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Remarks on the faithfulness of the Jones representations

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Abstract.

We consider the linear representations of the mapping class group of an n–punctured 2–sphere constructed by V. F. R. Jones using Iwahori–Hecke algebras of type A. We show that their faithfulness is equivalent to that of certain related Iwahori–Hecke algebra representation of Artin's braid group of $n-1$ strands. In the case of $n=6$, we provide a further restriction for the kernel using our previous result, as well as a certain relation to the Burau representation of degree 4.

§1. Introduction

A linear representation of a group is said to be faithful if it is injective as a homomorphism of the group into the corresponding group of linear transformations. In the seminal paper [7], V. F. R. Jones constructed a family of linear representations of $\mathcal{M}_0^n$, the mapping class group of an n–punctured 2–sphere with one parameter. Each of the family is obtained as a modification of the Iwahori–Hecke algebra representation of B_n, Artin's braid group of n strands, provided with a *rectangular* Young diagram with n boxes. Except for the trivial cases, which correspond to the Young diagrams $[1^n]$ or $[n]$, it remains open whether these Jones representations of $\mathcal{M}_0^n$ are faithful or not. It might be therefore possible that some of these Jones representations gives a *naturally defined faithful* linear representation of $\mathcal{M}_0^n$, which seems missing even after the work of Korkmaz [11], and Bigelow–Budney [3], who independently constructed a faithful linear representation of $\mathcal{M}_0^n$ as an *induced* representation of a faithful representation of a certain subgroup of finite index defined by modifying the Lawrence–Krammer representation of B_{n-1}, the faithfulness of which was established by the celebrated works of Bigelow [2] and Krammer [12]. On the other hand, if there exists a nontrivial element in

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the kernel of a Jones representation of $\mathcal{M}_0^n$, probably of infinite order, it might be expected to give a nontrivial knot with the same value of the Jones polynomial as the unknot.

In this note, we show, for arbitrary $n \geq 6$, that the faithfulness of the Jones representation of $\mathcal{M}_0^n$ corresponding to the rectangular Young diagram Y is equivalent to that of the Iwahori–Hecke algebra representation of the braid group B_{n-1} which corresponds to the *unique* Young diagram obtained from Y by removing a single box. We note that the faithfulness problem for the latter representation also remains open. Furthermore, in the case of $n = 6$, we apply our previous result [8] to obtain a restriction for the kernel of the Jones representation of $\mathcal{M}_0^6$ in terms of the mapping class group of genus 2 via the Birman–Hilden theory. The case $n < 6$ is mentioned in Remark 3.2 (1), and its details are discussed elsewhere [10].

§2. Preliminaries

We fix some notation and briefly recall necessary material to describe Jones' construction.

2.1. Mapping class groups of genus 0

Let D^2 be a 2–disk, P the set of distinct n points $p_1, p_2, \ldots, p_n$ in $\operatorname{Int} D^2$. We call P the set of "punctures". We denote by D_n the pair of spaces (D^2, P). The n–strand braid group B_n is defined as $\pi_0 \operatorname{Homeo}^+(D_n; \partial D_n)$, the mapping class group of D_n. Namely, it is the group of the orientation preserving homeomorphisms of D_n which restrict to the identity on the boundary, modulo the isotopy in the same class of homeomorphisms.

We choose a simple closed curve in $\operatorname{Int} D^2$ so that the disk component of its complement intersects with P at $P_0 = \{p_1, \ldots, p_{n-1}\}$. Identifying the bounding 2–disk with D^2 again, the inclusion defines $D_{n-1} \hookrightarrow D_n$, which induces an injective homomorphism

$$i : B_{n-1} \to B_n$$

by extending the homeomorphisms with the identity on $D_n \setminus D_{n-1}$.

Next, capping off a 2–disk on ∂D^2, we obtain a 2–sphere S^2. We choose a single point p_{n+1} in the interior of the added disk, and set $P_+ = P \cup \{p_{n+1}\}$. As in the case of D^2, we set $S_n = (S^2, P)$ and $S_{n+1} = (S^2, P_+)$, and define their mapping class groups as $\mathcal{M}_0^n = \pi_0 \operatorname{Homeo}^+(S_n)$ and $\mathcal{M}_0^{n+1} = \pi_0 \operatorname{Homeo}^+(S_{n+1})$, respectively. We denote by $\mathcal{M}_0^{n+1}(p_{n+1})$ the subgroup of $\mathcal{M}_0^{n+1}$ consisting of those mapping classes which fix p_{n+1}. By "forgetting" p_{n+1}, we obtain the natural surjective homomorphism

$$p : \mathcal{M}_0^{n+1}(p_{n+1}) \to \mathcal{M}_0^n$$

The inclusion $D_n \hookrightarrow S_{n+1}$ induces the homomorphism $j_n : B_n \to \mathcal{M}_0^{n+1}$ by again extending the homeomorphism with the identity.

