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Subgroups generated by two pseudo-Anosov elements in a mapping class group. I. Uniform exponential growth

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Abstract.

Suppose G acts acylindrically by isometries on a δ-hyperbolic graph Γ. We discuss subgroups generated by two hyperbolic elements in G and give sufficient conditions for them to be free of rank two.

We apply our results to the mapping class group $\mathrm{Mod}(S)$ of a compact orientable surface S and its action on the curve graph such that S is non-sporadic. There exists a constant Q, depending only on S, with the following property. If $a, b \in \mathrm{Mod}(S)$ are pseudo-Anosovs such that $\langle a, b\rangle$ is not virtually cyclic, then there exists $M > 0$, which depends on a, b, such that either $\langle a^n, b^m\rangle$ is free of rank two for all $n \geq Q, m \geq M$, or $\langle a^m, b^n\rangle$ is free of rank two for all $n \geq Q, m \geq M$ (Theorem 3.1).

At the end we ask a question in connection to the uniformly exponential growth of subgroups in a mapping class group (Question 3.4).

§1. δ-hyperbolic geometry and the Nielsen condition

In the paper, we expect that the readers are familiar to δ-hyperbolic geometry. We give definitions and references, and describe the idea of the argument without all details for standard facts and techniques. We recommend [BrHa, III, H] as a good reference book.

1.1. Nielsen condition for free generators

A geodesic space is called *δ-hyperbolic* for $\delta \geq 0$ if for any geodesics α, β, γ which form a triangle, α is contained in the δ-neighborhood of $\beta \cup \gamma$ ([Gr]). Let Γ be a δ-hyperbolic graph. Let a be an isometry of Γ. If there exist a point $x \in \Gamma$ and a constant $C > 0$ such that $d(x, a^n(x)) \geq Cn$ for any $n > 0$, then a is called *hyperbolic*.

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Suppose a is a hyperbolic isometry. If there exists a bi-infinite geodesic α such that $a(\alpha)$ is contained in the C-neighborhood of α for some $C \geq 0$, α is called a *quasi-axis* of a. By δ-hyperbolicity of Γ, it then follows that $a(\alpha)$ is in the 2δ-neighborhood of α (use [BrHa, III.H.3.3 Lemma]). If α and β are quasi-axes of a (they are geodesics by definition), then they are contained in the 2δ-neighborhood of each other. If $C = 0$ we say α is an *axis* of a.

For two points $x, y \in \Gamma$, we may denote a (non-unique) geodesic joining them by $[x, y]$. We may write the distance between the two points as $|x - y|$. For an isometry a, we define its *translation length*, (or *stable length*), $\mathrm{tr}(a)$, by

$$\mathrm{tr}(a) = \lim_{n \to \infty} \frac{|x - a^n(x)|}{n} \geq 0$$

for a point x. It is easy to see $\mathrm{tr}(a)$ does not depend on the choice of x. Also, $\mathrm{tr}(a^n) = |n|\mathrm{tr}(a)$. The isometry a is hyperbolic iff $\mathrm{tr}(a) > 0$.

For any point $z \in \Gamma$, we have $|z - a(z)| \geq \mathrm{tr}(a)$ by the triangle inequality. If a has an axis α, then $|z - a(z)| = \mathrm{tr}(a)$ for any point $z \in \alpha$. If α is a quasi-axis of a, then $|z - a(z)| \leq \mathrm{tr}(a) + 10\delta$ (otherwise, use that $a^n(\alpha)$ is in the 2δ-neighborhood of α for any $n > 0$ and show $|z - a^n(z)| \geq n(\mathrm{tr}(a) + \delta)$, which gives a contradiction if $\delta > 0$). This last inequality will be used, sometimes implicitly since we do not always spell out all details, in the rest of the paper. We recommend interested readers who want to know all details, in particular estimates regarding δ, first to imagine $\delta = 0$ and/or quasi-axes are axes, then try to modify the estimates and the arguments.

Let $C \geq 0$ be a constant. For geodesics α and β, we define *the C-overlap*, denoted by $\alpha \cap_C \beta$, by

$$\alpha \cap_C \beta = (\alpha \cap N_C(\beta)) \cup (\beta \cap N_C(\alpha)),$$

where $N_C(\alpha)$ is the C-neighborhood of α. Let $|\alpha \cap_C \beta|$ denote the diameter of this set.

