the first statement.

Assume now that φ vanishes on single commutators. As $(xy)^n x^{-n} y^{-n}$ can be expressed as the product of $\frac{n}{2} + c$ commutators, see [5], the right hand side of the formula is bounded above by $\frac{1}{2} D(\varphi)$. However, taking the supremum of the left hand side over all x and y, we get the defect $D(\varphi)$. Thus $D(\varphi) \leq \frac{1}{2} D(\varphi)$, showing that the defect vanishes, and φ is a homomorphism. Q.E.D.

Here is the main mechanism for the vanishing theorems.

Proposition 2.2. *Let $\Lambda \subset \Gamma$ be a subgroup with the property that there is an arbitrarily large number of conjugate embeddings $\Lambda_i \subset \Gamma$*

of Λ in Γ with the property that elements of Λ_i and of Λ_j commute with each other in Γ whenever $i \neq j$. Then every homogeneous quasi-homomorphism on Γ restricts to Λ as a homomorphism.

Note that homogeneous quasi-homomorphisms are constant on conjugacy classes, so that the restriction of φ to Λ_i is independent of i.

Proof. By the second part of Lemma 2.1 we only have to prove that $\varphi([x,y]) = 0$ for any $x, y \in \Lambda$. Let x_i, $y_i \in \Gamma$ be the images of x and y under the embedding $\Lambda_i \subset \Gamma$. We have the following equalities:

$$\begin{aligned} n\varphi([x,y]) &= \varphi([x_1,y_1]) + \ldots + \varphi([x_n,y_n]) \\ &= \varphi([x_1,y_1]\ldots[x_n,y_n]) \\ &= \varphi([x_1\ldots x_n, y_1\ldots y_n]) \ , \end{aligned}$$

where the first one comes from the constancy of φ on conjugacy classes, the second is the first part of Lemma 2.1 applied to the commuting embeddings, and the third follows directly from the commuting property of the embeddings $\Lambda_i \subset \Gamma$. On the right hand side φ is applied to a single commutator in Γ, and therefore the right hand side is bounded in absolute value by the defect of φ on Γ. However, if $\varphi([x,y]) \neq 0$, then the left hand side is unbounded because by assumption we can make n arbitrarily large. Thus $\varphi([x,y]) = 0$ for all $x, y \in \Lambda$. Q.E.D.

There are several ways in which this mechanism can be applied to obtain the vanishing of the stable commutator length in various groups. We try to give a general statement, in the hope of unifying several different applications of the argument.

Theorem 2.3. *Let Γ be a group in which every element can be decomposed as a product of some fixed number k of elements contained in distinguished subgroups $\Lambda \subset \Gamma$. If each Λ is perfect and has the property in Proposition 2.2, then the stable commutator length of Γ vanishes.*

Proof. As the subgroups Λ are assumed perfect, the restrictions of quasi-homomorphisms on Γ to Λ vanish, because by Proposition 2.2 they are homomorphisms. Therefore the value of a quasi-homomorphism on every element of Γ is bounded by $k-1$ times the defect. But every bounded homogeneous quasi-homomorphism is trivial. Q.E.D.

§3. Applications to discrete groups

3.1. The stable mapping class group

Let Γ_g^1 be the group of isotopy classes of diffeomorphisms with compact support in the interior of a compact surface Σ_g^1 of genus g with one

boundary component. Attaching a two-holed torus along the boundary defines the stabilization homomorphism $\Gamma_g^1 \to \Gamma_{g+1}^1$. The stable mapping class group Γ_∞ is defined as the limit

$$\Gamma_\infty = \varinjlim_g \Gamma_g^1 \ .$$

For $g \geq 3$ the groups Γ_g^1 are perfect, see [19], hence Γ_∞ is also perfect. Recall that in [10] it was proved that the stable commutator length is non-trivial on every Γ_g^1 with $g \geq 2$, see also [6, 7, 9, 15]. In contrast with this we have:

Theorem 3.1. *The stable commutator length for Γ_∞ vanishes identically.*

Proof. We apply Theorem 2.3 with $k = 1$. The subgroups Λ are the images of the Γ_g^1. In detail, every element of Γ_∞ is in the image of some $\Gamma_g^1 = \Lambda$. This has arbitrarily large numbers of commuting conjugate embeddings in Γ_∞ given by taking the boundary connected sum of an arbitrarily large number of copies of the surface of genus g with one boundary component. Any homogeneous quasi-homomorphism on Γ_∞ restricts to (the image of) Γ_g^1 as a homomorphism. However, for $g \geq 3$ this group is perfect, and so the homomorphism vanishes. Q.E.D.