Proposition 2.1. *For $n \geq 2$, there exists the following short exact sequence:*

$$0 \to \mathbb{Z} \to B_n \xrightarrow{j_n} \mathcal{M}_0^{n+1}(p_{n+1}) \to 1 \tag{1}$$

Here, the image of $\mathbb{Z}$ in B_n is generated by the single element D_∂, the Dehn twist about the simple closed curve parallel to the boundary, and coincides with Center(B_n) *if $n \geq 3$. If $n = 2$,* Center(B_2) *coincides with B_2 and the image of $\mathbb{Z}$ is an index 2 subgroup of* Center(B_2).

Remark 2.2. In the case of $n \geq 3$, this proposition is nothing but [3, Lemma 2.2]. In the case of $n = 2$, the proposition follows from the fact that $\mathcal{M}_0^3$ is isomorphic to the symmetric group of three letters.

Finally, we denote by $k : B_n \to \mathcal{M}_0^n$ the homomorphism induced by the inclusion $D_n \hookrightarrow S_n$, and obtain the commutative diagram:

$$\begin{tikzcd} & B_n \arrow[d, "j"'] \arrow[dr, "k"] & \\ B_n/\langle D_\partial \rangle \arrow[r, equal] & \mathcal{M}_0^{n+1}(p_{n+1}) \arrow[r, "p"'] & \mathcal{M}_0^n \end{tikzcd}$$

where $\langle D_\partial \rangle$ denotes the subgroup generated by D_∂.

2.2. Iwahori–Hecke algebra representations of B_n

We denote by $H(q,n)$ the Iwahori–Hecke algebra of type A_{n-1}. As its ground ring, we take $\mathbb{Q}(q)$, the quotient field of the polynomial ring $\mathbb{Q}[q]$ with q an indeterminate. Let σ be the right-handed half twist about an arbitrary embedded arc in IntD^2 joining two distinct points of P such that no interior points of the arc intersect with P. Such σ is unique up to conjugation in B_n. Then $H(q,n)$ can be defined as the quotient of the group ring $\mathbb{Q}(q)[B_n]$ of B_n by the two–sided ideal generated by the single element $(\sigma - 1)(\sigma + q)$. Hence the projection defines a natural multiplication–preserving mapping $B_n \to H(q,n)$. From any representation of $H(q,n)$, this mapping gives rise to a representation of B_n.

Here, we summarize the facts necessary in this note about $H(q,n)$. For details, we refer to [15].

Proposition 2.3. *As a $\mathbb{Q}(q)$–algebra, $H(q,n)$ is semisimple. Furthermore, all the irreducible representations of $H(q,n)$ are in one-to-one correspondence with all the Young diagrams with n boxes.*

Let Y be a Young diagram with n boxes. We denote by V_Y the representation space of the corresponding representation of $H(q,n)$, and by $\pi_Y : B_n \to \mathrm{GL}(V_Y)$ the representation obtained from V_Y as above. The identity element of $\mathrm{GL}(V_Y)$ will be denoted by I. As usual, we adopt the notation that $\pi_{[n]}$ denotes the one–dimensional scalar representation $\sigma \mapsto q \cdot I$, which coincides with the notation of [7]. Then the representation $\pi_{[1^n]}$ is the one–dimensional representation defined by $\sigma \mapsto -I$. We call this representation sgn.

Proposition 2.4. *Let Y be an arbitrary Young diagram with n boxes. We denote all the distinct Young diagrams obtained from Y by removing a single box by $Y_0^1, Y_0^2, \dots, Y_0^s$. Then the representation of B_{n-1} obtained as the composition of π_Y with the injection $i : B_{n-1} \to B_n$ is equivalent to the direct sum $\pi_{Y_0^1} \oplus \pi_{Y_0^2} \oplus \cdots \oplus \pi_{Y_0^s}$ as representations over $\mathbb{Q}(q)$.*

Next, let $\mathfrak{S}_n$ denote the permutation group of n letters, and $\nu : B_n \to \mathfrak{S}_n$ the homomorphism induced by the permutation of the puncture set P.