In the following argument, we often take $C = 10\delta$. By δ-hyperbolicity, if $|\alpha \cap_{10\delta} \beta|$ is finite, then the longest segment of α, the longest segment of β and the longest geodesics which are contained in $\alpha \cap_{10\delta} \beta$ all have length between $|\alpha \cap_{10\delta} \beta| - 20\delta$ and $|\alpha \cap_{10\delta} \beta| + 20\delta$, and those segments are in the 20δ-neighborhood of each other.

The following result is fundamental and classical. It has its origin in combinatorial group theory and was generalized in [Gr] to the setting of δ-hyperbolic geometry. It was used in [De] and [Ko] as an essential tool. For a proof, see [Ko, Lemma 2.4] or [Fu], where the convexity (see [FaMo]) of $\langle a^n, b^m \rangle$ is also discussed.

Proposition 1.1 (Nielsen condition). *Suppose a, b act as hyperbolic isometries on a δ-hyperbolic graph Γ with quasi-axes α, β, respectively. Suppose $|\alpha \cap_{10\delta} \beta| < \infty$. If $1 \le n, m \in \mathbb{Z}$ are such that*

$$\mathrm{tr}(a^n) \ge |\alpha \cap_{10\delta} \beta| + 100(\delta+1),\ \mathrm{tr}(b^m) \ge |\alpha \cap_{10\delta} \beta| + 100(\delta+1),$$

then a^n and b^m freely generate a rank-two free subgroup in $\mathrm{Isom}(\Gamma)$.

1.2. Quasi-geodesic axis

The existence of quasi-axes, which are geodesics by our definition, is not strictly necessary for Proposition 1.1 or for some other results in this paper. Indeed, we can use certain quasi-geodesics as discussed in section 2.2. As we do not use this for our main application to mapping class groups, uninterested readers may skip this section. First, a path α, parametrized by arclength, is called a (K, ε)-*quasi-geodesic* for $0 < K \le 1$ and $0 \le \varepsilon$ if for all t, s we have $K|t-s| - \varepsilon \le d(\alpha(t), \alpha(s))$. (A standard definition of quasi-geodesics also requires $d(\alpha(t), \alpha(s)) \le |t-s|/K + \varepsilon$ but this is trivially satisfied since α is parametrized by arclength.)

If a is a hyperbolic isometry of a δ-hyperbolic graph Γ, there exists a (K, ε)-quasi-geodesic α for some K, ε such that

(1) $a^n(\alpha)$ and α are in the 30δ-neighborhood of each other for any n. (Namely, α is almost invariant by a.)
(2) Let $p, q \in \alpha$. Then the subpath of α between p, q and any geodesic $[p, q]$ are in the 10δ-neighborhood of each other.

We call such a path α a *quasi-geodesic axis* of a in this paper. To be precise, we should use the term *quasi-geodesic quasi-axis*, but we make it shorter. One can easily show from (1) and (2) that any two quasi-geodesic axes of a are in the 30δ-neighborhood of each other. Note that (2) concerns only the path and not the element a. Also, the quasi-geodesic constants of α are not important for our purpose. What is useful for us is (2).

We briefly review how to find a quasi-geodesic axis for a. Fix a point $x \in \Gamma$. Choose $I > 0$ such that $\mathrm{tr}(a^I) \ge 1000\delta$. Set $y = a^I(x)$. Then, $|x - y| \ge 1000\delta$. Choose $N > 0$ such that $|a^N(x) - x| \ge 100|a^I(x) - x|$. Let m be the mid point of a geodesic $[x, a^N(x)]$. Define a path, which is invariant by a^I, by

$$\alpha = \cup_{n\in\mathbb{Z}} a^{nI}([m, a^I(m)]).$$

Since a is hyperbolic, α is a quasi-geodesic. Observe that α trivially satisfies (1) for the element a^I (not for the element a yet). We claim that α also satisfies (2). Firstly, by the way we chose I and N, most parts of the geodesics, except for a small portion near each end, $[x, a^N(x)]$

and $[y, a^N(y)] = a^I([x, a^N(x)])$ are in the 2δ-neighborhood of each other. This is because $|x - a^N(x)|$ is much larger than $|x - y|$. (Draw a geodesic rectangle joining $x, a^N(x), a^N(y)$ and y in this order. Then the rectangle is narrow.) Also, $|m - a^I(m)| \geq \mathrm{tr}(a^I) \geq 1000\delta$. Those two estimates imply (2) for α by δ-hyperbolic geometry. Note that $|m - a^I(m)|$ is much smaller than $|x - m| = |m - a^N(x)|$. (See Figure 1.)