Remark 3.2. Theorem 3.1 shows that the second bounded cohomology of mapping class groups does not stabilize. This contrasts sharply with the Harer stability theorem [13] for the ordinary group cohomology.

Remark 3.3. Theorem 3.1 fits in nicely with the form of the estimates for the stable commutator length obtained in [10, 7, 15]. The lower bounds given there for the stable commutator length of specific elements in Γ_g^1 go to zero for $g \to \infty$. A similar phenomenon seems to appear in the work of Calegari and Fujiwara [9].

Remark 3.4. The discussion in this subsection also applies to the stable mapping class groups for surfaces with several boundary components.

3.2. The braid group on infinitely many strands

Let B_n be the Artin braid group on n strands. Adding strands defines injective stabilization homomorphisms $B_n \longrightarrow B_{n+1}$. The braid group on infinitely many strands is

$$B_\infty = \bigcup_n B_n \ .$$

Theorem 3.5. *Any homogeneous quasi-homomorphism on the infinite braid group B_∞ is a homomorphism.*

Proof. We apply Proposition 2.2. The subgroups Λ are the B_n. Any two $x, y \in B_\infty$ are contained in some finite $B_n = \Lambda$. This has arbitrarily large numbers of commuting conjugate embeddings in B_∞ given by considering braids on n strands placed side by side. Any homogeneous quasi-homomorphism φ on B_∞ therefore restricts to B_n as a homomorphism. Thus $\varphi(xy) = \varphi(x) + \varphi(y)$, showing that φ is a homomorphism on B_∞. Q.E.D.

We have chosen this formulation of the result because the braid groups have infinite Abelianizations, so that it does not make sense to speak of the stable commutator lengths of elements. The stabilization is compatible with Abelianization, so that B_∞, like B_n for finite n, has infinite cyclic Abelianization. The conclusion of Theorem 3.5 is that every homogeneous quasi-homomorphism is a constant multiple of the Abelianization homomorphism.

For finite n, the braid groups do admit non-trivial homogeneous quasi-homomorphisms that are not proportional to the Abelianization map $B_n \longrightarrow \mathbb{Z}$, see [3, 12]. Thus Theorem 3.5 shows that the bounded cohomology of braid groups does not stabilize. The quotients B_n/C of finite braid groups modulo their centers have finite Abelianizations, so that all elements have powers that are products of commutators. Baader [3] proves some lower bounds for the stable commutator length of certain elements in B_n/C. These lower bounds tend to zero as $n \to \infty$. This of course fits with Theorem 3.5. The corresponding homogeneous quasi-homomorphisms defined on B_n/C, and, by composition, on B_n, do not extend to the braid group on infinitely many strands.

3.3. Automorphism groups of free groups

The canonical homomorphisms $F_n \longrightarrow F_n \star \mathbb{Z} = F_{n+1}$ give rise to injective homomorphisms $\mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(F_{n+1})$. Thus we define

$$\mathrm{Aut}_\infty(F) = \bigcup_n \mathrm{Aut}(F_n) \ .$$

Note that this is smaller than $\mathrm{Aut}(F_\infty)$.

Theorem 3.6. *The stable commutator length for $\mathrm{Aut}_\infty(F)$ vanishes identically.*

Proof. The Abelianization of $\mathrm{Aut}(F_n)$ is of order 2, and is stable. Thus, up to taking squares, all elements in $\mathrm{Aut}(F_n)$ and in $\mathrm{Aut}_\infty(F)$ are products of commutators. We again apply Proposition 2.2. The

subgroups Λ are the $\mathrm{Aut}(F_n)$. In detail, every element of $\mathrm{Aut}_\infty(F)$ is contained in some $\mathrm{Aut}(F_n) = \Lambda$. This has arbitrarily large numbers of commuting conjugate embeddings in $\mathrm{Aut}_\infty(F)$ given by embedding $F_n \star F_n \star \ldots \star F_n$ in some large F_N. Although the elements in the different copies of F_n do not commute, the induced embeddings of $\mathrm{Aut}(F_n)$ in $\mathrm{Aut}_\infty(F)$ do commute and are conjugate to each other. Any homogeneous quasi-homomorphism on $\mathrm{Aut}_\infty(F)$ restricts to $\mathrm{Aut}(F_n)$ as a homomorphism. As the Abelianization of $\mathrm{Aut}(F_n)$ is finite, the homomorphism is trivial. Q.E.D.