Proposition 2.5. *Let Y be a Young diagram with n boxes. Then the specialization of the representation π_Y of B_n at $q = 1$ descends via ν to the irreducible representation of $\mathfrak{S}_n$ over $\mathbb{Q}$ which corresponds to the same Young diagram Y.*

2.3. Jones' construction.

Let Y be an arbitrary Young diagram with n boxes, and π_Y the corresponding irreducible representation of B_n with the representation space V_Y. We denote by d $(= d_Y)$ the dimension of V_Y over $\mathbb{Q}(q)$. According to the analysis in [7], the Dehn twist D_∂ along the boundary curve is mapped under π_Y to the scalar $q^{rn(n-1)/d}$. Here, r is a certain non-negative integer defined as $\operatorname{rank}(I + \pi_Y(\sigma))$. A precise combinatorial description of r can be found in [7]. Important here is the fact that r is equal to 0 if and only if $Y = [1^n]$. We also remark that $rn(n-1)/d$ is always an integer.

We now adjust π_Y by rescaling so that the image of D_∂ becomes trivial, appealing to the well-known fact that the abelianization of B_n is $\mathbb{Z}$. For that purpose, we need the formal power $q^{1/d}$. To avoid confusion, we introduce another indeterminate corresponding to $q^{1/d}$. Let t be another indeterminate. We consider the $\mathbb{Q}$–algebra homomorphism

$\mathbb{Q}(q) \to \mathbb{Q}(t)$ defined by $q \mapsto t^d$. This homomorphism naturally gives rise to a structure of a $\mathbb{Q}(q)$–algebra on $\mathbb{Q}(t)$. From now on, we consider every representation over $\mathbb{Q}(q)$ also as that over $\mathbb{Q}(t)$ by coefficient extension.

We denote the representation obtained as the composition of the abelianization of B_n with the mapping $m \in \mathbb{Z} \mapsto (t^{-r})^m$ by

$$\alpha : B_n \to \mathrm{GL}(\mathbb{Q}(t)).$$

Clearly, α is trivial if and only if $r = 0$, *i.e.*, if and only if $Y = [1^n]$. We now consider the representation $\alpha \otimes_{\mathbb{Q}(q)} \pi_Y$. Its representation space is $M_Y = \mathbb{Q}(t) \otimes_{\mathbb{Q}(q)} V_Y$ which is a d-dimensional vector space over $\mathbb{Q}(t)$. Since the abelianization maps D_∂ to $n(n-1) \in \mathbb{Z}$, we have $\alpha \otimes_{\mathbb{Q}(q)} \pi_Y(D_\partial) = I$. Therefore, by Proposition 2.1, the representation $\alpha \otimes_{\mathbb{Q}(q)} \pi_Y$ descends via j_n to that of $\mathcal{M}_0^{n+1}(p_{n+1})$. The condition that this representation further descends to that of $\mathcal{M}_0^n$ is given in [7]:

Proposition 2.6. *Via the homomorphism $k : B_n \to \mathcal{M}_0^n$, the representation $\alpha \otimes_{\mathbb{Q}(q)} \pi_Y$ descends to that of $\mathcal{M}_0^n$ if and only if Y is* rectangular.

For a rectangular Young diagram Y with n boxes, we call the representation of $\mathcal{M}_0^n$ given by Proposition 2.6 the Jones representation of the n–punctured sphere corresponding to Y, and denote it by $\bar{\pi}_Y : \mathcal{M}_0^n \to \mathrm{GL}(M_Y)$, with $M_Y = V_Y \otimes \mathbb{Q}(t)$.

§3. The faithfulness of $\bar{\pi}_Y$

With the preparation above, we can now state our main result.

Theorem 3.1. *Suppose that Y is a rectangular Young diagram with n boxes, $\neq [n]$, $[1^n]$. Let Y_0 be the* unique *Young diagram obtained from Y by removing a single box. Assume further that $n \geq 6$. Then the representation*

$$\bar{\pi}_Y : \mathcal{M}_0^n \to \mathrm{GL}(M_Y)$$

is faithful if and only if the representation

$$\pi_{Y_0} : B_{n-1} \to \mathrm{GL}(V_{Y_0})$$

is faithful.

Remark 3.2. (1) For $n < 6$, there exists only a single case where the other assumptions of Theorem 3.1 on Y hold: $n = 4$, $Y = [2,2]$, and $Y_0 = [2,1]$. In this case, the theorem is not true. In fact, on the one hand, π_{Y_0} is the reduced Burau representation of B_3, and is classically known to be faithful [14]. On the other hand, we can verify that the kernel of $\bar{\pi}_Y$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and coincides with the kernel of the natural homomorphism $\mathcal{M}_0^4 \to \mathrm{PSL}(2,\mathbb{Z})$ described in [1]. In another word, $\bar{\pi}_{[2,2]}$ descends to a faithful representation of $\mathrm{PSL}(2,\mathbb{Z})$. We also note $\mathrm{PSL}(2,\mathbb{Z}) \cong B_3/\operatorname{Center}(B_3)$. We discuss the details in [10].