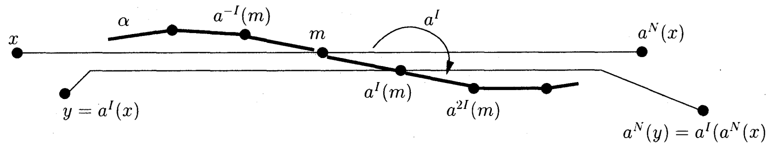

Fig. 1. Two geodesics $[x, a^N(x)]$ and $[y, a^N(y)]$ stay 2δ- close for the most part. The geodesic $[m, a^I(m)]$ is "pinched" between those two geodesics.

Now we claim that α satisfies (1) for the element a, namely, it is a quasi-geodesic axis for a. Let $L > 0$ be a large integer, which we decide later. By (2), the geodesic $[m, a^{LI}(m)]$ and the subpath α' of α between the two points m and $a^{LI}(m)$ are in the 10δ-neighborhood of each other. Therefore, the geodesic $a([m, a^{LI}(m)])$ and $a(\alpha')$ are also in the 10δ-neighborhood of each other. On the other hand, by δ-hyperbolic geometry, most parts of $[m, a^{LI}(m)]$ and $a([m, a^{LI}(m)])$ are in the 2δ-neighborhood of each other if $|m - a^{LI}(m)|$ is much larger than $|m - a(m)|$. (This follows from the same argument using a narrow geodesic rectangle as before.) We choose L this way. As a consequence, most parts of α' and $a(\alpha')$, except for a small segment at each end, are in the 30δ-neighborhood of each other. Replacing L by a larger integer, we find that α and $a(\alpha)$ are in the 30δ-neighborhood of each other. A similar argument works for $a^n(\alpha)$ for all n. This proves that α satisfies (1) for a.

§2. Acylindricity and free subgroups

2.1. Free subgroups

Suppose G acts on Γ. Bowditch [Bo] defined that the action of G is *acylindrical* if for any $R > 0$, there exist $K(R), L(R) \geq 1$ such that for any vertices $x, y \in \Gamma$ with $d(x, y) \geq L$, the following set has at most K elements.

$$\{g \in G | d(x, g(x)) \leq R, d(y, g(y)) \leq R\}.$$

We show one lemma (see Lemma 2.5).

Lemma 2.1. *Suppose G acts on a δ-hyperbolic graph Γ. If the action is acylindrical, then there exists an integer $P \geq 1$ such that $\mathrm{tr}(a^P) \geq 1$ for any element $a \in G$ which acts hyperbolically on Γ with a quasi-axis.*

Proof. If $\delta = 0$, then Γ is a tree. Therefore $\mathrm{tr}(a) \geq 1$. Set $P = 1$. Suppose $\delta > 0$. Fix a constant $R \geq 100\delta$. Let α be a (geodesic) quasi-axis of a. Take a point $x \in \alpha$. Let $y \in \alpha$ be a point with $|x - y| \geq L(2R)$. If $|a^i(x) - x| \leq R$ for some i, then $|a^i(y) - y| \leq 2R$. This is because $a^i(\alpha)$ is in the 2δ-neighborhood of α. Therefore, by the acylindricity, there is some I, with $1 \leq I \leq K(2R)$, such that $|a^I(x) - x| > R$. Since x lies on a quasi-axis of a, it then follows that $|a^{In}(x) - x| > n(R - 10\delta)$ for any $n \geq 1$. To verify this estimate, imagine first that the geodesic α is exactly invariant by a, namely, an axis. Then, clearly, $|a^{In}(x) - x| > nR$. Now, try to estimate the error terms using that α is only a quasi-axis. We leave the details to readers.

This implies that

$$\mathrm{tr}(a) \geq \frac{R - 10\delta}{I} \geq \frac{R - 10\delta}{K(2R)}.$$

Take P such that $P \geq \frac{K(2R)}{R-10\delta}$. Q.E.D.