One could also consider the outer automorphism groups $\mathrm{Out}(F_n)$, and of course quasi-homomorphisms on these induce quasi-homomorphisms on $\mathrm{Aut}(F_n)$ by composition with the projection. However, there is no natural way of stabilizing the outer automorphism groups.

Unlike for mapping class groups, no unbounded quasi-homomorphisms are known on the automorphism groups of free groups. The groups $\mathrm{Aut}(F_n)$ are analogous to the so-called extended mapping class groups, consisting of the isotopy classes of all diffeomorphisms of surfaces. The usual mapping class groups we considered in Theorem 3.1 are index 2 subgroups of the extended mapping class groups. A single Dehn twist has non-zero stable commutator length in a mapping class group [10, 7, 15], but is conjugate to its inverse in the extended mapping class group [1], so that in this latter group its stable commutator length must vanish. By analogy with this phenomenon, it may be more promising to look for quasi-homomorphisms on the special automorphism group $S\,\mathrm{Aut}(F_n)$, rather than on $\mathrm{Aut}(F_n)$, where by $S\,\mathrm{Aut}(F_n)$ we denote the automorphisms preserving an orientation on the Abelianization $\mathbb{Z}^n$ of F_n. Theorem 3.6 also applies to the stabilized special automorphism group $S\,\mathrm{Aut}_\infty(F)$.

§4. Applications to diffeomorphism groups

In this section we apply the algebraic vanishing result to some groups of diffeomorphisms. If M is a smooth closed manifold, we consider the identity component $G = \mathrm{Diff}_0(M)$ of the full diffeomorphism group. If M is open, we consider the identity component $G = \mathrm{Diff}_0^c(M)$ of the group of compactly supported diffeomorphisms. (Sometimes this notation is redundant because the group of compactly supported diffeomorphisms may be connected.) In both cases we assume that the diffeomorphisms are of class C^r with $1 \leq r \leq \infty$ and $r \neq 1 + \dim(M)$. The classical results of Herman, Thurston, Epstein and Mather then ensure that G is a perfect group; see [4] and the references quoted there.

4.1. The full diffeomorphism groups

We would like to prove that the stable commutator length vanishes on the diffeomorphism groups. Unfortunately, we can only achieve this goal in the following special cases:

Theorem 4.1. *The stable commutator length vanishes for the following diffeomorphism groups:*

(1) $\mathrm{Diff}_0^c(D)$ *for a ball* D,
(2) $\mathrm{Diff}_0^c(D \times M)$ *for any* M *and a ball* D *of positive dimension,*
(3) $\mathrm{Diff}_0(S^n)$ *for any sphere,*
(4) $\mathrm{Diff}_0(\Sigma^n)$ *for any exotic sphere of dimension* $\neq 4$.

Proof. Let D be an n-dimensional ball, and $G = \mathrm{Diff}_0^c(D)$ the group of diffeomorphisms of D with compact support in the interior of D. Fix an exhaustion of D by smaller nested balls D_i so that each D_i has closure contained in the interior of D_{i+1}. If G_i is the group of diffeomorphisms of D_i with compact support in the interior of D_i, then we have injective homomorphisms $G_i \to G_{i+1}$ induced by the inclusion $D_i \subset D_{i+1}$, and

$$G = \bigcup_i G_i \ .$$

We now apply Theorem 2.3 with $k = 1$. The subgroups Λ are the G_i. In detail, every element of G is contained in some G_i. This has arbitrarily large numbers of commuting conjugate embeddings in G, because we can embed arbitrarily large numbers of disjoint (smaller) balls in the annulus $D \setminus D_i$. Any homogeneous quasi-homomorphism on G then restricts to G_i as a homomorphism. But this group is perfect, and so the homomorphism vanishes. This completes the proof of the first claim. The same argument works for $D \times M$ using the exhaustion by $D_i \times M$.