(2) If n is prime, there exist no Young diagrams satisfying the assumptions of the theorem. Therefore, we cannot expect that the Jones representations here would provide faithful representations of B_n for *all* n, different from the representations obtained from the Lawrence–Krammer representations.

Proof of Theorem 3.1. Via the injective homomorphism $i : B_{n-1} \to B_n$, we consider B_{n-1} as a subgroup of B_n. Since Y_0 is the unique Young diagram obtained from Y by removing a single box, the restriction of π_Y to B_{n-1} coincides with π_{Y_0} by Proposition 2.4. Hence we have the following commutative diagram:

$$\begin{tikzcd}
 & B_{n-1} \arrow[dl, "j_{n-1}"'] \arrow[d, "i"] \arrow[ddr, bend left, "\alpha_0 \otimes \pi_{Y_0}"] & \\
 & B_n \arrow[d, "k"] \arrow[dr, "\alpha \otimes \pi_Y"] & \\
\mathcal{M}_0^n(p_n) \arrow[r] & \mathcal{M}_0^n \arrow[r, "\bar{\pi}_Y"'] & \mathrm{GL}(M_Y)
\end{tikzcd}$$

where α_0 denotes the restriction of α to B_{n-1}.

Suppose now that $\bar{\pi}_Y$ is faithful. For $x \in B_{n-1}$, $\bar{x}$ denotes the corresponding element $j_{n-1}(x)$ in $\mathcal{M}_0^n(p_n)$. If $\pi_{Y_0}(x) = 1$, then

$$\bar{\pi}_Y(\bar{x}) = \pi_{Y_0}(x) \cdot \alpha_0(x) = \alpha(x) \cdot I.$$

Therefore, by the faithfulness of $\bar{\pi}_Y$, $\bar{x}$ lies in the center of $\mathcal{M}_0^n$. On the other hand, by Gillette–Buskirk [5], it is known that the center of $\mathcal{M}_0^n$ is trivial. Hence, we have $\bar{x} = 1$, that is, $x \in \operatorname{Ker} j_{n-1}$. Now, by Proposition 2.1 for $n-1$, $x \in \operatorname{Center}(B_{n-1})$. Then the faithfulness of π_{Y_0} reduces to its faithfulness on $\operatorname{Center}(B_{n-1})$, which can be seen as follows. Since $Y \neq [1^n]$, we can see that α, and therefore α_0, is non-trivial, as noted in Section 2.3. Since α_0 factors through the abelianization of B_{n-1}, it is faithful on $\operatorname{Center}(B_{n-1})$. Together with the fact that $\alpha_0 \otimes \pi_{Y_0}$ is trivial on $\operatorname{Ker} j_{n-1} = \operatorname{Center}(B_{n-1})$, this implies that π_{Y_0} is faithful on $\operatorname{Center}(B_{n-1})$. This completes the proof that π_{Y_0} is faithful.

Suppose next that π_{Y_0} is faithful. Assume that $f \in \mathcal{M}_0^n$ and $\bar{\pi}_Y(f) = 1$. By Proposition 2.5, the specialization of $\bar{\pi}_Y$ at $t = 1$ descends to the irreducible representation of the symmetric group $\mathfrak{S}_n$ corresponding to Y. By the assumption $Y \neq [1^n]$, $[n]$, we can see that this specialization is a faithful representation of $\mathfrak{S}_n$. In fact, since $n \geq 5$, it is a classical fact that the alternating group $\mathfrak{A}_n$ is a simple finite group. Therefore, the kernel of the specialization is either $\mathfrak{S}_n$, $\mathfrak{A}_n$, or $\{1\}$. But since $Y \neq [1^n]$, $[n]$, we can conclude that the kernel is not $\mathfrak{S}_n$ nor $\mathfrak{A}_n$. Hence the specialization is faithful on $\mathfrak{S}_n$. Therefore, the permutation of the set of punctures P induced by f is trivial. In particular, we have $f \in \mathcal{M}_0^n(p_n)$. Hence, there exists some $x \in B_{n-1}$ such that $\bar{x} = f$. Then we have

$$\alpha(x) \cdot \pi_{Y_0}(x) = \bar{\pi}_Y(f) = I.$$

So, we have $\pi_{Y_0}(x) = \alpha(x)^{-1} \cdot I$, and then by the assumption that π_{Y_0} is faithful, $x \in \mathrm{Center}(B_{n-1})$. Hence, $f = \bar{x} = 1 \in \mathcal{M}_0^n$. Therefore, $\mathrm{Ker}\,\bar{\pi}_Y = \{1\}$. This completes the proof of Theorem 3.1. Q.E.D.