The following lemma is a generalization of a result by Koubi ([Ko, Lemma 5.4]). He discusses the case such that the action of G is (uniformly) proper and $\mathrm{tr}(a) = \mathrm{tr}(b)$. The commutator of two elements is defined by

$$[f, g] = f^{-1}g^{-1}fg.$$

Lemma 2.2. *Let Γ be a δ-hyperbolic graph and G a group acting acylindrically on Γ with constants $K(R), L(R)$. Suppose $a, b \in G$ act hyperbolically with quasi-axes $\alpha, \beta \subset \Gamma$, respectively.*

(1) *If $a^n b \neq b a^n$ for all $n \neq 0$ or $b^n a \neq a b^n$ for all $n \neq 0$, then*

$$|\alpha \cap_{10\delta} \beta| < 4PK(20\delta)L(20\delta)\max(\mathrm{tr}(a), \mathrm{tr}(b)) + 100\delta,$$

where P is the constant from Lemma 2.1.

(2) *If $\mathrm{tr}(a) = \mathrm{tr}(b)$ and for all $n \neq 0$, $a^n \neq b^{\pm n}$, then we have the same inequality as above.*

Proof. 1. To argue by contradiction, assume that the inequality is false. It suffices to show that $a^n b = b a^n$ for some $n \neq 0$ and also $b^n a = a b^n$ for some $n \neq 0$. Set $K = K(20\delta), L = L(20\delta)$. For concreteness, suppose $\mathrm{tr}(b) \leq \mathrm{tr}(a)$. By our assumption, since $|\alpha \cap_{10\delta} \beta|$ is much larger than 2δ, the set $\alpha \cap_{10\delta} \beta$ looks like a narrow tube.

Let $\ell \subset \alpha$ be the longest segment which is contained in $\alpha \cap_{10\delta} \beta$. Then, by our assumption, $|\ell| \geq 4PKL\mathrm{tr}(a) + 80\delta$. Take a point $p \in \ell$ such that the following points are in $N_{2\delta}(\ell)$:

$$p, a(p), a^2(p), \cdots, a^{4PKL}(p).$$

(See Figure 2. For simplicity, we put those points on ℓ in the figure.) To see that we can take such a point p, as usual, first imagine that α is invariant by a. Then most parts of ℓ and $a(\ell)$ coincide, therefore, one can take p such that all the above points are on ℓ. Now, in general, most parts of ℓ and $a(\ell)$ are in the 2δ-neighborhood of each other by δ-hyperbolic geometry, hence a required point p exists.

Set

$$x = a^{PKL}(p),\ y = a^{2PKL}(p).$$

Since $y = a^{PKL}(x)$, it follows from Lemma 2.1 that $d(x, y) \geq PKL\mathrm{tr}(a) \geq KL \geq L$.

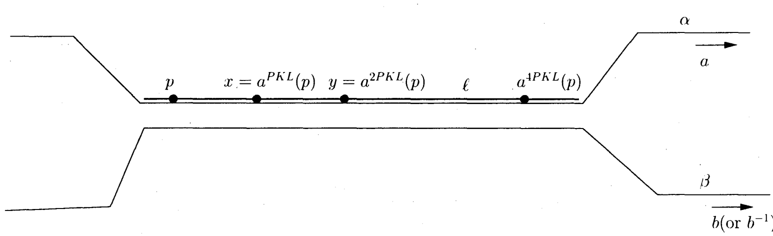

Fig. 2. Apply the acylindricity to the pair x, y.

Claim. For each $i, (1 \leq i \leq PKL)$,

$$d(x, [b, a^i](x)) \leq 20\delta,\ d(y, [b, a^i](y)) \leq 20\delta.$$

We first consider the special case that $\delta = 0$, namely, Γ is a tree. Then, $\alpha \cap_{10\delta} \beta$ coincides the segment $\alpha \cap \beta$, and also the segment ℓ, therefore, all above points $a^n(p), 1 \leq n \leq 4PKL$, are in $\alpha \cap \beta$. We want to show $x = [b, a^i](x)$, but this is obvious since when we apply a^i, b, a^{-i}, then b^{-1} to x, the point moves within ℓ. Thus, $[b, a^i](x) = x$. If $\delta > 0$, we can show that the point moves in the 10δ-neighborhood of ℓ when we apply a^i, b, a^{-i} followed by b^{-1} to x. Therefore, we get $d(x, [b, a^i](x)) \leq 20\delta$ by estimating the error terms from the tree case using triangle inequality. We leave the details to readers. (See Figure 3.) We can show $d(y, [b, a^i](y)) \leq 20\delta$ in the same way. This proves the claim.