Now let $G = \mathrm{Diff}_0(S^n)$. We can compose any $g \in G$ with a suitable $f_1^{-1} \in G$ supported in an open ball to achieve that $f_1^{-1}g$ has a fixed point. It follows that we can write $f_1^{-1}g = f_2 f_3$, with f_3 having support in a ball around the fixed point, and f_2 having support in the complement of a (smaller) ball around the fixed point. Because the manifold is a sphere, this complement is again a ball. Using $g = f_1 f_2 f_3$, we apply Theorem 2.3 with $k = 3$. Let $\Lambda = \mathrm{Diff}^c(D)$ be the group of compactly supported diffeomorphisms of a ball. This is perfect and admits infinitely many conjugate commuting embeddings in G.

The argument given for spheres also works for exotic spheres in dimensions $n \neq 4$, because they are all twisted spheres obtained from two standard balls by gluing along the boundary [14]. Q.E.D.

After this paper had been submitted, Tsuboi [20] proved a much stronger result. He proved that $\mathrm{Diff}_0(M)$ is uniformly perfect for every closed manifold, with the possible exception of even-dimensional manifolds having no handle decomposition without handles of middle dimension. The simplest M for which the question is unresolved, is T^2. Because of Tsuboi's results, I have removed some material from the original version of this paper, which was concerned with the continuity, with respect to the C^0 topology, of potential homogeneous quasi-homomorphisms on diffeomorphism groups. That discussion is rather delicate technically, and I hope to return to it elsewhere, unless it is rendered completely empty by a generalization of Tsuboi's work.

4.2. Diffeomorphism groups preserving a symplectic or volume form

The first statement in Theorem 4.1 can be phrased for the compactly supported diffeomorphism group of Euclidean space $\mathbb{R}^n$ instead of a ball D. However, the two cases are rather different if we consider diffeomorphism groups preserving a symplectic or volume form ω, and we fix our conventions so that $\mathbb{R}^n$ has infinite (symplectic) volume and a ball has finite volume.

Let $G = \mathrm{Diff}_0^c(\mathbb{R}^n, \omega)$ be the identity component of the group of compactly supported diffeomorphisms preserving ω. This is not perfect because the Calabi invariant

$$\mathrm{Cal}\colon \mathrm{Diff}_0^c(\mathbb{R}^n, \omega) \longrightarrow \mathbb{R} \tag{1}$$

is a non-trivial homomorphism. (See for example [4] for the definition of Cal.) By results of Thurston and Banyaga, see [4], its kernel is a perfect group. It follows that Cal is the Abelianization homomorphism. So far there is no difference between the finite and infinite volume cases, and we have the same statements if we replace $(\mathbb{R}^n, \omega)$ by a ball of finite volume.

Theorem 4.2. *The stable commutator length vanishes on the kernel of the Calabi homomorphism in* $\mathrm{Diff}_0^c(\mathbb{R}^n, \omega)$. *On* $\mathrm{Diff}_0^c(\mathbb{R}^n, \omega)$ *every homogeneous quasi-homomorphism is a constant multiple of the Calabi invariant.*

Proof. As in the previous proof we can write the kernel of Cal as

$$G = \bigcup_i G_i \ ,$$

with each G_i consisting of those elements of G supported in a ball of radius i, say. Then each element of G is contained in some G_i, but this

G_i has arbitrarily large numbers of commuting conjugate embeddings given by disjoint Hamiltonian displacements of D_i in $\mathbb{R}^n$. Therefore we can apply Theorem 2.3 to conclude that every homogeneous quasi-homomorphism vanishes on Ker Cal.

Applying the same argument to $G = \mathrm{Diff}_0^c(\mathbb{R}^n, \omega)$ with each G_i equal to $\mathrm{Diff}_0^c(D_i, \omega)$, we conclude that the restriction to G_i of any homogeneous quasi-homomorphism φ on G is a homomorphism, and is therefore a constant multiple of the Calabi invariant. But any two balls in $\mathbb{R}^n$ are both contained in some larger ball, and therefore the restriction of φ is the same multiple of Cal on all balls. Q.E.D.

This argument clearly does not work in the finite volume case because the number of commuting conjugate embeddings one can construct is bounded in terms of the available volume. The abstract isomorphism type of the group of symplectic or volume-preserving diffeomorphisms of a ball is always the same. In particular it does not depend on the size or volume of the ball. Therefore one can always find arbitrarily large numbers of commuting embeddings of such a group in the corresponding group of a larger ball, or in itself, but these embeddings are conjugate only in the full diffeomorphism group, and not in the diffeomorphism group preserving ω.