Remark 3.3. In the course of the proof above, the use of the result in [5], to show $x \in \mathrm{Center}(B_{n-1})$ under the assumption that $\bar{\pi}_Y$ be faithful and $\pi_{Y_0}(x) = 1$, is not necessary as follows. Since $\bar{x} \in \mathrm{Center}(\mathcal{M}_0^n)$, it holds for any $\tau \in B_{n-1}$ that $[\bar{x}, \bar{\tau}] = 1$ in $\mathcal{M}_0^n$. Hence $[x, \tau] \in \mathrm{Ker}\, j_{n-1} = \mathrm{Center}(B_{n-1})$. On the other hand, we see $\pi_{Y_0}[x, \tau] = [\pi_{Y_0}(x), \pi_{Y_0}(\tau)] = [1, \pi_{Y_0}(\tau)] = 1$. Since we have already seen the faithfulness of π_{Y_0} on $\mathrm{Center}(B_{n-1})$, we have $[x, \tau] = 1$ for any $\tau \in B_{n-1}$, *i.e.*, $x \in \mathrm{Center}(B_{n-1})$.

§4. Hyperelliptic mapping class group

In order to consider the case $n = 6$ further, we need to consider the Jones representation of $\mathcal{M}_0^n$, with n even, as a representation of hyperelliptic mapping class group, as follows. Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let $\mathcal{M}_g$ be its mapping class group. The hyperelliptic mapping class group $\mathcal{H}_g$ of genus g is defined as the subgroup of $\mathcal{M}_g$ consisting of those elements which commute with the class of a fixed hyperelliptic involution $\iota : \Sigma_g \to \Sigma_g$. We identify the quotient orbifold of Σ_g by the action of ι with S_{2g+2} where the singular locus of the orbifold corresponds to P, the set of punctures. Due to Birman–Hilden [4], the natural projection $\Sigma_g \to S_{2g+2}$ induces the following short exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{H}_g \xrightarrow{q} \mathcal{M}_0^{2g+2} \to 1$$

where the image of $\mathbb{Z}/2\mathbb{Z}$ in $\mathcal{H}_g$ is generated by the class of ι. Provided with an arbitrarily *rectangular* Young diagram Y with $(2g+2)$ boxes, we obtain, following Jones [7], the representation of $\mathcal{H}_g$

$$\rho_Y : \mathcal{H}_g \to \mathrm{GL}(M_Y)$$

as the composition of $\bar{\pi}_Y$ with the above homomorphism q. As a corollary to Theorem 3.1, we have:

Corollary 4.1. $\operatorname{Ker}\rho_Y \cong \mathbb{Z}/2\mathbb{Z}$ *if and only if* π_{Y_0} *is faithful.*

§5. The case of $g = 2$

In the case of $g = 2$, it is classically known that $\mathcal{H}_2 = \mathcal{M}_2$. Also, because of the "column-row symmetry", the representation of $\mathcal{M}_2$ obtained as above is essentially unique (c.f. [8, Section 2.2]). So, we take $Y = [2^3]$, and consider the representation

$$\rho = \rho_{[2^3]} : \mathcal{M}_2 \to \mathrm{GL}(M_{[2^3]}).$$

We can now say a little bit about $\operatorname{Ker}\rho$, and hence about $\operatorname{Ker}\bar{\pi}_{[2^3]}$. Let

$$\rho_0 : \mathcal{M}_2 \to \mathrm{Sp}(4, \mathbb{Z})$$

be the symplectic representation induced by the action of $\mathcal{M}_2$ on the homology group $H_1(\Sigma_2; \mathbb{Z})$. The Torelli group $\mathcal{I}_2$ is defined as $\operatorname{Ker}\rho_0$.

Theorem 5.1. $\operatorname{Ker}\rho = \mathbb{Z}/2\mathbb{Z} \oplus (\operatorname{Ker}\rho \cap \mathcal{I}_2)$. *Here,* $\mathbb{Z}/2\mathbb{Z}$ *is generated by the class of the hyperelliptic involution* ι. *In particular,* ρ *is* faithful on $\mathcal{I}_2$ *if and only if* $\pi_{[2^2,1]}$ *is a faithful representation of* B_5.