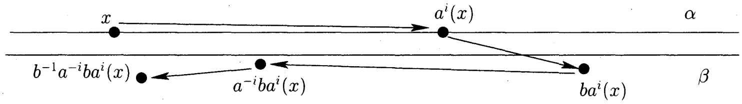

Fig. 3. How commutators act near $\alpha \cap_{10\delta} \beta$.

Since $|x - y| \geq L = L(20\delta)$, by the acylindricity of the action, it follows from the claim that there are at most K distinct elements in the set $[b, a^i], (1 \leq i \leq PKL)$. By the pigeon-hole principle, $[b, a^i] = [b, a^j]$ for some $i \neq j, (1 \leq i, j \leq PKL)$. It follows that $a^i b^{-1} a^{-i} = a^j b^{-1} a^{-j}$, therefore, $a^{i-j} b^{-1} = b^{-1} a^{i-j}$. We get $[b, a^n] = 1$ for some $n \neq 0$. The same argument applies to the elements $[a, b^i]$ since $\mathrm{tr}(b) \leq \mathrm{tr}(a)$, therefore we also get $[a, b^n] = 1$ for some $n \neq 0$ as well.

2. This is similar to 1. Assume that the inequality is false. Take a segment $\ell \subset \alpha$ as before. Then, as we said, the set $\alpha \cap_{10\delta} \beta$ looks like a narrow tube. Therefore, it makes sense to talk about the direction of the action by a and b along this tube, and furthermore, the direction of a coincides with the direction of one of b or b^{-1}. We did not need this consideration in 1 since we used commutators. Now, take points $p, x, y \in \ell$ as before and apply the same argument to the set of elements $\{b^n a^n : 1 \leq n \leq PKL\}$ (if the action of a, b along ℓ have the opposite direction) or $\{b^{-n} a^n : 1 \leq n \leq PKL\}$ (if the actions have the same direction). Then we conclude that there must be $n \neq m$ with $b^n a^n = b^m a^m$, or $b^{-n} a^n = b^{-m} a^m$, respectively. This is a contradiction. Q.E.D.

As a consequence, a generalization of [Ko, Proposition 5.5] follows.

Proposition 2.3. *Let Γ be a δ-hyperbolic graph, and G a group acting acylindrically by isometries on Γ. Then there exists a constant $N > 0$, which depends on δ and the set of the acylindricity constants $K(R), L(R)$, with the following property.*

(1) *Suppose $a, b \in G$ act by hyperbolic isometries with quasi-axes such that $a^n b \neq b a^n$ for all $n \neq 0$, or $b^n a \neq a b^n$ for all $n \neq 0$. Then there exists a constant $M > 0$, which depends on a, b, such that either $\langle a^n, b^m \rangle$ is free of rank two for all $n \geq N, m \geq M$, or $\langle a^m, b^n \rangle$ is a free group of rank two for all $n \geq N, m \geq M$.*

(2) *Suppose $a \in G$ acts by a hyperbolic isometry with a quasi-axis. Let $c \in G$ be such that for any $n > 0$, $c a^n c^{-1} \neq a^{\pm n}$. Then the subgroup $\langle a^n, c a^m c^{-1} \rangle$ is free of rank two for all $n, m \geq N$.*

Proof. Let $K = K(20\delta), L = L(20\delta)$ be the acylindricity constants evaluated at 20δ. Let P be the constant from Lemma 2.1. Set

$$N = 4PKL + 200P(\delta + 1).$$

1. Let α, β be quasi-axes of a, b, respectively. Suppose $\mathrm{tr}(b) \le \mathrm{tr}(a)$. Set $M = N\frac{\mathrm{tr}(a)}{\mathrm{tr}(b)}$. We will show that a^n, b^m generate a free group of rank two if $n \ge N, m \ge M$.

By Lemma 2.2 (1), we have

$$|\alpha \cap_{10\delta} \beta| < 4PKL\mathrm{tr}(a) + 100\delta.$$

It follows that if $n \ge N, m \ge M$ then

$$\mathrm{tr}(a^n) \ge |\alpha \cap_{10\delta} \beta| + 100(\delta + 1),\ \mathrm{tr}(b^m) \ge |\alpha \cap_{10\delta} \beta| + 100(\delta + 1)$$

by the way we chose N, M, P. Note that the first inequality follows from our assumption and

$$\begin{aligned} \mathrm{tr}(a^n) \ge N\mathrm{tr}(a) &= 4PKL\mathrm{tr}(a) + 200P(\delta + 1)\mathrm{tr}(a) \\ &\ge 4PKL\mathrm{tr}(a) + 200(\delta + 1). \end{aligned}$$

The second inequality also holds by the way we chose M. Now we can apply Proposition 1.1 to a^n, b^m. If $\mathrm{tr}(b) \ge \mathrm{tr}(a)$, then set $M = N\frac{\mathrm{tr}(b)}{\mathrm{tr}(a)}$ and argue in the same way switching the roles of a and b.