§5. Comparison with results on quasi-norms

In this section we compare our results on the non-existence of homogeneous quasi-homomorphisms on various stable groups to results about existence and non-existence of quasi-norms. The following definitions are due to Burago, Ivanov and Polterovich [8].

Definition 5.1. A function $q\colon G \longrightarrow \mathbb{R}$ on a group is called a quasi-norm if it is almost subadditive, almost invariant under conjugacy, and unbounded.

A group G is called unbounded if it admits a quasi-norm, and is called bounded otherwise.

Here almost subadditive and almost invariant under conjugation means that there is a constant c such that

$$q(xy) \leq q(x) + q(y) + c$$

and

$$|q(xyx^{-1}) - q(y)| \leq c$$

for all x and $y \in G$.

If φ is a non-trivial homogeneous quasi-homomorphism, then its absolute value is a quasi-norm. Therefore boundedness of a group is an even stronger property than the non-existence of homogeneous quasi-homomorphisms. A bounded group has finite Abelianization and vanishing stable commutator length. If the Abelianization is trivial, then the group is uniformly perfect [8].

Burago, Ivanov and Polterovich [8] prove boundedness of many diffeomorphism groups by an argument that is rather stronger than ours. In particular, they prove boundedness for the groups we considered in Theorem 4.1. An easy example that illustrates the difference between the vanishing mechanism in [8] and the one considered here is the following.

Example 5.2. Consider the group $G = \mathrm{Diff}_0^c(\mathbb{R}^n, \omega)$ from Theorem 4.2. Then the function q which assigns to every $g \in G$ the (symplectic) volume of its support is a quasi-norm which is non-trivial on the kernel of the Calabi homomorphism. Therefore this kernel is an unbounded group which nevertheless has vanishing stable commutator length.

This also brings out another aspect of the difference between the finite and infinite volume cases mentioned earlier. If we take $G = \mathrm{Diff}_0^c(D, \omega)$ with D of finite (symplectic) volume, then q is bounded and therefore not a quasi-norm.

This idea of a quasi-norm defined by support size actually applies to all the discrete groups we considered in Section 3:

Theorem 5.3. *The following groups are unbounded:*

(1) *the stable mapping class group* Γ_∞,
(2) *the braid group* B_∞ *on infinitely many strands,*
(3) *the stable automorphism group of free groups* $Aut_\infty(F)$, *and*
(4) *the stable special automorphism group of free groups* $S\,\mathrm{Aut}_\infty(F)$.

Proof. Perhaps the case of the braid group is easiest to visualize. Define $q\colon B_\infty \longrightarrow \mathbb{Z}$ by sending a braid x to the smallest number k of strands needed to express it. These need not be the first k strands, but can be any k strands. The function q is clearly invariant under conjugation and is unbounded. It is also subadditive, because a set of strands used to express x and a set of strands used to express y can together be used to express xy. Thus q is a quasi-norm.

On the stable mapping class group define $q\colon \Gamma_\infty \longrightarrow \mathbb{Z}$ by mapping an element x to the smallest genus g of a compact surface with one boundary component on which a diffeomorphism representing the

isotopy class x can be supported. Again this does not have to be the embedding of Σ_g^1 from the definition of the stabilization procedure, but can be any such subsurface of the infinite genus surface. By definition, this function is conjugacy-invariant. It is also not hard to see that it is unbounded, for example by considering the action of mapping classes on homology. To check (almost) subadditivity, one has to consider several cases depending on how compact surfaces with one boundary component supporting representatives for x and y sit in the infinite genus surface. In all cases the union of these two subsurfaces can be enlarged slightly, without increasing the genus, to obtain a compact surface with one boundary component supporting a representative for xy. The genus of this surface is at most $q(x) + q(y)$. Therefore q is in fact subadditive and a quasi-norm.

The argument for $Aut_\infty(F)$ and $S\,\mathrm{Aut}_\infty(F)$ is completely analogous. Q.E.D.

Acknowledgements: I am very grateful to all the colleagues with whom I discussed the subject matter of this paper, including in particular N. A'Campo, D. Burago, D. Calegari, K. Fujiwara, V. Markovic, S. Matsumoto, S. Morita, L. Polterovich, P. Py and T. Tsuboi. Special thanks are due to P. Py for an inspiring conversation about the work of Burago, Ivanov and Polterovich, which got me started in the first place.

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