By the definition $\rho = \bar{\pi}_{[2^3]} \circ q$, we have $\operatorname{Ker}\bar{\pi}_{[2^3]} = q(\operatorname{Ker}\rho)$. Therefore, Theorem 5.1 implies immediately a restriction of the kernel for the case $n = 6$:

Corollary 5.2. $\operatorname{Ker}\bar{\pi}_{[2^3]} \subset q(\mathcal{I}_2) \cong \mathcal{I}_2$.

Proof of Theorem 5.1. It suffices to prove the isomorphism in the first part of the theorem since the rest of the theorem is then a direct consequence of Corollary 4.1. We construct the isomorphism directly. Set $K = \operatorname{Ker}\rho$, and $K_0 = K \cap \mathcal{I}_2$. Note that ι lies in K by the definition of ρ. We then define a mapping $h_0 : \mathbb{Z} \oplus K_0 \to K$ by $h_0(a, x) = \iota^a \cdot x$. Since $\iota^2 = 1$, h_0 descends to a well-defined mapping

$$h : \mathbb{Z}/2\mathbb{Z} \oplus K_0 \to K.$$

Hereafter, the multiplication in $\mathbb{Z}/2\mathbb{Z}$ is written additively, and each element of $\mathbb{Z}/2\mathbb{Z}$ will be denoted by its representative in $\mathbb{Z}$, so that $1 + 1 = 0$ in $\mathbb{Z}/2\mathbb{Z}$. Since ι lies in Center $(\mathcal{M}_2)$, the mapping h is actually a homomorphism of group. Suppose next that $h(a,x)=1$, *i.e.*, $\iota^a\cdot x=1$. Then, by taking the image under ρ_0, we have $(-I)^a\cdot\rho_0(x)=I$. Since $\rho_0(x)=I$, we have $(-I)^a=1$. Therefore, we have $a=0$ in $\mathbb{Z}/2\mathbb{Z}$, and hence $x=1$. This shows that the homomorphism h is injective.

Next, to prove that h is surjective, we consider the specialization of ρ at $t=-1$. By our previous result [8], this specialization is trivial on $\mathcal{I}_2$ and can be described as follows. Let sgn denote the one-dimensional representation of $\mathcal{M}_2$ which sends the Dehn twist along every non-separating simple closed curve to -1. It is easy to see that sgn is trivial on $\mathcal{I}_2$, and therefore descends to a representation of $\mathrm{Sp}(4,\mathbb{Z})$, denoted by $\overline{\mathrm{sgn}}$. Now, let λ denote the linear representation of $\mathrm{Sp}(4,\mathbb{Z})$ which is induced by the natural action on $\Lambda^2 H_1(\Sigma_2;\mathbb{Z})/\omega\mathbb{Z}$ where ω denotes the symplectic class in $\Lambda^2 H_1(\Sigma_2;\mathbb{Z})$. Then the specialization of ρ at $t=-1$ is equivalent to $(\overline{\mathrm{sgn}}\otimes\lambda)\circ\rho_0$ ([8, Lemma 2.1]).

Claim 5.3. As a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$, the kernel of $\overline{\mathrm{sgn}}\otimes\lambda$ coincides with $\{\pm I\}$.

This claim implies the surjectivity of h as follows. Note that we have an obvious relation

$$K\subset \mathrm{Ker}\,(\text{the specialization of } \rho \text{ at } t=-1)=\mathrm{Ker}\,((\overline{\mathrm{sgn}}\otimes\lambda)\circ\rho_0).$$

Taking the images of the both ends under ρ_0, we have $\rho_0(K)\subset\{\pm I\}$ by the claim. On the other hand, it is easy to see that $\rho_0(\iota)=-I$. Therefore, recalling that $\iota\in K$ again, we have the equality $\rho_0(\mathrm{Ker}\,\rho)=\{\pm I\}$.

Now, let z be an arbitrary element of K. Then, we have either $\rho_0(z)=I$, or $\rho_0(z)=-I$. In the case of $\rho_0(z)=I$, we have $z\in H_0$ and hence $z=h(0,z)$. In the case of $\rho_0(z)=-I$, take $x=\iota z$. We then have $\rho_0(x)=\rho_0(\iota)\cdot\rho_0(x)=(-I)^2=I$, and hence $x\in H_0$. Since $z=\iota x$, we have $z=h(1,x)$. This shows that h is surjective. Therefore, we have proven that h is an isomorphism. This finishes the proof of Theorem 5.1 Q.E.D.