2. Let α be a quasi-axis of a. Put $b = cac^{-1}$. Then $\beta = c\alpha$ is a quasi-axis of b and $\mathrm{tr}(a) = \mathrm{tr}(b)$. By the hypothesis, for all $n \ne 0$, $a^n \ne b^{\pm n}$. By Lemma 2.2 (2), we have

$$|\alpha \cap_{10\delta} \beta| < 4PKL\mathrm{tr}(a) + 100\delta.$$

We then argue in the same way as the previous case. Since $\mathrm{tr}(a) = \mathrm{tr}(b)$, $M = N$ in the previous argument, therefore, the subgroup generated by a^n, b^m is free of rank two for all $n, m \ge N$. Q.E.D.

It is more difficult to analyze a subgroup normally generated by two, or even one, elements. See an interesting paper by T. Delzant [De] which contains positive results and a warning example.

2.2. Hyperbolic isometries without quasi-axes

Although we have been assuming that hyperbolic isometries have quasi-axes, this assumption is not necessary for our purpose if we only assume the acylindricity of the action. In this section, we state results without assuming quasi-axes for potential application in the future since the existence of quasi-axes is a strong assumption. Uninterested readers may skip this section as it is unnecessary for our application to mapping class groups.

Our main goal is to drop the assumption on quasi-axes from Proposition 2.3 as follows.

Proposition 2.4. *Suppose that G acts acylindrically by isometries on a δ-hyperbolic graph Γ. Then there exists a constant $N > 0$, which depends on δ and the set of the acylindricity constants $K(R), L(R)$, such that the conclusion of Proposition* 2.3 (1), (2) *holds without the assumption that a, b have quasi-axes.* (*This constant N is maybe larger than the one which is obtained in Proposition* 2.3).

Proof. A hyperbolic isometry always has a quasi-geodesic axis (see Section 1.2). Basically, we use quasi-geodesic axes instead of quasi-axes for hyperbolic isometries. We then adapt the original argument to the new setting. We may first need to modify the statements since some of the original ones concern quasi-axes. Namely, we restate and prove Proposition 1.1, prove Lemma 2.1, restate and prove Lemma 2.2, and finally modify the proof of Proposition 2.3.

We only outline the arguments. As for Proposition 1.1, replace quasi-axes α, β by quasi-geodesic axes α, β and also $\alpha \cap_{10\delta} \beta$ by $\alpha \cap_{1000\delta} \beta$ in all places in the statement. We did not give a proof of Proposition 1.1 and only referred to [Ko] and [Fu]. The arguments there work with minor modification using the properties (1) and (2) of the quasi-geodesic axes.

We reprove Lemma 2.1 without using quasi-axes. This result says that the existence of such P is a consequence of the acylindricity.

Lemma 2.5. *Suppose G acts by isometries on a δ-hyperbolic graph Γ. If the action is acylindrical, then there exists an integer $P \geq 1$ such that for any element $a \in G$ which acts hyperbolically on Γ, we have* $\mathrm{tr}(a^P) \geq 1$.

The proof is essentially same as for Lemma 2.1. We use a quasi-geodesic axis α instead of a quasi-axis. The key property is (2), which says that a sub-path of α is at most 10δ-close to a geodesic with the common end points with the sub-path.

Proof. We may assume $\delta > 0$. Let α be a quasi-geodesic axis for a. Set $R = 1000\delta$. Take a point $x \in \alpha$, and let $y \in \alpha$ be a point with $|x - y| \geq L(1500\delta)$. Since α is a quasi-geodesic axis, it follows that if $|a^i(x) - x| \leq R$, then $|a^i(y) - y| \leq R + 500\delta = 1500\delta$. Therefore, by acylindricity (look at the set of elements $a^i, 1 \leq i \leq K(1500\delta)$, and a pair of points x, y), there exists some I, with $1 \leq I \leq K(1500\delta)$, such that $|a^I(x) - x| > 1000\delta = R$. From this, since again α is a quasi-geodesic axis, for any $n \geq 1$, $|a^{In}(x) - x| > 500\delta n$. Now, as before, we get

$$\mathrm{tr}(a) \geq \frac{500\delta}{K(1500\delta)}.$$

Take $P \geq \frac{K(1500\delta)}{500\delta}$. Q.E.D.