We now prove Claim 5.3 to complete the proof of Theorem 5.1. Most essential is Lemma 5.4 below, the proof of which is postponed until Appendix. It is clear that $\{\pm I\}\subset\mathrm{Ker}\,(\overline{\mathrm{sgn}}\otimes\lambda)$. To show the converse, suppose $X\in\mathrm{Ker}\,(\overline{\mathrm{sgn}}\otimes\lambda)$. By the definition of $\overline{\mathrm{sgn}}$, we see that $\lambda(X)$ is equal to either I or $-I$. On the other hand, it is easy to see that $\lambda(-I)=I$, and thus the representation λ descends to a representation of $\mathrm{PSp}(4,\mathbb{Z})=\mathrm{Sp}(4,\mathbb{Z})/\{\pm I\}$, denoted by

$$\bar\lambda:\mathrm{PSp}(4,\mathbb{Z})\to\mathrm{GL}(V).$$

Here, V denotes $\Lambda^2 H_1(\Sigma_2;\mathbb{Z})/\omega\mathbb{Z}$. Then, for the element $\overline{X} \in \mathrm{PSp}(4,\mathbb{Z})$ corresponding to X, we have $\bar{\lambda}(\overline{X}) \in \mathrm{Center}\,(\mathrm{GL}(V))$. Now the lemma is in order.

Lemma 5.4. (1) $\mathrm{PSp}(4,\mathbb{Z})$ *has no nontrivial centers.*
(2) *As a representation of* $\mathrm{PSp}(4,\mathbb{Z})$, $\bar{\lambda}$ *is faithful.*

Then the part (2) of the lemma implies that $\overline{X}$ lies in Center ($\mathrm{PSp}(4,\mathbb{Z})$). Next, by the part (1), we have $\overline{X} = I$ in $\mathrm{PSp}(4,\mathbb{Z})$. Hence, we have $X \in \{\pm I\}$. This completes the proof of Claim 5.3. Q.E.D.

Remark 5.5. By the celebrated theorem of Mess [16], the Torelli group $\mathcal{I}_2$ is an infinitely generated free group. Therefore, we can see, by Corollary 5.2, that $\mathrm{Ker}\,\bar{\pi}_{[2^3]}$ is, if non-trivial, a free group. On the other hand, our previous results on the non-triviality of $\rho_{[2^3]}$ [8, 9] could be considered as showing the non-triviality of $\bar{\pi}_{[2^3]}$, and hence of $\pi_{[2^2,1]}$.

Remark 5.6. It seems quite difficult to determine whether or not the representation $\pi_{[2^2,1]}$, and therefore $\rho|_{\mathcal{I}_2}$, is faithful. As an example, let us consider the restriction of $\pi_{[2^2,1]}$ to B_4. By Proposition 2.4, this restriction decomposes as $\pi_{[2,1^2]} \oplus \pi_{[2,2]}$. Then, by a result of Long [13], this direct sum is faithful if and only if either one of the summands is faithful. On the other hand, it is easy to see that the representation $\pi_{[2,2]}$ is *not faithful* since $\pi_{[2,2]}$ can be expressed as the composition of $\pi_{[2,1]}$ with a certain homomorphism $B_4 \to B_3$ with non-trivial kernel. Therefore, we can see that $\pi_{[2^2,1]}$ is faithful on B_4 if and only if $\pi_{[2,1^2]}$ is faithful. Now, it is well-known that $\pi_{[2,1^2]}$ is equivalent to the tensor product of sgn and the reduced Burau representation of B_4 (see [7, Note 5.7]). One can easily check that tensoring sgn does not affect the kernel, and so we can conclude that $\pi_{[2^2,1]}$ is faithful on B_4 if and only if the reduced Burau representation of B_4 is faithful. In particular, the unfaithfulness of the reduced Burau representation of B_4 implies the unfaithfulness of $\pi_{[2^2,1]}$ and $\rho|_{\mathcal{I}_2}$. We note that the reduced Burau representation of B_4 is the only representation among all the reduced Burau representations the faithfulness of which remains open.

§ Appendix—the proof of Lemma 5.4

It seems that Lemma 5.4 is well-known to experts, but because we are unable to provide any suitable reference in the literature, we include a proof here. We will use some well-known properties of $\mathrm{PSp}(4,\mathbb{Z}/p\mathbb{Z}) = \mathrm{Sp}(4,\mathbb{Z}/p\mathbb{Z})/\{\pm I\}$, for the details of which we refer to [6]. Let p be a prime number. Taking the mod p reduction $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ for each matrix entry, we obtain a homomorphism of group

$$k_p : \mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/p\mathbb{Z}).$$

Since $k_p(-I) = -I$, k_p induces a homomorphism $\bar{k}_p : \mathrm{PSp}(4, \mathbb{Z}) \to \mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z})$. Note that $\mathrm{Ker}\, k_p$ consists of those matrices in $\mathrm{Sp}(4, \mathbb{Z})$ for which every diagonal entry is equal to 1 mod p and every off-diagonal entry is equal to 0 mod p. Therefore, we can observe that the intersection of $\mathrm{Ker}\, k_p$'s, when p varies among infinitely many arbitrary primes, consists of the single matrix I. It is also well-known that $\mathrm{Sp}(4, \mathbb{Z}/p\mathbb{Z})$ is generated by transvections, and every transvection in $\mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z})$ comes from the one in $\mathrm{Sp}(4, \mathbb{Z})$ via k_p. Therefore, k_p and hence $\bar{k}_p$ are surjective.