Next, we modify Lemma 2.2. Namely, we replace quasi-axes α, β for a, b by quasi-geodesic axes in the assumption. The inequality in the conclusion should be replaced, for example, by

$$|\alpha \cap_{10000\delta} \beta| < 4PK(20\delta)L(20\delta)\max(\mathrm{tr}(a), \mathrm{tr}(b)) + 10000\delta,$$

where P is the constant from Lemma 2.5. The proof is same after an appropriate modification regarding constants.

Having done all this, the argument for the proposition is same as for Proposition 2.3 with minor modifications. For example, replace all $\alpha \cap_{10\delta} \beta$ by $\alpha \cap_{10000\delta} \beta$ and define the constant $N = 4PKL + 20000P(\delta+1)$, where, P is the constant from Lemma 2.5. We omit details. Q.E.D.

§3. Application to mapping class groups

3.1. Two pseudo-Anosov maps

Let S be a compact orientable surface. Following [MM], we call S *sporadic* when S is a sphere with $p \leq 4$ punctures or a torus with $p \leq 1$ puncture. Mapping class groups $\mathrm{Mod}(S)$ are already well-understood in this case. Namely (see [Iv, 9.2]), $\mathrm{Mod}(S)$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$ if S is a torus with ≤ 1 puncture; $\mathrm{Mod}(S)$ is commensurable with $\mathrm{PSL}(2, \mathbb{Z})$ when S is a sphere with 4 punctures; and $\mathrm{Mod}(S)$ is finite when S is a sphere with ≤ 3 punctures.

The following result is well-known (for example, see [Iv1]) except for a uniform bound on one of the exponents by Q.

Theorem 3.1. [1]

Let S be a compact orientable surface and $\mathrm{Mod}(S)$ *its mapping class group such that S is non-sporadic. Then there exists a constant $Q(S)$ with the following property.*

(1) *Suppose $a, b \in \mathrm{Mod}(S)$ are pseudo-Anosovs such that for all $n, m \neq 0$, $a^n b^m \neq b^m a^n$. Then the subgroup $\langle a^n, b^m \rangle$ is free of rank two*

[1] In [Fu], this result will be improved such that in (1), $\langle a^n, b^m \rangle$ is free of rank two for all $n, m \geq Q(S)$, where we may need to take a larger constant for $Q(S)$ than in Theorem 3.1.

for all sufficiently large $m > 0$ and all $n \geq Q$, or $\langle a^m, b^n \rangle$ is free of rank two for all sufficiently large $m > 0$ and all $n \geq Q$.

(2) *Suppose $a \in \mathrm{Mod}(S)$ is pseudo-Anosov and $c \in \mathrm{Mod}(S)$ is such that for any $n > 0$, $ca^nc^{-1} \neq a^{\pm n}$. Then the subgroup $\langle a^n, ca^mc^{-1} \rangle$ is free of rank two for all $n, m \geq Q$.*

Regarding the assumption on two pseudo-Anosov elements a, b in (1), it is equivalent to requiring that the subgroup generated by a, b is not *virtually* cyclic, namely, it does not contain a cyclic subgroup of finite index (see, for example, [Iv, Theorem 7.4.I]).

Proof. Let $C(S)$ be the curve graph of S (see [Iv]). $\mathrm{Mod}(S)$ acts on $C(S)$ by isometries. It is known that $C(S)$ is δ-hyperbolic, and an element $a \in \mathrm{Mod}(S)$ acts as a hyperbolic isometry if and only if it is pseudo-Anosov [MM, Theorem 1.1 and Proposition 4.6]. The action of $\mathrm{Mod}(S)$ is acylindrical, and there is a quasi-axis for a pseudo-Anosov element [Bo, Theorem 1.3 and 1.4].

Apply Proposition 2.3 to the action of $\mathrm{Mod}(S)$ on $C(S)$ and let N be the constant from the proposition. Set $Q(S) = N$.

1. We apply Proposition 2.3 (1) to a, b and obtain M. Then the claim holds for all $m \geq M$ and $n \geq Q$.

2. Apply Proposition 2.3 (2) to a and c. The claim holds for all $n, m \geq Q$. Q.E.D.

We remark that the existence of P in Lemma 2.1 for this setting was already known in [MM] before [Bo].

3.2. Exponential growth rate

Definition 3.2 (Growth rate (see [Har])). Let G be a finitely generated group. For a finite generating set A, and for an integer $n \geq 0$, let $b(G, A; n)$ be the number of elements in G whose word length in terms of A are at most n.

The *exponential growth rate* of (G, A), $\omega(G, A)$, is defined as

$$\omega(G, A) = \limsup_{n \to \infty} (b(G, A; n))^{\frac{1}{n}}.$$

The group G is said to be of *exponential growth* if $\omega(G, A) > 1$ for some (and consequently all) A.

The *minimal growth rate* of a finitely generated group G of exponential growth is defined to be

$$\omega(G) = \inf \omega(G, A),$$

where the infimum is taken over all finite generating sets A. G is said to be of *uniformly exponential growth* if $\omega(G) > 1$.

Let F_k be a free group of rank $k > 1$. By computation, $\omega(F_k, A_k) = 2k - 1$ for a free generating set A_k. It is a non-trivial fact ([Har, Proposition 13]) that

$$\omega(F_k) = 2k - 1.$$

If G is a non-elementary word-hyperbolic group, then G contains a free group of rank two as a subgroup, therefore clearly G has exponential growth. Moreover, Koubi [Ko] showed that G has uniformly exponential growth. The key result in his argument is Proposition 5.5 [Ko], which we generalized in Proposition 2.3. In his case, Γ is a Cayley graph of G, therefore the action is proper, in particular, acylindrical.

Corollary 3.3. *Let S be a compact orientable surface and* $\mathrm{Mod}(S)$ *its mapping class group such that S is non-sporadic. Let $Q(S) > 0$ be the constant from Theorem 3.1. Suppose $G <$* $\mathrm{Mod}(S)$ *is a finitely generated subgroup which is not virtually cyclic.*

Let Σ be a finite generating set of G. If Σ contains a pseudo-Anosov element a, then there is an element $c \in \Sigma$ such that the subgroup $\langle a^n, ca^m c^{-1} \rangle$ is free of rank two for all $n, m \geq Q$. In particular,

$$\omega(G, \Sigma) \geq 3^{\frac{1}{Q+2}}.$$

Proof. There must be an element $c \in \Sigma$ such that $ca^n c^{-1} \neq a^{\pm n}$ for all $n > 0$, since otherwise, Σ would generate a virtually cyclic group. Apply Theorem 3.1 (2) to a and c. Then, $\langle a^Q, ca^Q c^{-1} \rangle$ is free of rank two. The word length of the two elements $a^Q, ca^Q c^{-1}$ in terms of Σ is at most $Q+2$. Thus, the inequality on $\omega(G, \Sigma)$ follows from $\omega(F_2, A_2) = 3$, where A_2 is a free generating set. Q.E.D.

It is not clear if every finitely generated, non-virtually-abelian subgroup $G < \mathrm{Mod}(S)$ has uniformly exponential growth. [2] We ask the following question.

Question 3.4 (Short pseudo-Anosov elements)**.** Let $G < \mathrm{Mod}(S)$ be a finitely generated, non-virtually-cyclic subgroup which contains a pseudo-Anosov element. Does there exist a constant U which depends only on G, or even just on the surface S, such that if Σ is a finite generating set of G, then there is a pseudo-Anosov element $a \in G$ whose word length in terms of Σ is at most U ?

A positive answer to this question would imply $\omega(G) \geq 3^{\frac{1}{UQ+2}}$ by Corollary 3.3. It seems the answer is not known even for $G = \mathrm{Mod}(S)$,

[2] After this paper, an affirmative answer is announced by J. Mangahas in December 2007, using Theorem 3.1(2) combined with her result concerning non pseudo-Anosov generators. The question 3.4 seems to be still open.

although Mod(S) has uniformly exponential growth if Mod(S) is not virtually abelian [AAS]. This is because Mod(S) has a surjective homomorphism to $\mathrm{Aut}(H_1(\pi_1(S)), \mathbb{Z}))$, which has uniformly exponential growth as it is linear and not virtually nilpotent [EMO]. The kernel of this homomorphism is called the *Torelli* group. We do not know if this subgroup has uniformly exponential growth (see the footnote 2).

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