Now we prove the first part of the lemma. For each $X \in \mathrm{Sp}(4, \mathbb{Z})$, we denote by $\overline{X}$ the corresponding element of $\mathrm{PSp}(4, \mathbb{Z})$. Clearly, every element of $\mathrm{PSp}(4, \mathbb{Z})$ has an expression of this form. Let us choose an arbitrary element in $\mathrm{Center}\,(\mathrm{PSp}(4, \mathbb{Z}))$ and denote it by $\overline{Z}$ with $Z \in \mathrm{Sp}(4, \mathbb{Z})$. Since the homomorphism $\bar{k}_p$ is surjective, $\bar{k}_p(\overline{Z})$ lies in $\mathrm{Center}\,(\mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z}))$. We appeal to the following classical theorem.

Proposition A. *The finite group* $\mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z})$ *is simple for every prime* $p \geq 3$.

In particular, for $p \geq 3$, $\mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z})$ has no nontrivial centers. Therefore, we have $\bar{k}_p(\overline{Z}) = I$ in $\mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z})$. In other words, we have $k_p(Z) = I$, or $-I$ for $p \geq 3$. Therefore, either one of Z or $-Z$ is contained in infinitely many $(\mathrm{Ker}\, k_p)$ s'. Then the observation above implies that $Z = I$, or $-I$. In either case, we have $\overline{Z} = I$ in $\mathrm{PSp}(4, \mathbb{Z})$. This proves the first part of the lemma.

Next, we proceed to (2). Recall that the representation $\bar{\lambda}$ of $\mathrm{PSp}(4, \mathbb{Z})$ is induced by the natural action λ of $\mathrm{Sp}(4, \mathbb{Z})$ on the free abelian group $V = \Lambda^2 H_1(\Sigma_2; \mathbb{Z})/\omega\mathbb{Z}$, via the symplectic representation ρ_0. Let V_p denote $V \otimes \mathbb{Z}/p\mathbb{Z}$. We consider the representation $\lambda \otimes 1_{\mathbb{Z}/p\mathbb{Z}} : \mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{GL}(V_p)$. Here, $1_{\mathbb{Z}/p\mathbb{Z}}$ denotes the identity on $\mathbb{Z}/p\mathbb{Z}$. Clearly, we have $\mathrm{Ker}\, k_p \subset \mathrm{Ker}\,(\lambda \otimes 1_{\mathbb{Z}/p\mathbb{Z}})$, and thus the representation $\lambda \otimes 1_{\mathbb{Z}/p\mathbb{Z}}$ descends to that of $\mathrm{Sp}(4, \mathbb{Z}/p\mathbb{Z})$, denoted by

$$\lambda_p : \mathrm{Sp}(4, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{GL}(V_p).$$

Furthermore, recalling that $\lambda(-I) = I$ in $\mathrm{GL}(V)$, we see that λ_p descends to a representation of $\mathrm{PSp}(4, \mathbb{Z}/p\mathbb{Z})$ denoted by $\bar{\lambda}_p$. It is easy to check that $\bar{\lambda}_p$ is nontrivial for arbitrary prime p, for instance by computing the image under $\lambda_p \circ k_p \circ \rho_0$ of the Dehn twist along any non-separating simple closed curve. Then, by Proposition A again, $\bar{\lambda}_p$

is a faithful representation of $\mathrm{PSp}(4,\mathbb{Z}/p\mathbb{Z})$ for $p \geq 3$. Therefore, for arbitrary $X \in \operatorname{Ker}\lambda$, we have $k_p(X) = I$, or $-I$. Now the same argument as in (1) implies $\overline{X} = I$ in $\mathrm{PSp}(4,\mathbb{Z})$. This proves (2), and hence the lemma. Q.E.D.

We remark that the same proof as above works for $\mathrm{PSp}(2g,\mathbb{Z})$ with general $g \geq 2$, and its nontrivial linear representation defined naturally on an arbitrary free abelian subquotient of the tensor product of copies of the fundamental representation.

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